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CHAMP: a Cherednik algebra MAGMA package

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Abstract

We present a computer algebra package based on MAGMA for performing computations in rational Cherednik algebras with arbitrary parameters and in Verma modules for restricted rational Cherednik algebras. Part of this package is a new general Las Vegas algorithm for computing the head and the constituents of a module with simple head in characteristic zero, which we develop here theoretically. This algorithm is very successful when applied to Verma modules for restricted rational Cherednik algebras and it allows us to answer several questions posed by Gordon in some specific cases. We can determine the decomposition matrices of the Verma modules, the graded G-module structure of the simple modules, and the Calogero–Moser families of the generic restricted rational Cherednik algebra for around half of the exceptional complex reflection groups. In this way we can also confirm Martino's conjecture for several exceptional complex reflection groups.

Supplementary materials are available with this article.

Contents

Introduction

Based on the computer algebra system MAGMA [**6**] we developed a package, called CHAMP, which provides an environment for performing computations in rational Cherednik algebras as introduced by Etingof and Ginzburg [**9**] and in Verma modules for restricted rational Cherednik algebras as introduced by Gordon [**17**]. It is freely available at http://thielul.github.io/CHAMP/ and consists of around 16 000 lines of code at the moment. It is designed to be highly flexible so that it is possible to work with arbitrary parameters (including indeterminates of a rational function field and thus covering the generic setting), with arbitrary reflection groups over arbitrary fields (including fields of positive characteristic as long as all reflections are diagonalizable), and with arbitrary realizations of the irreducible representations of the reflection groups (see § 8). The development of this package was motivated by questions posed

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by Gordon [**17**, § 7] (see § 2.4) and by Martino's conjecture [**25**] (see § 2.6), which relates Calogero–Moser families with Rouquier families coming from Hecke algebras (see [**7**, **24**], and [**8**]). For *exceptional* complex reflection groups almost nothing was known about this. Using the theoretical methods developed here and their implementation in Champ, we can make significant progress (see § 7 for a summary and the online supplementary material available from the publisher's website for all results).

In § 1, we introduce rational Cherednik algebras over general base rings and deduce the Poincaré–Birkhoff–Witt (PBW) theorem in this generality by using properties of rewrite systems. We discuss an efficient algorithm for performing computations in these algebras, that is, for expressing products in the PBW basis. This has been implemented in Champ and allows us for example to explicitly compute Poisson brackets, which have a variety of applications (see [**5**]). In § 2, we discuss an efficient algorithm for computing Verma modules for restricted rational Cherednik algebras. This allows us to construct and handle Verma modules even of dimension around 3000 in Champ. As there is so far no algorithm capable of decomposing such high-dimensional modules over a field of characteristic zero, it is one of the central advances in this article that we theoretically develop a very general strategy for doing this (see §§ 4–6). We say 'strategy' here, as our theory yields a so-called Las Vegas algorithm, meaning that it does not have to be successful but if it is we get the correct result. We have implemented this algorithm, with a lot of technical extensions, in Champ. Our idea is to use *finite field specializations* (which are compositions of decomposition morphisms in the sense of Geck and Rouquier [**15**]) to transport the modules to an algebra over a finite field (see § 4), then apply the MeatAxe [**20**], and use a method for *reconstructing* the head of the original module: the latter is the essential part of our approach (see § 5) and culminates in an algorithm we call ModFinder. To apply this algorithm to Verma modules for restricted rational Cherednik algebras we first have to ensure the existence of 'integral structures' of these algebras. This is an interesting theoretical problem which has not been considered before. In § 4, we develop some theory around this problem and present an algorithmic partial (but for us sufficient) solution. Despite the uncertainty in the success of this algorithm, it turned out to be extremely efficient and successful for Verma modules. Namely, we are able to compute for all the exceptional complex reflection groups

$$\mathrm{G}_4, \mathrm{G}_5, \mathrm{G}_6, \mathrm{G}_7, \mathrm{G}_8, \mathrm{G}_9, \mathrm{G}_{10}, \mathrm{G}_{12}, \mathrm{G}_{13}, \mathrm{G}_{14}, \mathrm{G}_{15}, \mathrm{G}_{16}, \mathrm{G}_{20}, \mathrm{G}_{22}, \mathrm{G}_{23} = \mathrm{H}_3, \mathrm{G}_{24}$$

the decomposition matrices of the Verma modules, the structure of the simple modules as graded G-modules, and the Calogero–Moser families of the associated generic restricted rational Cherednik algebra, and thus the answers to Gordon's questions in these cases†. Nothing was known about this before. Moreover, we confirm in this way the generic part of Martino's conjecture for these groups. As Champ was designed to handle arbitrary parameters (including generic points of subschemes), we are also able to do the same for *all* parameters for the groups G_4, G_{12}, G_{13}, G_{20}, G_{22}, and $\mathrm{G}_{23} = \mathrm{H}_3$, and confirm the complete form of Martino's conjecture in these cases. For the groups G_4, G_6, G_8, G_{13}, G_{14}, and G_{20}, we furthermore give an explicit description of the 'exceptional locus', which could not be determined so far. It coincides precisely with the union of Chlouveraki's essential hyperplanes of cyclotomic Hecke algebras [**8**], except for G_8, where we surprisingly have one additional 'exceptional' hyperplane (this was discovered before by Bonnafé using entirely different methods).

All results are listed explicitly in tabular form in the online supplementary material available from the publisher's website and are easily accessible from within Champ for future work (see § 8.5). In § 7 we summarize them along with some observations.

We hope that our package and our results will enable us to better understand problems about rational Cherednik algebras, like the precise connection between Calogero–Moser families and

†The reader should check the web site http://thielul.github.io/CHAMP/ and [**33**] for further results obtained after publication of this article.

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Rouquier families, and the recent Calogero–Moser cell conjecture by Bonnafé and Rouquier [5]. We expect that our method for computing the heads and decomposition matrices of Verma modules can be applied to many more examples outside of rational Cherednik algebras.

1. *Computing in rational Cherednik algebras*

We start by reviewing rational Cherednik algebras (see also [5, 9, 17], and [32]) in this section and explain how they can be treated computationally. Instead of the complex numbers as base rings, we consider a very general setup here to be able to treat generic parameters algebraically and to introduce analogous problems with modular reflection groups. We argue that the PBW theorem follows in this generality from the fact that there exists a terminating confluent rewrite system for rational Cherednik algebras. As a by-product, this formalizes an algorithm for computing in these algebras and proves its correctness.

1.1. *Rational Cherednik algebras*

Throughout, we fix a field K and a finite reflection group $\Gamma := (G, V)$ over K. This means that G is a non-trivial finite group, V is a finite-dimensional faithful KG-module, and G is generated by the set Ref_Γ of elements $s \in G$ which act as reflections on V, that is, those elements whose fixed space $\mathrm{H}_s := \mathrm{Ker}(\mathrm{id}_V - s)$ is of codimension one. We denote the action of $g \in G$ on $v \in V$ by ${}^g v$. For $s \in \mathrm{Ref}_\Gamma$, we denote by $\alpha_s^\vee$ a *root* of s, that is, a non-zero element of $\mathrm{Im}(\mathrm{id}_V - s)$, and by α_s we denote a *coroot* of s, that is, an element of V^* whose kernel is equal to H_s. Both roots and coroots of reflections are unique up to scalars and our constructions will not depend on their choice.

We assume that all reflections in G are diagonalizable. This is equivalent to $\langle \alpha_s^\vee, \alpha_s \rangle \neq 0$ for all $s \in \mathrm{Ref}_\Gamma$, where $\langle \cdot , \cdot \rangle$ is the canonical pairing between V and V^*. As all reflections in Γ are of finite order, this is certainly satisfied if Γ is *non-modular*, that is, if the characteristic of K is coprime to the order of G. In the modular case, the general orthogonal groups in their natural representation in case K is of characteristic not equal to 2, the symmetric group S_n in the representation attached to the partition $(n-1, 1)$ in case K is of characteristic not equal to 2, and some modular reductions of exceptional complex reflection groups satisfy this property, for example (see [32]).

In addition to Γ, we furthermore fix a commutative K-algebra R, an element $t \in R$, and a map $c : \mathscr{C}_\Gamma \to R$ from the set $\mathscr{C}_\Gamma$ of conjugacy classes of reflections of Γ to R. The *rational Cherednik algebra* of Γ in (t, c) is defined as the quotient $\mathrm{H}_{t,c}$ of $R\langle V \oplus V^* \rangle \rtimes RG$ by the ideal $\mathrm{I}_{t,c}$ generated by the relations

$$[x, x'] = 0 \quad \text{for all } x, x' \in V^*, \tag{1.1}$$

$$[y, y'] = 0 \quad \text{for all } y, y' \in V, \tag{1.2}$$

and

$$[y, x] = t\langle y, x \rangle + \sum_{s \in \mathrm{Ref}_\Gamma} (y, x)_s c(s) s \quad \text{for all } x \in V^*, y \in V, \tag{1.3}$$

where

$$(y, x)_s := \frac{\langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle}{\langle \alpha_s^\vee, \alpha_s \rangle} \in K. \tag{1.4}$$

Here, we denote by $R\langle V \rangle$ the tensor algebra of V^* over R and by $R[V]$ we denote the symmetric algebra of V^*, that is, the quotient of $R\langle V \rangle$ by the ideal generated by the elements $xx' - x'x$ for $x, x' \in V^*$. Furthermore, $R\langle V \oplus V^* \rangle \rtimes RG$ denotes the semi-direct product of the tensor algebra of $V^* \oplus V$ over R with the group algebra over R. As we assumed that all reflections are diagonalizable, we have $\langle \alpha_s^\vee, \alpha_s \rangle \neq 0$, so that the last relation is always well defined. Note that it is also independent of the choice of the roots and coroots.

