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ON SKEW LINEAR GROUPS ASSOCIATED WITH CERTAIN SOLUBLE-BY-FINITE GROUPS

BY

B. A. F. WEHRFRITZ AND M. SHIRVANI

ABSTRACT. Let O_1 denote the class of groups G such that every group ring of G (over a field) is an Ore domain. Several approaches to the correction of the proof of a result concerning subrings generated by certain O_1-groups are given.

In our book [4] on skew linear groups the proof of 4.4.6 is incorrect, or more precisely the proof is a correct proof, but not quite of what is claimed. Specifically in the notation of [4] we assume only that the group $G/A \in O_1$, the class of groups whose group algebras over fields are Ore domains, while the proof requires slightly more. For example it would suffice to assume that $G/A \in O$, the class of groups whose crossed products over division rings are Ore domains. Since the results of [4] Section 4.4 have not appeared in print in full elsewhere, we feel the need to publish a correction.

We give three approaches to repairing the error. The first is the most economical but it uses a non-trivial and as yet unpublished theorem of Kropholler, Linnell and Moody (see [2]), which is turn based on the recent remarkable induction theorem of Moody (for an accessible proof of the latter result see [1]). The second is the simplest but fails to keep tight control of the matrix degree, and the third uses only material from [4] and ultimately depends on the very elementary result [4], 1.4.3.

In [4] the result 4.4.6 is applied only to groups in the class $\mathcal{X}$, where a group lies in the class $\mathcal{X}$ if and only if it has a normal series

$$(*) \qquad \langle 1 \rangle = G_0 \leqq G_1 \leqq \cdots \leqq G_m \leqq G$$

of finite length such that G/G_m is torsion-free, polycyclic-by-finite, each G_{i+1}/G_i is abelian, and each G/G_i is residually torsion-free polycyclic-by-finite. All three approaches depend on utilizing a little more about the class $\mathcal{X}$ to get by with some weakened verison of [4], 4.4.6. In particular both 1. and 5. below are weakened variants of this. The question remains whether the original formulation of [4], 4.4.6 is correct. Since we know of no O_1-group that is not also an O-group, the question seems rather academic to us at present. We choose not to state the basic lemmas below in terms of

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the classes O_1 and O. In this way some of our results theoretically gain in generality. Whether they do in practise is far from clear and possibly the form of the formulation in [4] is all that will ever be required. In any event the proofs are not lengthened.

The following implies [4] 4.4.6 with $G/A \in O$.

1. *Let F be a field and G a group with an abelian normal subgroup A such that the group algebras FG and $F \cdot G/A$ are Ore domains, say with division rings of quotients D and D_1 respectively. Suppose any skew group ring of G/A over a field extension of F is a domain. Let $\mathfrak{a}$ denote the augmentation ideal of A in FA and set $Q = FG \backslash \mathfrak{a}G$. Then the natural map of FG onto $F \cdot G/A$ extends to a homomorphism φ of the subring $FG[Q-1]$ of D onto D_1.*

Proof. Necessarily A is torsion-free. There is a discrete valuation domain J with maximal ideal $\mathfrak{m}$ lying between FA and its quotient subfield of D that is normalized by G and satisfies $FA \cap \mathfrak{m} = \mathfrak{a}$, see [4], 4.4.2. Set $R = J[G] \leqq D$ and $\mathfrak{p} = \mathfrak{m}G$. Then R is a crossed product of J by G/A and $\mathfrak{p}$ is an ideal of R.

Now R is an Ore domain, e.g. by [4], 4.4.3. Also $R/\mathfrak{p}$ is a skew group ring of G/A over the field extension $J/\mathfrak{m}$ of F, so by hypothesis $R/\mathfrak{p}$ is a domain. Then the set $\mathcal{C}_R(\mathfrak{p})$ of elements regular modulo $\mathfrak{p}$ is Ore by [4] 4.4.5 and we can form the ring $S = R\mathcal{C}_R(\mathfrak{p})^{-1}$ of quotients, regarding it as a subring of D. It follows that $D_2 = S/\mathfrak{p}S$ is the division ring of quotients of $R/\mathfrak{p}$.

Clearly $Q \subseteq R \backslash \mathfrak{p} = \mathcal{C}_R(\mathfrak{p})$, so $FG[Q^{-1}] \subseteq S$. Also $F \cdot G/A$ is naturally embedded in $R/\mathfrak{p}$ and hence D_1 is embedded in D_2 in an obvious way. It is easy to see that the natural projection θ of S onto D_2 maps FG onto $F \cdot G/A$, and hence maps $FG[Q^{-1}]$ onto D_1. Let φ denote the restriction of θ to $FG[Q^{-1}]$.

2. *Assume the notation of the proof of 1. Then $\bigcap_{i=0}^{\infty}(\mathfrak{p}S)^i = \{0\}$. In particular $\bigcap_i (\ker \varphi)^i = \{0\}$.*

It follows from 2. that $GL(n, S) \cap (1_n + (\mathfrak{p}S)^{n \times n})$ are residually torsion-free nilpotent if char $F = 0$ and residually nilpotent p-groups if char $F = p > 0$, cf. [4], 4.4.7.