1.2. *The PBW theorem*

Let $\mathbf{y} := (y_i)_{i=1}^n$ be a basis of V with dual basis $\mathbf{x} := (x_i)_{i=1}^n$. We denote by F_n the set of finite sequences $\alpha := (\alpha_1, \ldots, \alpha_l)$ in $[1, n] := \{1, \ldots, n\}$ and define for such a sequence the expression $\mathbf{x}_\alpha := \prod_{i=1}^l x_{\alpha_i} \in R\langle V\rangle$. Then $(\mathbf{x}_\alpha)_{\alpha \in \mathrm{F}_n}$ is an R-basis of $R\langle V\rangle$. An R-basis of $R[V]$ is formed by the elements $\mathbf{x}^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$ with $\alpha \in \mathbb{N}^n$. The choice of a basis provides us with a natural R-linear section of the quotient morphism $R\langle V\rangle \twoheadrightarrow R[V]$ by mapping $\mathbf{x}^\alpha \in R[V]$ to $\prod_{i=1}^n x_i^{\alpha_i} \in R\langle V\rangle$. In the same way, we have a natural R-linear section of $R\langle V^*\rangle \twoheadrightarrow R[V^*]$. As an R-module, the semi-direct product $R\langle V \oplus V^*\rangle \rtimes RG$ is isomorphic to $R\langle V \oplus V^*\rangle \otimes_R RG$. The two sections above can thus be put together to yield an R-linear section $s_\mathbf{y}$ of the quotient morphism $R\langle V \oplus V^*\rangle \rtimes RG \twoheadrightarrow R[V \oplus V^*] \rtimes RG$. The image $N_\mathbf{y}$ of $s_\mathbf{y}$ is the free R-submodule of $R\langle V \oplus V^*\rangle \rtimes RG$ with basis $\mathbf{x}^\alpha \mathbf{y}^\beta g$ and we get a commutative diagram

$$\begin{tikzcd}
 & N_\mathbf{y} \arrow[d, hook, dashed] & \\
 & R\langle V \oplus V^*\rangle \rtimes RG \arrow[dl, two heads] \arrow[dr, two heads] & \\
R[V \oplus V^*] \rtimes RG \arrow[uur, bend left, dashed, "s_\mathbf{y}", "\cong"'] \arrow[rr, dashed, "\pi"'] & & \mathrm{H}_{t,c}
\end{tikzcd}$$

where the dashed arrows are morphisms of R-modules only and π is the composition of $s_\mathbf{y}$ with the quotient morphism. This morphism is actually independent of the choice of $\mathbf{y}$ and is called the *PBW morphism*. It is clear from the relations (1.1) and (1.2) that π is surjective, so that the elements $\mathbf{x}^\alpha \mathbf{y}^\beta g$ generate $\mathrm{H}_{t,c}$ as an R-module. The essence of the *PBW theorem* for rational Cherednik algebras is that π is in fact an isomorphism (equivalently, the restriction of the quotient morphism $R\langle V \oplus V^*\rangle \rtimes RG \twoheadrightarrow \mathrm{H}_{t,c}$ to $N_\mathbf{y}$ is injective for one, and then any, basis $\mathbf{y}$). Hence, the elements $\mathbf{x}^\alpha \mathbf{y}^\beta g$ with $\alpha, \beta \in \mathbb{N}^n$ form an R-basis of $\mathrm{H}_{t,c}$. We call such a basis a *PBW basis*. One sometimes prefers to use that $R[V \oplus V^*] \rtimes RG$ is as an R-module isomorphic to $R[V] \otimes_R RG \otimes_R R[V^*]$, so that we have a triangular decomposition of $\mathrm{H}_{t,c}$ and a basis of the form $\mathbf{x}^\alpha g \mathbf{y}^\beta$. This fact is used in § 2.3.

The PBW theorem was originally proven by Etingof and Ginzburg [**9**] in the case $K = R = \mathbb{C}$. Their proof, however, seems to be not easily extendable to our general setting. Ram and Shepler [**28**] instead gave a proof in the same case, which is formalized and extended in [**32**]. The advantage of this approach is not only that it can be adapted to give a proof of the PBW theorem over general base rings but that it also provides the theoretical foundation of our computational approach to rational Cherednik algebras. To explain this, let us first formalize the role of $N_\mathbf{y}$ in the PBW theorem.

1.3. *Normal forms and rewrite systems*

DEFINITION 1. Let A be an algebra over a commutative ring R and let $I \trianglelefteq A$ be an ideal. A *weak normal form* of A/I is an R-submodule $N \subseteq A$ such that any element of A is modulo I equivalent to an element of N, that is, the restriction $\pi|_N$ of the quotient morphism $\pi : A \twoheadrightarrow A/I$ to N is still surjective. For $a \in A$, we call the elements in $\mathscr{N}_N(a) := \pi|_N^{-1}(\pi(a)) = \pi^{-1}(\pi(a)) \cap N$ the *normal forms* of a with respect to N, and similarly we define $\mathscr{N}_N(\overline{a}) := \pi|_N^{-1}(\overline{a}) = \pi^{-1}(\overline{a}) \cap N$ for $\overline{a} \in A/I$. If every element of A has a unique normal form with respect to N, that is, $\pi|_N : N \twoheadrightarrow A/I$ is an isomorphism of R-modules, we say that N is a *normal form* of A/I.

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Finding a normal form for a quotient of a (commutative) polynomial ring by an ideal is one of the central problems of computational commutative algebra and it can be solved via Gröbner bases, as explained in the following example.

EXAMPLE 1. Let $A := K[\mathbf{X}]$ be the polynomial ring over a field K in the variables $\mathbf{X} := (X_i)_{i=1}^n$. Let $\prec$ be a monomial order on A. Let $I \trianglelefteq A$ be an ideal and let $G := \{g_1, \ldots, g_s\}$ be a Gröbner basis of I with respect to $\prec$, that is, $\mathrm{LT}(I) = \mathrm{LT}(G)$, where $\mathrm{LT}(-)$ denotes the ideal generated by the leading terms. Let

$$C(I) := \{\mathbf{X}^\alpha \mid \alpha \in \mathbb{N}^n \text{ and } \mathbf{X}^\alpha \text{ is not divisible by some } \mathrm{LT}(g) \text{ for } g \in G\} \subseteq A.$$

Then $N_I := \langle C(I) \rangle_K \subseteq A$ is a normal form of A/I (see [**11**, § 1.2]).

We can reformulate the PBW theorem as stating that the R-submodule $N_{\mathbf{y}}$ is a normal form for $\mathrm{H}_{t,c} = (R\langle V \oplus V^* \rangle \rtimes RG)/\mathrm{I}_{t,c}$. We will show this by proving that there exists a terminating confluent rewrite system having $N_{\mathbf{y}}$ as the set of normal forms. To this end, let us first recall some basic notions about rewrite systems (see [**4**]).

DEFINITION 2. A *rewrite system* is a pair $\mathscr{A} := (A, \to)$ consisting of a set A and a binary relation $\to$ on A. We write $a \to b$ if $(a, b) \in \to$. This relation is called the *rewrite relation* of $\mathscr{A}$. The reflexive–transitive closure of $\to$ is denoted by $\twoheadrightarrow$. An element $a \in A$ is *reducible* if there is some $b \in A$ with $a \neq b$ and $a \to b$. Otherwise it is called *irreducible* (or in *normal form*). A *normal form* of an element $a \in A$ is an irreducible element $b \in A$ with $a \twoheadrightarrow b$. We denote by $\mathscr{N}_{\mathscr{A}}(a)$ the set of normal forms of a. The rewrite system $\mathscr{A}$ is (*uniquely*) *normalizing* if every element $a \in A$ has a (unique) normal form. It is called *terminating* if there does not exist an infinite chain $a_1 \to a_2 \to \ldots$. It is called *locally confluent* if

$$\text{for all } a, b, c \in A(c \leftarrow a \to b \Rightarrow \exists\, d \in A(c \twoheadrightarrow d \twoheadleftarrow b)).$$

This condition is precisely the commutativity of the diagram

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[dr, dashed, two heads] & & \bullet \arrow[dl, dashed, two heads] \\ & \bullet & \end{tikzcd}$$

where the vertices denote the corresponding elements of A and the dashed arrows indicate the existence condition. Finally, $\mathscr{A}$ is called *confluent* if

$$\text{for all } a, b, c \in A(c \twoheadleftarrow a \twoheadrightarrow b \Rightarrow \exists d \in A(c \twoheadrightarrow d \twoheadleftarrow b)).$$

Very helpful for proving confluence of a rewrite system is Newman's lemma, which states that a terminating rewrite system is confluent if and only if it is locally confluent (see [**4**, Theorem 1.2.1]). Let us record some further elementary facts about rewrite systems.