Proof. Certainly $\bigcap \mathfrak{m}^i = \{0\}$ and G normalizes $\mathfrak{m}$, so $\bigcap \mathfrak{p}^i = \{0\}$. Now $\mathfrak{p}S$ is an ideal of S and hence $(\mathfrak{p}S)^i = \mathfrak{p}^i S$ for $i = 1, 2, \ldots$. By [4] 4.4.4 we have $\mathcal{C}_R(\mathfrak{p}) = \mathcal{C}_R(\mathfrak{p}^i)$ and consequently $R \bigcap \mathfrak{p}^i S = \mathfrak{p}^i$. Therefore $R \cap \bigcap \mathfrak{p}^i S = \bigcap \mathfrak{p}^i = \{0\}$ and $\bigcap (\mathfrak{p}S)^i = \bigcap \mathfrak{p}^i S = (R \cap \bigcap \mathfrak{p}^i S)S = \{0\}$.

3. *With the notation above θ maps the group $U(S)$ of units of S onto D_2^*. In particular if N is a normal subgroup of D^* then $(N \cap U(S))\theta \cap D_1^*$ is a normal subgroup of D_1^*.*

Proof. $U(S)\theta \supseteq (\mathcal{C}_R(\mathfrak{p}) \cdot \mathcal{C}_R(\mathfrak{p})^{-1})\theta = ((R/\mathfrak{p})\backslash\{0\})((R/\mathfrak{p})\backslash\{0\})^{-1} = D_2^*$.

The first approach. In our notation, in [2] Kropholler, Linnell and Moody prove the following

4. (Kropholler, Linnell, Moody). $\langle P, L \rangle \mathfrak{A}\mathfrak{F} \cap \mathfrak{F}^{-S} \subseteq O$.

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In particular torsion-free, soluble-by-finite groups are $\mathcal{O}$-groups. Therefore $\mathcal{X} \subseteq \mathcal{O}$ and consequently the whole of [4] stands after replacing [4] 4.4.6 by 1. above, or more economically after replacing $G/A \in \mathcal{O}_1$ in [4] 4.4.6 by $G/A \in \mathcal{O}$ and leaving subsequent proofs unchanged. This includes [4] 4.5.8 where one uses the remark 3. above and [4] 4.4.8c) and d) where one uses 2. above. Note that the subclass $\mathfrak{S}\mathfrak{F} \cap R(\mathfrak{G} \cap \mathfrak{F}^{-S} \cap \mathfrak{N})$ of $\mathcal{X}$ considered by Lichtman in [3] is contained in $\mathcal{O}$ by the elementary result 1.4.3 of [4], so for this class [2] is not involved.

The second approach. Let G be an $\mathcal{X}$-group and consider the series (*) above. Then G/G_m contains a poly-C_∞ normal subgroup H/G_m of finite index. Then $H \in \langle P, L\rangle C_\infty \subseteq \mathcal{O}$ by [4], 1.4.3, a result very much more elementary than either [2] or [4], 1.4.23. More generally if K is a normal subgroup of G with G/K torsion-free and if $A = G_1 \cap K$, then also $H/A \in \langle P, L\rangle C_\infty \subseteq \mathcal{O}$. Hence we can apply [4] 4.4.6, or more strictly 1. above, but to H rather than G. We have then only to lift our conclusions up from a normal subgroup of finite index. Our second approach is just a simple-minded way of carrying out this lifting.

Thus let F be a field, D the division ring of quotients of FG and E the subring of D of quotients of FH. Then G normalizes E and the subring $E[G]$ of D has finite dimension over E. Specifically $E[G] = \bigotimes_{t\in T} tE$ for T any transversal of H to G and $(E[G] : E) = (G : H)$. In particular $E[G]$ is a division subring of D containing FG and therefore $D = E[G] = \bigotimes_T tE = GE = FG(FH\backslash\{0\})^{-1}$.

Since 1. above applies to H, induction on m in (*) gives information about the subgroups of $GH(n, E)$, exactly as in Section 4.4 of [4]. But if $d = (G : H)$ then $GL(n, D)$ is isomorphic to a subgroup of $GL(nd, E)$ and we obtain information about the subgroups of $GL(n, D)$. This is fine for qualitative results (for example the residual finiteness of the finitely generated subgroups of $GL(n, D)$, see [4] 4.4.8b), but does not produce the full conclusion in quantitative results (such as [4] 4.4.8a) that involve n.

The third approach. Here we generalize 1. above to cover arbitrary $\mathcal{X}$-groups, but not to the extent claimed in [4], 4.4.6.

5. *Let F be a field and G a group with an abelian normal subgroup A and a normal subgroup $H \supseteq A$ of finite index. Suppose that the four group algebras $FG \supseteq FH$ and $F \cdot G/A \supseteq F \cdot H/A$ are Ore domains with, respectively, division rings of quotients $D \supseteq E$ and $D_1 \supseteq E_1$. Assume that every skew group ring of H/A over a field extension of F is a domain. Let $\mathfrak{a}$ denote the augmentation ideal of A in FA and set $Q = FH\backslash\mathfrak{a}H$. Then the natural map of FG onto $F \cdot G/A$ extends to a homomorphism φ of the subring $FG[Q^{1}]$ of D onto D_1.*

This, and also $\bigcap_i (\ker\varphi)^i = \{0\}$, see 6. below, can be proved directly from the statement of 1., but for the proof of 7. below it will be clearer if we work from the proof of 1. from the start.

PROOF. Let T be a transversal of H to G, and let $x \longrightarrow \bar{x}$ denote the natural projection of G onto G/A. Then as above $D = E[G] = \bigoplus_{t \in T} tE = FG(FH\backslash\{0\})^{-1}$ and $D_1 = E_1[\bar{G}] = \bigoplus_t \bar{t}E_1 = F\bar{G}(F\bar{H}\backslash\{0\})^{-1}$. Choose J as in the proof of 1. and set $R_0 = J[G]$, $R = J[H]$, $\mathfrak{p}_0 = \mathfrak{m}G$ and $\mathfrak{p} = \mathfrak{m}H$. As in 1. the set $\mathcal{C}_R(\mathfrak{p})$ is Ore in R and, cf. [4] 4.5.3, we can form the rings of quotients $S = R_0\mathcal{C}_R(\mathfrak{p})^{-1}$ and $S_1 = (R_0/\mathfrak{p}_0)\cdot \mathcal{C}_{R/\mathfrak{p}}(0)^{-1}$. Further there is a natural map θ of S onto S_1 and $\ker\theta = \mathfrak{p}S = \bigoplus_t t\mathfrak{p}\mathcal{C}_R(\mathfrak{p})^{-1}$.

Clearly $FG \subseteq R_0$ and $R/\mathfrak{p}$ is a domain with $\mathfrak{p} \cap FH = \mathfrak{a}H$, so $Q \subseteq \mathcal{C}_R(\mathfrak{p})$ and $FG[Q^{-1}] \subseteq S$. Further $F \cdot G/A \subseteq R_0/\mathfrak{p}_0$ and $(F \cdot H/A)\backslash\{0\} \subseteq (R/\mathfrak{p})\backslash\{0\} = \mathcal{C}_{R/\mathfrak{p}}(0)$, so D_1 is embedded in S_1. Finally θ maps FG onto $F \cdot G/A$ in the obvious way and hence maps $FG[Q^{-1}]$ onto D_1. Let φ be the restriction of θ to $FG[Q^{-1}]$.

6. *With the notation above* $\bigcap_{i=0}^{\infty}(\mathfrak{p}S)^i = \{0\}$ *and* $\bigcap_{i=0}^{\infty}(\ker\varphi)^i = \{0\}$.

PROOF. As in the proof of 2. we have $(\mathfrak{p}\mathcal{C}_R(\mathfrak{p})^{-1})^i = \mathfrak{p}^i\mathcal{C}_R(\mathfrak{p})^{-1}$ and $\bigcap_i(\mathfrak{p}\mathcal{C}_R(\mathfrak{p})^{-1})^i = \{0\}$. Also G normalizes $\mathfrak{p}$ and $\mathcal{C}_R(\mathfrak{p})$, so $\bigcap_i(\mathfrak{p}S)^i = \bigoplus_t t\left(\bigcap_i(\mathfrak{p}\mathcal{C}_R(\mathfrak{p})^{-1}\right)^i = \{0\}$.

7. *In the notation above θ maps the group $U(S)$ of units of S onto $U(S_1)$. In particular if N is a normal subgroup of D^* then $(N \cap U(S))\theta \cap D_1^*$ is a normal subgroup of D_1^*.*

We are not claiming that $(U(FG[Q^{-1}]))\varphi = D_1^*$.

PROOF. By [4] 4.5.3, see final paragraph on page 157, we have $U(S)\theta = U(S_1)$. The result follows.

8. *Let $G \in \mathcal{X}$. Choose H as in the second approach above. Then G and H satisfy* 5. *above for any field F. Since $D = FG \cdot (FH\backslash\{0\})^{-1}$ we can choose the denominator r of* [4], *page* 147, *to lie in $FH\backslash\{0\}$ instead of just in $FG\backslash\{0\}$. Choose K and A as on page* 147 *of* [4]. *Then $FH \cap \mathfrak{a}G = AH$, $r \in Q = FH\backslash\mathfrak{a}H$, and* [4], 4.4.1 *follow from* 5. *above. The remaining results of* [4] *involving $\mathcal{X}$ follows exactly as in* [4] , *with* [4], 4.4.8c *and* d) *using* 6. *and* [4], 4.5.8 *using* 7.

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Queen Mary College
London E1 4NS
U.K.

University of Alberta
Edmonton T6G 2G1
Canada