LEMMA 1.1. *The following hold for a rewrite system* $\mathscr{A} := (A, \to)$*:*
(i) *if* $\mathscr{A}$ *is terminating, then* $\mathscr{A}$ *is normalizing;*
(ii) *if* $\mathscr{A}$ *is confluent, then any element of* A *has at most one normal form;*
(iii) $\mathscr{A}$ *is uniquely normalizing if and only if it is normalizing and confluent.*

Proof. Assertions (i) and (ii) are easy to see. If $\mathscr{A}$ is uniquely normalizing, it is normalizing by definition. To see that $\mathscr{A}$ is confluent, let $a, b, c \in A$ with $c \twoheadleftarrow a \twoheadrightarrow b$. Let $\widetilde{c}$ be a normal

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form of c and let $\widetilde{b}$ be a normal form of b. We then have $a \twoheadrightarrow b \twoheadrightarrow \widetilde{b}$ and $a \twoheadrightarrow c \twoheadrightarrow \widetilde{c}$. Since $\widetilde{b}$ and $\widetilde{c}$ are irreducible, they are both normal forms of a. But then $\widetilde{b} = \widetilde{a} = \widetilde{c}$, where $\widetilde{a}$ is the unique normal form of a. This shows that $\mathscr{A}$ is confluent. The other direction is evident. $\square$

We want to establish a rewrite system on an algebra with respect to an ideal. Such a rewrite system should satisfy some natural compatibility conditions. We propose the following definition (there seems to be no established general theory yet).

DEFINITION 3. Let A be an algebra over a commutative ring R and let $I \trianglelefteq A$ be an ideal. A *rewrite system* for A/I is a rewrite system $\mathscr{A} := (A, \to)$ on A satisfying the following properties:

(i) if $a \to b$, then $a \equiv b \bmod I$ for all $a, b \in A$;
(ii) if $a \in A$ is irreducible, also ra is irreducible for all $r \in R$;
(iii) if $a, b \in A$ are irreducible, also $a + b$ is irreducible.

We can now relate the two notions of normal forms in Definitions 1 and 2. The following two lemmas are the key to the PBW theorem.

LEMMA 1.2. *Let A be an algebra over a commutative ring R, let $I \trianglelefteq A$ be an ideal, and let $\mathscr{A} := (A, \to)$ be a rewrite system for A/I. The following hold:*

(i) *if $a \twoheadrightarrow b$, then $a \equiv b \bmod I$ for all $a, b \in A$;*
(ii) *if $\mathscr{A}$ is normalizing, then*
$$N_{\mathscr{A}} := \bigcup_{a \in A} \mathscr{N}_{\mathscr{A}}(a) \subseteq A$$
is a weak normal form of A/I with $\mathscr{N}_{\mathscr{A}}(a) \subseteq \mathscr{N}_{N_{\mathscr{A}}}(a)$ for all $a \in A$.

Proof. The first assertion follows immediately from Definition 3(i) and the fact that $\equiv$ is both reflexive and transitive. Furthermore, Definition 3(ii) and Definition 3(iii) imply that $N_{\mathscr{A}}$ is an R-submodule of A and it is then a weak normal form of A/I due to (i). $\square$

LEMMA 1.3. *Let A be an algebra over a commutative ring R, let $I \trianglelefteq A$ be an ideal, and let $\mathscr{A} := (A, \to)$ be a normalizing rewrite system for A/I. The following are equivalent:*

(i) *$N_{\mathscr{A}}$ is a normal form of A/I;*
(ii) *$a \twoheadrightarrow 0$ for all $a \in I$.*

In this case $\mathscr{A}$ is uniquely normalizing and $\mathscr{N}_{\mathscr{A}}(a) = \mathscr{N}_{N_{\mathscr{A}}}(a)$ for all $a \in A$.

Proof. Suppose that $N_{\mathscr{A}}$ is a normal form of A/I. Then $\mathscr{N}_{N_{\mathscr{A}}}(a)$ is a singleton for all $a \in A$. Since $\mathscr{A}$ is normalizing and $\mathscr{N}_{\mathscr{A}}(a) \subseteq \mathscr{N}_{N_{\mathscr{A}}}(a)$, this implies $\mathscr{N}_{\mathscr{A}}(a) = \mathscr{N}_{N_{\mathscr{A}}}(a)$ and so $\mathscr{N}_{\mathscr{A}}(a)$ is also a singleton. Hence, $\mathscr{A}$ is uniquely normalizing. Moreover, if $a \in I$, then $\mathscr{N}_{\mathscr{A}}(a) = \mathscr{N}_{N_{\mathscr{A}}}(a) = \pi^{-1}(\pi(a)) \cap N_{\mathscr{A}} = \pi^{-1}(0) \cap N_{\mathscr{A}} = I \cap N_{\mathscr{A}} = \{0\}$. Hence, $a \twoheadrightarrow 0$ for all $a \in I$.

Now, suppose that (ii) holds. To show that $N_{\mathscr{A}}$ is a normal form, we show that the restriction $\pi|_{N_{\mathscr{A}}}$ of the quotient morphism $\pi : A \twoheadrightarrow A/I$ to $N_{\mathscr{A}}$ is injective. If $\widetilde{a}$ is an element of the kernel of this morphism, then $\widetilde{a} \in N_{\mathscr{A}} \cap I$ and so $\widetilde{a}$ is an irreducible element contained in I. But the assumption that $a \twoheadrightarrow 0$ for all $a \in I$ implies that whenever $a \in I$ is irreducible, then already $a = 0$. Hence, $\widetilde{a} = 0$ and so $N_{\mathscr{A}}$ is a normal form of A/I. $\square$

REMARK 1. If $\mathscr{A}$ is normalizing and satisfies $a \twoheadrightarrow 0$ for all $a \in I$, then it follows from Lemmas 1.3 and 1.1(iii) that $\mathscr{A}$ is confluent. The condition $a \twoheadrightarrow 0$ for all $a \in I$ might, however, be stronger than confluence. In other words, confluence of $\mathscr{A}$ alone might not be sufficient for making $N_{\mathscr{A}}$ into a normal form for A/I.

1.4. *Monomial rewrite systems*

Defining rewrite relations for A/I is much more intricate than it seems at first; in particular, when it comes to verifying confluence and the property $a \twoheadrightarrow 0$ for all $a \in I$. Usually, one would tend to define rewrite relations on *symbolic monomials* of A, which we understand as symbolic concatenations of elements of A symbolizing a product, and then extend these relations to *symbolic expressions*, that is, symbolic monomials involving parentheses and addition and subtraction symbols. But this approach leads to the following major issue. Let $a \in A$ be an irreducible element and let $b \in A$ be a reducible element. In A we have of course $a = a + b - b$ but as symbolic expressions a and $a + b - b$ are distinct. Since b is reducible and we extended the rewrite rules by linearity, also $a + b - b$ is reducible. This is a contradiction, since in A this symbolic term becomes equal to a, which is irreducible. Because of this, one has to be very careful when defining rewrite relations for A/I. We can avoid this problem by defining rewrite rules on basis elements of A and then extending these linearly. We formalize this in the following definition.

DEFINITION 4. Let $\mathbf{a} := (a_\lambda)_{\lambda \in \Lambda}$ be an R-basis of A. In this context we call the elements a_λ also *monomials* of A and by *terms* we understand multiples ra_λ with $r \in R \setminus \{0\}$. If $a \in A$, we say that a term ra_λ is a *term of* a if it occurs in the basis representation of a. Now, suppose that $\to$ is a subset of $(a_\lambda)_{\lambda \in \Lambda'} \times A$ for some subset $\Lambda' \subseteq \Lambda$, that is, $\to$ relates some monomials of A with elements of A. We extend $\to$ to a relation $\to'$ as follows:

(i) if $a \in A$ and ra_λ is a term of a with $a_\lambda \to b$, then $a \to' a - ra_\lambda + rb$;

(ii) if $a_\lambda \to b$ and $a_\mu = xa_\lambda y$ for some $\lambda, \mu \in \Lambda$ and $x, y \in A$, then $a_\mu \to' xby$.

The first extension rule should be understood as removing the term ra_λ from a and replacing it by rb. The second extension rule means that we can apply rules to 'submonomials' of monomials. We call the rules defined by $\to$ the *elementary rules* of the resulting rewrite system and call rewrite systems defined like this *monomial rewrite systems*.

It is easy to see that a monomial rewrite system on an algebra A satisfies Definitions 3(ii) and (iii). So, what remains to be verified to establish it as a rewrite system for A/I is Definition 3(i) on elementary rules (note that I is a two-sided ideal). Suppose that in this case we can furthermore show that the resulting rewrite system $\mathscr{A}$ for A/I is terminating and that $a \twoheadrightarrow 0$ for all $a \in I$ holds. Then we know from Lemma 1.1(i) that $\mathscr{A}$ is normalizing and so it follows from Lemma 1.3 that $\mathscr{A}$ is already uniquely normalizing. Furthermore, the module theoretic notion of normal forms in Definition 1 coincides with the rewrite system theoretic one in Definition 2.

THEOREM 1.4. *Define the monomial rewrite system $\mathscr{A}_{t,c,\mathbf{y}}$ on $R\langle V \otimes V^* \rangle \rtimes RG$ with respect to the R-basis $\mathbf{x}_\alpha \mathbf{y}_\beta g$ by the following elementary rules:*

$$x_j x_i \to x_i x_j \quad \text{for } j > i, \tag{1.5}$$

$$y_j y_i \to y_i y_j \quad \text{for } j > i, \tag{1.6}$$

$$y_i x_j \to x_j y_i + t\langle y_i, x_j \rangle + \sum_{s \in \mathrm{Ref}_\Gamma} (y, x)_s c(s) s \quad \text{for all } i, j. \tag{1.7}$$

This rewrite system is terminating and satisfies $a \twoheadrightarrow 0$ for all $a \in \mathrm{I}_{t,c}$. It is thus a uniquely normalizing rewrite system for $\mathrm{H}_{t,c}$.

Proof. This is a tedious but straightforward computation (see [**32**, § 16]). □

It is obvious that $N_{\mathscr{A}_{t,c,\mathbf{y}}} = N_\mathbf{y}$ and, as Lemma 1.3 implies that $N_\mathbf{y}$ is a normal form for $\mathrm{H}_{t,c}$, this proves the PBW theorem.

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REMARK 2. The proof of the PBW theorem is given in [**32**, § 16] by the same arguments for the much more general Drinfeld–Hecke algebras (see also [**28**]). The class of such algebras includes for example the symplectic reflection algebras by Etingof and Ginzburg [**9**]. With the straightforward adaptations of the algorithms we discuss in the next section we can thus compute in these algebras, too.

1.5. *Computing in rational Cherednik algebras*

Our approach to the PBW theorem using rewrite systems directly gives us a first algorithm for computing in rational Cherednik algebras. As the semi-direct product is usually not supported by computer algebra systems, we switch to a 'cover' which is supported, namely the tensor algebra $R\langle \mathbf{x} \cup \mathbf{y} \cup \mathbf{g} \rangle$, where $\mathbf{g} := (g_k)_{k=1}^r$ is a system of generators of G. We equip this algebra with the same rewrite rules as in Theorem 1.4 and the additional monomial rewrite rules

$$g_k x_i \to {}^{g_k}x_i g_k \quad \text{for all } i \text{ and } k, \tag{1.8}$$

$$g_k y_i \to {}^{g_k}y_i g_k \quad \text{for all } i \text{ and } k. \tag{1.9}$$

This yields a confluent terminating rewrite system on $R\langle \mathbf{x} \cup \mathbf{y} \cup \mathbf{g} \rangle$. It does not take care of the relations in the group, so to get PBW basis expressions we have to rewrite the 'group algebra part' of each monomial of the normal form of an element uniquely as a word in the generators $\mathbf{g}$. We can do this by choosing unique representations for every element of G.

Although straightforward, this algorithm is very inefficient as the elements in the tensor algebra can become very large and as we apply just one rule at a time. There is a much more efficient way to compute in rational Cherednik algebras[†]. Namely, the PBW theorem implies that $\mathrm{H}_{t,c}$ is as an R-module isomorphic to the group algebra $R[V \oplus V^*]G$ of G over the commutative ring $R[V \oplus V^*]$, so that we can consider $\mathrm{H}_{t,c}$ as $R[V \oplus V^*]G$ with a modified multiplication. Working in $R[V \oplus V^*]G$ instead of $R\langle \mathbf{x} \cup \mathbf{y} \cup \mathbf{g} \rangle$ is much more efficient as the commutativity of the $\mathbf{x}$ and the $\mathbf{y}$ is already inherent so that we do not need rewrite rules for this, and we do not have to rewrite group elements. Moreover, Lemma 1.5 below provides an explicit commutator formula which combines several rewrite rules and thus allows much faster computation of products. The idea for computing a product ab in $\mathrm{H}_{t,c} \cong R[V \oplus V^*]G$ is then to multiply each term of a with each term of b using the commutator formula in Lemma 1.5 and sum up the result. This is made precise in Algorithm 1. Here, we denote by $a_g(\mathbf{x}, \mathbf{y}) \in R[V \oplus V^*]$ the coefficient of g of an element $a \in R[V \oplus V^*]G$, so $a = \sum_{g \in G} a_g(\mathbf{x}, \mathbf{y})g$. Although we have six nested loops in this algorithm, it is still very efficient. In the implementation in CHAMP, we also make use of a database of commutators which is updated during run time. This leads to an additional speed-up. The most time-consuming part of the algorithm is the computation of the action of elements of G on polynomials in $R[V \oplus V^*]$.

LEMMA 1.5. *For any $\mu \in \mathbb{N}^n$, the following relation holds in $\mathrm{H}_{t,c}$:*

$$[y_i, x_1^{\mu_1} \dots x_n^{\mu_n}] = [y_i, x_1^{\mu_1} \dots x_n^{\mu_n}]_0 + [y_i, x_1^{\mu_1} \dots x_n^{\mu_n}]_t, \tag{1.10}$$

where

$$[y_i, x_1^{\mu_1} \dots x_n^{\mu_n}]_0 := \sum_{s \in \mathrm{Ref}_\Gamma} [y_i, x_1^{\mu_1} \dots x_n^{\mu_n}]_s s \tag{1.11}$$

with

$$[y_i, x_1^{\mu_1} \dots x_n^{\mu_n}]_s := c(s) \sum_{j=1}^n (y_i, x_j)_s x_1^{\mu_1} \dots x_{j-1}^{\mu_{j-1}} \left(\sum_{l=0}^{\mu_j - 1} x_j^l \, {}^s(x_j^{\mu_j - l - 1}) \right) {}^s(x_{j+1}^{\mu_{j+1}} \dots x_n^{\mu_n}) \tag{1.12}$$

[†]The use of the group algebra instead of the tensor algebra was suggested and already used by Bonnafé.

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and

$$[y_i, x_1^{\mu_1} \dots x_n^{\mu_n}]_t := t \sum_{j=1}^{n} \mu_j x_1^{\mu_1} \dots x_{j-1}^{\mu_{j-1}} x_j^{\mu_j - 1} x_{j+1}^{\mu_{j+1}} \dots x_n^{\mu_n} \langle y_i, x_j \rangle. \tag{1.13}$$

Proof. This is a straightforward proof by induction and we omit it here. □

Algorithm 1: Computation of products in rational Cherednik algebras.

Data: Elements $a = \sum_{g \in G} a_g(\mathbf{x}, \mathbf{y}) g$ and $b = \sum_{h \in G} b_h(\mathbf{x}, \mathbf{y}) h$ of $R[V \oplus V^*]G$
Result: The product $c := ab$ in $\mathrm{H}_{t,c} \cong R[V \oplus V^*]G$

$c := 0$;
for $g \in G$ with $a_g(\mathbf{x}, \mathbf{y}) \neq 0$ **do**
  $d := 0$; //this will be $(a_g(\mathbf{x}, \mathbf{y})g)b$ in the end
  $e := \sum_{h \in G} {}^g b_h(\mathbf{x}, \mathbf{y}) gh$; //$e = gb \in \mathrm{H}_{t,c}$
  //now we compute $a_g(\mathbf{x}, \mathbf{y})e = a_g(\mathbf{x}, \mathbf{y})gb$
  for t a term of $a_g(\mathbf{x}, \mathbf{y})$ **do**
    $m_t :=$ the monomial of t, so $m_t = \mathbf{x}^\alpha \mathbf{y}^\nu$ for some $\alpha, \nu \in \mathbb{N}^n$;
    $k_t :=$ the coefficient of t;
    $E := e$; //this will be $\mathbf{y}^\nu e$ in the end
    for $i := 1$ to n **do**
      for $j := 1$ to ν_i **do**
        $l := 0$; //this will be $y_i E$
        for $h \in G$ with $E_h(\mathbf{x}, \mathbf{y}) \neq 0$ **do**
          for u a term of $E_h(\mathbf{x}, \mathbf{y})$ **do**
            $m_u :=$ the monomial of u, so $m_u = \mathbf{x}^\mu \mathbf{y}^\beta$ for some $\mu, \beta \in \mathbb{N}^n$;
            $k_u :=$ the coefficient of u;
            $l := l + k_u(\mathbf{x}^\mu y_i \mathbf{y}^\beta + [y_i, \mathbf{x}^\mu]_t \mathbf{y}^\beta + \sum_{s \in \mathrm{Ref}_\Gamma} [y_i, \mathbf{x}^\mu]_s {}^s\mathbf{y}^\beta sh)$;
            //the second summand above is simply the PBW expression
            //for $k_u y_i \mathbf{x}^\mu \mathbf{y}^\beta h =$ from Lemma 1.5
          end
        end
        $E := l$;
      end
    end
    $d := d + k_t \mathbf{x}^\alpha E$; //$d := d + te$
  end
  $c := c + d$; //$c := c + a_g(\mathbf{x}, \mathbf{y})gb$
end
return c;

1.6. *Poisson brackets*

One of the motivations for devising and implementing algorithms for computing in rational Cherednik algebras is that this allows us to explicitly compute Poisson brackets of central elements of $\mathrm{H}_{0,c}$. We give a non-standard (but equivalent) definition of the Poisson bracket here, as this is more efficient for computations (see [5, 5.4.A] for the usual definition). Let $\widetilde{R} := D \otimes_K R$, where $D := K[\varepsilon]/\langle \varepsilon^2 \rangle$ is the ring of dual numbers. We denote the image of ε in D again by ε. The map $c : \mathscr{C}_\Gamma \to R$ can of course also be considered as mapping to $\widetilde{R}$

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and so the $\widetilde{R}$-algebra $\mathrm{H}_{\varepsilon,c}$ is defined. By the PBW theorem, we have $\mathrm{H}_{\varepsilon,c} \cong D \otimes_K \mathrm{H}_{0,c}$ as R-modules and we have a canonical embedding $\widetilde{\cdot} : \mathrm{H}_{0,c} \hookrightarrow \mathrm{H}_{\varepsilon,c}$ of R-modules. There is a canonical surjective R-module morphism $\widetilde{R} \to R$ sending $\varepsilon \otimes 1$ to 1 and $1 \otimes r$ to r. This map induces a surjective R-module morphism $q : \mathrm{H}_{\varepsilon,c} \twoheadrightarrow \mathrm{H}_{0,c}$. Now, the Poisson bracket of $a, b \in \mathrm{Z}(\mathrm{H}_{0,c})$ is defined as $\{a, b\} := q([\widetilde{a}, \widetilde{b}])$. As the implementation of rational Cherednik algebras in Champ supports general base rings and parameters, we are also able to compute Poisson brackets in Champ.

2. *Restricted rational Cherednik algebras*

Besides the capability of performing computations in rational Cherednik algebras, it is one aim of Champ to compute representation theoretic properties of *restricted* rational Cherednik algebras. These algebras (which were first seriously studied by Gordon [**17**]) are finite-dimensional quotients of $\mathrm{H}_{0,c}$ by a centrally generated ideal and they possess (partially established, partially conjectural) relations to Hecke algebras. These relations are one reason for studying (restricted) rational Cherednik algebras. In this section, we will review the basic properties of these algebras, explain what representation theoretic problems we are interested in, and address some computational issues. We include a quick review of Martino's conjecture to be very precise about what we computed and to ensure that these computations yield proofs of this conjecture in the cases under consideration.

2.1. *Restricted rational Cherednik algebras*

The $\mathbb{N}$-graded ring

$$\mathrm{Z}_\Gamma := K[V]^G \otimes_K K[V^*]^G \subseteq K[V \oplus V^*]^G$$

of *bi-invariants* maps under the PBW morphism into the center of $\mathrm{H}_{0,c}$ and embeds the scalar extension Z_Γ^R as a central subalgebra of $\mathrm{H}_{0,c}$. For $K = R = \mathbb{C}$, this was proven by Etingof and Ginzburg [**9**], and Gordon's proof [**17**] in this case also works without modifications in our general setting. We can thus view $\mathrm{H}_{0,c}$ as a Z_Γ^R-algebra. Note that since R is a flat K-module, the scalar extension Z_Γ^R is simply given by replacing K by R above. As the extension $K[V]^G \subseteq K[V]$ is finite (see [**18**, 12.27]), the PBW theorem implies that $\mathrm{H}_{0,c}$ is a finite Z_Γ^R-module. The finiteness implies (see [**32**, §§ 6 and 17]) that we have a decomposition

$$\mathrm{Simp}(\mathrm{H}_{0,c}) = \coprod_{\mathfrak{m} \in \mathrm{Max}(\mathrm{Z}_\Gamma^R)} \mathrm{Simp}(\mathrm{H}_{0,c}(\mathfrak{m})) \tag{2.1}$$

of the set of isomorphism classes of simple modules, where Max denotes the maximal ideal spectrum and $\mathrm{H}_{0,c}(\mathfrak{m}) := \mathrm{H}_{0,c}/\mathfrak{m}\mathrm{H}_{0,c}$ is the *specialization* of $\mathrm{H}_{0,c}$ in $\mathfrak{m} \in \mathrm{Max}(\mathrm{Z}_\Gamma^R)$. This decomposition follows essentially from the fact that maximal ideals and left primitive ideals coincide in $\mathrm{H}_{0,c}$, as it is a polynomial identity (PI) ring. The advantage is that on the right-hand side we have finite-dimensional algebras over fields, which might be easier to study than $\mathrm{H}_{0,c}$ itself.

Let

$$\mathfrak{a}_\Gamma^R := (\mathrm{Z}_\Gamma^R)_+ = (R[V]_+^G \otimes_R R[V^*]^G) + (R[V]^G \otimes_R R[V^*]_+^G)$$

be the augmentation ideal of Z_Γ^R. The quotient $\overline{\mathrm{H}}_c := \mathrm{H}_{0,c}/\mathfrak{a}_\Gamma^R \overline{\mathrm{H}}_c$ is called the *restricted rational Cherednik algebra* of Γ in c. Note that $\mathrm{Z}_\Gamma^R/\mathfrak{a}_\Gamma^R \cong R$, so $\mathfrak{a}_\Gamma^R$ is maximal if and only if R is a field. In this case, $\overline{\mathrm{H}}_c$ is one of the specializations in the decomposition (2.1).

Recall that the *coinvariant algebra* $K[V]_G$ of Γ is the quotient of $K[V]$ by the *Hilbert ideal* $\mathfrak{h}_\Gamma$, which is the ideal in $K[V]$ generated by the augmentation ideal $K[V]_+^G$ of $K[V]^G$. It follows

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at once from the PBW theorem that the PBW morphism induces an R-module isomorphism

$$R[V]_G \otimes_R RG \otimes_R R[V^*]_G \cong \overline{\mathrm{H}}_c, \tag{2.2}$$

implying that $\overline{\mathrm{H}}_c$ is a free R-module with

$$\dim_R \overline{\mathrm{H}}_c = \dim_R R[V]_G \cdot |G| \cdot \dim_R R[V^*]_G.$$

In case both $K[V]^G$ and $K[V^*]^G$ are polynomial (this holds for example in the non-modular setting by a theorem by Bourbaki–Chevalley–Serre as Γ is a reflection group), the extensions $K[V]^G \subseteq K[V]$ and $K[V^*]^G \subseteq K[V^*]$ are free of dimension equal to $|G|$. This implies that in this case $\dim_R \overline{\mathrm{H}}_c = |G|^3 = \dim_{\mathrm{Z}_\Gamma^R} \mathrm{H}_{0,c}$.

2.2. *Computing in restricted rational Cherednik algebras*

Fix a Gröbner basis of the Hilbert ideal of Γ with respect to some monomial order. As in Example 1, this allows us to compute a monomial basis $(\overline{\mathbf{x}}^\lambda)_{\lambda\in\Lambda}$ of the coinvariant algebra $K[V]_G$, where $\Lambda \subseteq \mathbb{N}^n$ is some finite subset and $\overline{\mathbf{x}} := (\overline{x}_i)_{i=1}^n$ are the images of the $x_i \in K[V]$ in $K[V]_G$. Similarly, we obtain a monomial basis $(\overline{\mathbf{y}}^\sigma)_{\sigma\in\Sigma}$ of $K[V^*]_G$. Then, by the above, $\overline{\mathrm{H}}_c$ is a free R-module with basis $(\overline{\mathbf{x}}^\lambda\overline{\mathbf{y}}^\sigma g)_{\lambda\in\Lambda,\sigma\in\Sigma,g\in G}$ and we call a basis of this form a *PBW basis* of $\overline{\mathrm{H}}_c$. Algorithm 1 can easily be modified to compute PBW basis representations of products in $\overline{\mathrm{H}}_c$: we just have to work in the group algebra $(R[V]_G \otimes_R R[V^*]_G)G$. This is again supported by Champ.

2.3. *Representation theory*

Now, we turn our attention to representation theoretic problems of $\overline{\mathrm{H}}_c$, which are originally due to Gordon [**17**]. First of all, note that $R\langle V \oplus V^*\rangle \rtimes RG$ is naturally a $\mathbb{Z}$-graded R-algebra by putting V^* in degree 1, G in degree 0, and V in degree -1. The elements in (1.1)–(1.3) defining the ideal $\mathrm{I}_{0,c}$ are all homogeneous, so that $\mathrm{H}_{0,c}$ inherits this $\mathbb{Z}$-grading. Since the Hilbert ideals are homogeneous, it follows moreover that the restricted rational Cherednik algebra $\overline{\mathrm{H}}_c$ also inherits this $\mathbb{Z}$-grading.

Gordon [**17**] observed that the triangular decomposition (2.2) of $\overline{\mathrm{H}}_c$ governs its representation theory by employing a general theory of Holmes and Nakano [**19**]. First note that due to the PBW theorem both the R-algebras $\overline{\mathrm{H}}_{c,m} := RG$ and $\overline{\mathrm{H}}_{c,r} := RG \ltimes R[V^*]_G$ naturally embed as subalgebras in $\overline{\mathrm{H}}_c$. This is the 'middle part' and the 'right Borel subalgebra' of the triangular decomposition (2.2), respectively. Mapping elements of V to zero yields a surjective algebra morphism $q_{c,r} : \overline{\mathrm{H}}_{c,r} \twoheadrightarrow \overline{\mathrm{H}}_{c,m}$ and by $q_{c,r*}$ we denote the induced inflation functor ${}_{\overline{\mathrm{H}}_{c,m}}(\mathsf{gr})\mathsf{mod} \to {}_{\overline{\mathrm{H}}_{c,r}}(\mathsf{gr})\mathsf{mod}$. The key tool is now the *Verma functor*

$$\Delta_c := \overline{\mathrm{H}}_c \otimes_{\overline{\mathrm{H}}_{c,r}} q_{c,r*}(-) : {}_{\overline{\mathrm{H}}_{c,m}}(\mathsf{gr})\mathsf{mod} \to {}_{\overline{\mathrm{H}}_c}(\mathsf{gr})\mathsf{mod}$$

between categories of finitely generated (graded) modules. It is not hard to see that

$$\Delta_c(W) \cong R[V]_G \otimes_R W \tag{2.3}$$

as R-modules provided that W is free as an R-module (see [**19**] or [**32**, § 18]).

Now, suppose that R is a field and that KG splits (the latter holds for example if K is of characteristic zero by a theorem by Benard [**3**]). Although Holmes and Nakano [**19**] assumed for their theory an algebraically closed base field, their arguments also work when the algebra is just split (see [**32**, § 18]) and show that for each simple KG-module λ the corresponding Verma module $\Delta_c(\lambda) := \Delta_c(\lambda^R)$ of $\overline{\mathrm{H}}_c$ is an indecomposable module with simple head $\mathrm{L}_c(\lambda)$ and that $(\mathrm{L}_c(\lambda))_{\lambda\in\mathrm{Simp}(KG)}$ is a system of representatives of the simple $\overline{\mathrm{H}}_c$-modules. The Verma module $\Delta_c(\lambda)$ is naturally graded and it has been proven in [**19**] that its radical is a

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graded submodule. Hence, $\mathrm{L}_c(\lambda)$ is naturally graded, too. Arguments by Bonnafé and Rouquier [**5**, Proposition 9.2.5] furthermore show that $\overline{\mathrm{H}}_c$ itself splits. There is now a natural correspondence between simple KG-modules and simple $\overline{\mathrm{H}}_c$-modules and so the distribution of simple $\overline{\mathrm{H}}_c$-modules into the blocks of $\overline{\mathrm{H}}_c$ yields a partition CM_c of the set of simple KG-modules whose members are called the *Calogero–Moser c-families*.

2.4. *Gordon's questions*

Gordon formulated in [**17**, § 7] the following questions concerning the representation theory of $\overline{\mathrm{H}}_c$ for a parameter c with values in an extension field R of K.

(i) What is the graded G-character of the simple modules $\mathrm{L}_c(\lambda)$? This includes knowing their dimensions and their Poincaré series.

(ii) What are the composition factors of the Verma modules $\Delta_c(\lambda)$?

(iii) What are the Calogero–Moser c-families?

These questions are so far only studied for $K = R = \mathbb{C}$ and we cannot go into details about what is already known in this case (see [**9**, **10**, 16.2 and 16.4], [**17**, 6.4 and 7.3], [**1**, § 3.3], [**25**, **26**], and [**32**]). The point is that almost nothing is known for *exceptional* complex reflection groups and this was one reason for the development of Champ.

2.5. *The generic situation*

The above problems are formulated for parameters c with values in an extension field R of K, that is, for points of the affine K-scheme $\mathfrak{R}_\Gamma := \mathbb{A}_K^{\#\mathscr{C}_\Gamma}$. This infinite amount of parameters would be a serious issue for a computational approach, but the following two facts allow us to reduce this to finitely many problems. First of all, it is proven in [**31**] that decomposition morphisms are generically trivial. This means essentially that once we know the solution to § 2.4(i) and § 2.4(ii) for the generic point $\mathbf{c}$ of $\mathfrak{R}_\Gamma$, that is, $\mathbf{c}$ is the family of indeterminates of the rational function field $K((c_s)_{s\in\mathscr{C}_\Gamma})$, then we know the solution for all c in a non-empty open subset of $\mathfrak{R}_\Gamma$. This *generic situation* is really the starting point of computational considerations and is supported by Champ. Similarly, it is proven in [**5**] (see also [**32**, § 11]) that blocks show the same behavior, meaning that once we know the generic Calogero–Moser families $\mathrm{CM}_{\mathbf{c}}$, we know them for all c in a non-empty open subset of $\mathfrak{R}_\Gamma$. After the generic situation is understood, we have to determine the locus of 'exceptional parameters' and continue the above process. This is exactly how we proceed for the groups G_4, G_{13}, and G_{20} to compute the answers to Gordon's questions for all parameters.

2.6. *Martino's conjecture*

Before we discuss our approach to the computational solution of Gordon's questions, let us first explain why the Calogero–Moser families are interesting. To this end, we need a different type of parameters for rational Cherednik algebras due to Ginzburg *et al.* [**16**]. Let $\mathscr{A}_\Gamma$ be the set of G-orbits of reflection hyperplanes of Γ. For a reflection hyperplane H of Γ the stabilizer subgroup G_H is cyclic of some order e_H prime to the characteristic of K. This order is constant along the G-orbit Ω of H, so that we can denote it by e_Ω. We denote by $\boldsymbol{\Omega}_\Gamma$ the set of pairs (Ω, j) with $\Omega \in \mathscr{A}_\Gamma$ and $1 \leqslant j \leqslant e_\Omega - 1$, and denote by $\overline{\boldsymbol{\Omega}}_\Gamma$ the set of pairs (Ω, j) with $0 \leqslant j \leqslant e_\Omega - 1$. Let $\overline{\mathfrak{K}}_\Gamma$ be the affine K-scheme $\mathbb{A}_K^{\#\overline{\boldsymbol{\Omega}}_\Gamma}$. For $k \in \overline{\mathfrak{K}}_\Gamma(R)$, we now define a function $c_k : \mathscr{C}_\Gamma \to R$ by

$$c_k(s) := \sum_{j=0}^{e_{\Omega_s - 1}} \det(s)^j (k_{\Omega_s, j+1} - k_{\Omega_s, j}), \tag{2.4}$$

where Ω_s is the G-orbit of the reflection hyperplane of s and we consider the index j always modulo e_{Ω_s}. We set $\mathrm{H}_{0,k} := \mathrm{H}_{0,c_k}$. It is not hard to see that (2.4) yields a surjective

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R-linear map $\Phi_\Gamma(R) : \overline{\mathfrak{K}}_\Gamma(R) \to \mathfrak{K}_\Gamma(R)$, and that this defines a surjective K-scheme morphism $\Phi_\Gamma : \overline{\mathfrak{K}}_\Gamma \to \mathfrak{K}_\Gamma$. We can thus think of $\overline{\mathfrak{K}}_\Gamma$ as an artificial extension of the parameter space for restricted rational Cherednik algebras of Γ. On $\overline{\mathfrak{K}}_\Gamma$, we define an involution $(\cdot)^\sharp$ by $k^\sharp := (k_{\Omega,-j})$. The closed subscheme $\overline{\mathfrak{K}}_\Gamma^0$ of $\overline{\mathfrak{K}}_\Gamma$ consisting of all k with $k_{\Omega,0} = 0$ is stable under this involution and we call its points *Cherednik parameters of GGOR type* for Γ. Here, GGOR stands for Ginzburg–Guay–Opdam–Rouquier who used these type of parameters [16]. Note that Φ_Γ restricts to an isomorphism between $\overline{\mathfrak{K}}_\Gamma^0$ and $\mathfrak{K}_\Gamma$, so that this can be considered as a reparametrization of $\mathfrak{K}_\Gamma$.

Now assume that K is of characteristic zero and that Γ is irreducible. Chlouveraki's [8] essential hyperplanes define a union $\overline{\mathscr{E}}_\Gamma$ of hyperplanes in $\overline{\mathfrak{K}}_\Gamma$ defined by integral equations, and attached to any point $k \in \overline{\mathfrak{K}}_\Gamma$ is a partition Rou_k of the simple KG-modules whose members are called the *Rouquier k-families*. We cannot go into details about Rouquier families here (see [7, 8, 24], and in particular [5] for the most general discussion) and just note how we can define them for a general base field K of characteristic zero instead of just $K = \mathbb{C}$. To this end, we have to choose a realization Γ' of Γ over the complex numbers, which is possible as Γ admits a realization over its character field. When doing this, we have to keep track of the orbits of hyperplanes of reflections to avoid changing the parameters. Then Chlouveraki's theory defines the essential hyperplanes in $\overline{\mathfrak{K}}_{\Gamma'}$ and the Rouquier k-families for any $k \in \overline{\mathfrak{K}}_{\Gamma'}(\mathbb{C})$. These families are already uniquely determined by the essential hyperplanes k lies on. This and the fact that the essential hyperplanes are defined by integral equations allow us to transport the essential hyperplanes to $\overline{\mathfrak{K}}_\Gamma$ and to define Rouquier families for any point of $\overline{\mathfrak{K}}_\Gamma$. We remark that for the definition of Rouquier families we tacitly assume the validity of some standard assumptions about Hecke algebras (see [8, 4.2.3]), which are not known to hold for all exceptional complex reflection groups. The interest in Calogero–Moser families is now justified by the following conjecture.

Conjecture 1 [25]. Assume that K is of characteristic zero and that Γ is irreducible. The following hold:
(i) $\mathrm{Rou}_{k^\sharp}$ is a refinement of $\mathrm{CM}_k := \mathrm{CM}_{c_k}$ for any $k \in \overline{\mathfrak{K}}_\Gamma$;
(ii) there is a non-empty open subset U of $\overline{\mathfrak{K}}_\Gamma$ such that $\mathrm{Rou}_{k^\sharp} = \mathrm{CM}_k$ for all $k \in U$.

We call the first part of the conjecture the *special parameter conjecture* and the second part the *generic parameter conjecture*. Because restricted rational Cherednik algebras and cyclotomic Hecke algebras always split, it is enough to consider some particular realization of each type of complex reflection groups in the Shephard–Todd classification and then prove the conjecture for K-points. Furthermore, we note that for $k \in \overline{\mathfrak{K}}_\Gamma$, the k-cyclotomic Hecke algebra is naturally isomorphic to the k^0-cyclotomic Hecke algebra, where k^0 is obtained from k by setting $k_{\Omega,0}$ to zero for all Ω (this follows from [5, 2.1.13]). Hence, we can equivalently consider the conjecture just for points of $\overline{\mathfrak{K}}_\Gamma^0$ as originally formulated by Martino. Due to the behavior of Calogero–Moser families explained in § 2.5 and due to the behavior of Rouquier families explained in § 2.6, the generic parameter conjecture is equivalent to $\mathrm{Rou}_{\mathbf{k}} = \mathrm{CM}_{\mathbf{k}}$, where $\mathbf{k}$ is the generic point of $\overline{\mathfrak{K}}_\Gamma^0$.

Martino's conjecture is known to be true for symmetric and imprimitive complex reflection groups by [1, 25], and [26]. The generic parameter conjecture is known to be true for G_4 by [1], and also for G_5, G_6, G_8, G_{10}, G_{23}, G_{24}, and G_{26} by [30]. It was shown in [30], however, that the generic parameter conjecture fails for G_{25}. In all cases where this conjecture is known to hold, it was proven by determining the Calogero–Moser families and comparing them to the Rouquier families, which have been determined by Chlouveraki [8]. So far, there is no theoretical explanation for this connection, and the counter-example in case G_{25} makes it even harder to understand the situation.

2.7. *Euler families*

Bonnafé and Rouquier [**5**] have pointed out a neat argument as to why there could exist a connection between Calogero–Moser families and Rouquier families at all. First of all, the *Euler element* of $\mathrm{H}_{0,c}$, introduced in [**16**], is defined as

$$\mathrm{eu}_c = \sum_{i=1}^{n} y_i x_i + \sum_{s \in \mathrm{Ref}_\Gamma} \frac{1}{\varepsilon_s - 1} c(s) s = \sum_{i=1}^{n} x_i y_i + \frac{\varepsilon_s}{\varepsilon_s - 1} c(s) s, \tag{2.5}$$

where as usual $(y_i)_{i=1}^n$ is a basis of V with dual basis $(x_i)_{i=1}^n$ and where ε_s denotes the non-trivial eigenvalue of s. The definition does not depend on the choice of a basis. This element is known to be central and its image in $\overline{\mathrm{H}}_c$ is again a non-trivial central element. Let Ω_λ^c be the central character of the simple $\overline{\mathrm{H}}_c$-module $\mathrm{L}_c(\lambda)$. Then the values of these characters on the Euler element yield a partition Eu_c of the simple KG-modules which is coarser than CM_c. We call its members the *Euler c-families*. It is proven in [**5**, 10.2.2] that for $k \in \overline{\mathfrak{K}}_\Gamma$ the equality $\Omega_\lambda^k(\mathrm{eu}_k) = c_\lambda(k^\sharp)$ holds, where $c_\lambda(k^\sharp)$ is a constant multiple of the 'q-logarithm' of the value of the central character of the simple module belonging to λ of the $k^\sharp$-cyclotomic Hecke algebra on the central element π coming from the center of the braid group of Γ (see [**5**, § 2.2]). These values define similarly a partition $\Pi_{k^\sharp}$ of the simple KG-modules which is coarser than $\mathrm{Rou}_{k^\sharp}$ and equal to Eu_k. We thus have

$$\mathrm{CM}_k \leqslant \mathrm{Eu}_k = \Pi_{k^\sharp} \geqslant \mathrm{Rou}_{k^\sharp}, \tag{2.6}$$

where $\leqslant$ denotes refinement. Of course, this does not explain why $\mathrm{CM}_k \geqslant \mathrm{Rou}_{k^\sharp}$ should hold.

2.8. *Verma families*

Next to the Euler families, there is another type of families giving a further approximation of the Calogero–Moser families. Namely, for a fixed simple KG-module λ we collect all constituents of $\Delta_c(\lambda)$. For each of these constituents S_μ, we again collect all constituents of $\Delta_c(\mu)$ etc. This process stabilizes and gives us a partition Ver_c of the simple KG-modules whose members we call the *Verma c-families*. As Verma modules are indecomposable, these are always contained in a family coming from a block of $\overline{\mathrm{H}}_c$, that is, each Verma family is contained in a Calogero–Moser family, so $\mathrm{Ver}_c \leqslant \mathrm{CM}_c$. We thus get a tower

$$\mathrm{Ver}_c \leqslant \mathrm{CM}_c \leqslant \mathrm{Eu}_c \tag{2.7}$$

giving us approximations of CM_c from two sides. The Euler families are easily computable using the characters of the simple KG-modules (see [**5**] or [**30**]). The Verma families in turn can be computed in many cases by the methods we discuss in the next sections. Usually, the above tower collapsed, that is, the Verma families were equal to the Euler families and thus equal to the Calogero–Moser families.

REMARK 3. Recently, Bonnafé and Rouquier [**5**, § 13.4] haven proven that in case K is of characteristic zero, the Verma families are in fact equal to the Calogero–Moser families (we observed this property before in our explicit computations). This is now the theoretical foundation showing that the key to determining the Calogero–Moser families are the Verma families.

3. *Computations with Verma modules*

After we have discussed the main problems we are interested in, namely Gordon's questions, let us go back to computational issues. Clearly, we first have to find an explicit description of the Verma modules for any computational approach to these problems. We discuss this here along with some aspects about efficient computation of Verma modules. The main problem

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to solve Gordon's questions is then to be able to compute decomposition matrices of Verma modules. We discuss an abstract strategy in § 3.3, which we will turn into a serious method in the following three sections.

3.1. *Computing Verma modules*

Let $\rho : G \to \mathrm{End}_K(W)$ be a finite-dimensional K-representation. Then the Verma module $\Delta_c(\rho)$ is uniquely determined by the action of the generators $\mathbf{x} \cup \mathbf{y} \cup \mathbf{g}$ of $\overline{\mathrm{H}}_c$, where $\mathbf{g}$ is a generating system of G and, as $\Delta_c(\rho)$ is free and finitely generated as an R-module, these actions are described by some matrices. In this way, a Verma module can be represented in the computer once we have chosen bases and understood the action. To this end, we choose besides a basis $\mathbf{y} := (y_i)_{i=1}^n$ of V with dual basis $\mathbf{x} := (x_i)_{i=1}^n$ and a generating system $\mathbf{g} := (g_i)_{i=1}^r$ of G also a monomial basis $\overline{\mathbf{x}}^\Lambda := (\overline{\mathbf{x}}^\lambda)_{\lambda \in \Lambda}$ of $K[V]_G$ as described in § 2.2. Furthermore, we fix a basis $\mathbf{w} := (w_k)_{k=1}^d$ of W. Then an R-basis of $\Delta_c(\rho) \cong R[V]_G \otimes_R W$ is formed by the elements $\overline{\mathbf{x}}^\lambda \otimes w_k$ and, with respect to this basis, we now describe the action of the generators.

First, let us consider the action of x_i on $\Delta_c(\rho)$. We have

$$x_i.(\overline{\mathbf{x}}^\mu \otimes w_k) = (x_i \overline{\mathbf{x}}^\mu) \otimes w_k.$$

Hence, if the basis representation of $x_i \overline{\mathbf{x}}^\mu \in K[V]_G$ in the basis $\overline{\mathbf{x}}^\Lambda$ is

$$x_i \overline{\mathbf{x}}^\mu = \sum_{\lambda \in \Lambda} \alpha_\lambda^{i,\mu} \overline{\mathbf{x}}^\lambda,$$

then

$$x_i.(\overline{\mathbf{x}}^\mu \otimes w_k) = \sum_{\lambda \in \Lambda} \alpha_\lambda^{i,\mu} \overline{\mathbf{x}}^\lambda \otimes w_k \tag{3.1}$$

is the basis representation of $x_i.(\overline{\mathbf{x}}^\mu \otimes w_k) \in \Delta_c(\rho)$ in the basis $\overline{\mathbf{x}}^\Lambda \otimes \mathbf{w}$. So, we actually just need to understand the action of the x_i on the coinvariant algebra $K[V]_G$ and this can computationally be solved using Gröbner bases.

Now, let us consider the action of g_i on $\Delta_c(\rho)$. We have

$$g_i.(\overline{\mathbf{x}}^\mu \otimes w_k) = (g_i \overline{\mathbf{x}}^\mu) \otimes w_k = ({}^{g_i}\overline{\mathbf{x}}^\mu g_i) \otimes w_k = {}^{g_i}\overline{\mathbf{x}}^\mu \otimes {}^{g_i}w_k.$$

Hence, if the basis representation of ${}^{g_i}\overline{\mathbf{x}}^\mu \in K[V]_G$ in the basis $\overline{\mathbf{x}}^\Lambda$ is

$${}^{g_i}\overline{\mathbf{x}}^\mu = \sum_{\lambda \in \Lambda} \beta_\lambda^{i,\mu} \overline{\mathbf{x}}^\lambda$$

and the basis representation of ${}^{g_i}w_k$ in the basis $\mathbf{w}$ is

$${}^{g_i}w_k = \sum_{t=1}^d \gamma_t^{i,k} w_t,$$

then the basis representation of $g_i.(\overline{\mathbf{x}}^\mu \otimes w_k)$ in the basis $\overline{\mathbf{x}}^\Lambda \otimes \mathbf{w}$ is

$$g_i.(\overline{\mathbf{x}}^\mu \otimes w_k) = \left(\sum_{\lambda \in \Lambda} \beta_\lambda^{i,\mu} \overline{\mathbf{x}}^\lambda \right) \otimes \left(\sum_{t=1}^d \gamma_t^{i,k} w_t \right) = \sum_{\lambda \in \Lambda} \sum_{t=1}^d \beta_\lambda^{i,k} \gamma_t^{i,k} \overline{\mathbf{x}}^\lambda \otimes w_t. \tag{3.2}$$

So, to understand the action of g_i in $\Delta_c(\rho)$, we need to understand the action of g_i on the coinvariant algebra $K[V]_G$ and on the KG-module W. The first can again be computationally achieved using Gröbner bases; the second is no problem when we have an explicit realization of ρ.

Now, we come to the hardest part, namely the action of y_i on $\Delta_c(\rho)$. This is the point where the structure of the restricted rational Cherednik algebra enters the game. Namely, to write the element $y_i(\overline{\mathbf{x}}^\mu \otimes w_k) = (y_i\overline{\mathbf{x}}^\mu) \otimes w_k$ in the basis $\overline{\mathbf{x}}^\Lambda \otimes \mathbf{w}$, we first have to rewrite $y_i\overline{\mathbf{x}}^\mu$ in the PBW basis of $\overline{\mathrm{H}}_c$. Recall from Lemma 1.5 that

$$[y_i, \mathbf{x}^\mu] = \sum_{t=1}^{n} \sum_{s\in\mathrm{Ref}_\Gamma} \sum_{l=0}^{\mu_t-1} c(s)(y_i, x_t)_s x_1^{\mu_1} \dots x_{i-1}^{\mu_{t-1}} x_t^l ({}^s x_t)^{\mu_t-l-1} ({}^s x_{t+1})^{\mu_{t+1}} \dots ({}^s x_n)^{\mu_n} s. \quad (3.3)$$

Using this formula, we get

$$\begin{aligned} y_i(\overline{\mathbf{x}}^\mu \otimes w_k) & \\ &= (y_i\overline{\mathbf{x}}^\mu) \otimes w_k \\ &= (\overline{\mathbf{x}}^\mu y_i + [y_i, \overline{\mathbf{x}}^\mu]) \otimes w_k \\ &= (\overline{\mathbf{x}}^\mu y_i) \otimes w_k + [y_i, \overline{\mathbf{x}}^\mu] \otimes w_k \\ &= \sum_{t=1}^{n} \sum_{s\in\mathrm{Ref}_\Gamma} \sum_{l=0}^{\mu_t-1} c(s)(y_i, x_t)_s x_1^{\mu_1} \dots x_{t-1}^{\mu_{t-1}} x_t^l ({}^s x_t)^{\mu_t-l-1} ({}^s x_{t+1})^{\mu_{t+1}} \dots ({}^s x_n)^{\mu_n} s \otimes w_k \\ &= \sum_{t=1}^{n} \sum_{s\in\mathrm{Ref}_\Gamma} \sum_{l=0}^{\mu_t-1} c(s)(y_i, x_t)_s x_1^{\mu_1} \dots x_{t-1}^{\mu_{t-1}} x_t^l ({}^s x_t)^{\mu_t-l-1} ({}^s x_{t+1})^{\mu_{t+1}} \dots ({}^s x_n)^{\mu_n} \otimes {}^s w_k \\ &= \sum_{s\in\mathrm{Ref}_\Gamma} \sum_{t=1}^{n} \sum_{l=0}^{\mu_t-1} c(s)(y_i, x_t)_s x_1^{\mu_1} \dots x_{t-1}^{\mu_{t-1}} x_t^l ({}^s x_t)^{\mu_t-l-1} ({}^s x_{t+1})^{\mu_{t+1}} \dots ({}^s x_n)^{\mu_n} \otimes {}^s w_k, \end{aligned} \quad (3.4)$$

where we used that $(\overline{\mathbf{x}}^\mu y_i) \otimes w_k = 0$ by definition of $\Delta_c(\rho)$. This expression is not yet a basis expression in the basis $\overline{\mathbf{x}}^\Lambda \otimes \mathbf{w}$ but, by rewriting the elements on the left-hand side of the tensor products in the basis $\overline{\mathbf{x}}^\Lambda$ as above using Gröbner bases and rewriting the elements on the right-hand side in the basis $\mathbf{w}$, immediately gives a basis expression. Hence, with the formulas in (3.1), (3.2), and (3.4), we can explicitly compute the Verma module $\Delta_c(\rho)$ and represent it in this way in a computer. Note, however, that it still needs an explicit method, like Gröbner bases, to rewrite elements in the coinvariant algebra in terms of a chosen (monomial) basis.

3.2. *X-tables*

Some parts of formula (3.4) occur multiple times. In particular, if one wants to consecutively compute Verma modules for different KG-representations, one can split off these parts to increase efficiency. We propose the following approach. Fix $i \in [1, n]$, $s \in \mathrm{Ref}_\Gamma$, and $\mu \in \Lambda$. Let $X_\mu^{(i,s)} = (X_{\mu,\eta}^{(i,s)})_{\eta\in\Lambda}$ be such that $X_{\mu,\eta}^{(i,s)}$ is the coefficient of $\overline{\mathbf{x}}^\eta$ in the basis representation of

$$\sum_{t=1}^{n} \sum_{l=0}^{\mu_t-1} (y_i, x_t)_s x_1^{\mu_1} \dots x_{t-1}^{\mu_{t-1}} x_t^l ({}^s x_t)^{\mu_t-l-1} ({}^s x_{t+1})^{\mu_{t+1}} \dots ({}^s x_n)^{\mu_n} \in K[V]_G$$

in the basis $\overline{\mathbf{x}}^\Lambda$. We can consider $X_\mu^{(i,s)}$ as a row vector and, by varying μ, we get a matrix $X^{(i,s)} \in \mathrm{Mat}_{\Lambda\times\Lambda}(K)$ satisfying

$$y_i(\overline{\mathbf{x}}^\mu \otimes w_k) = \sum_{s\in\mathrm{Ref}_\Gamma} c(s) \sum_{\eta\in\Lambda} X_{\mu,\eta}^{(i,s)} \overline{\mathbf{x}}^\eta \otimes {}^s w_k. \quad (3.5)$$

Note that the matrix $X^{(i,s)}$ is independent of the representation ρ and even of c, so that it can be used again for further computations. For the computation of $X^{(i,s)}$, we can define for fixed $\mu \in \Lambda$ the following two expressions, indexed by $t \in [1,n]$:

$$p_\mu^{\mathrm{start}}(t) := x_1^{\mu_1} \dots x_{t-1}^{\mu_{t-1}}, \tag{3.6}$$

$$p_\mu^{\mathrm{end}}(t) := x_{t+1}^{\mu_{t+1}} \dots x_n^{\mu_n}. \tag{3.7}$$

Then, for $s \in \mathrm{Ref}_\Gamma$, the row vector $X_\mu^{(i,s)}$ can be determined by computing the basis representation of the element

$$\sum_{s \in \mathrm{Ref}_\Gamma} \sum_{t=1}^{n} (y_i, x_t)_s p_\mu^{\mathrm{start}}(t) \left(\sum_{l=0}^{\mu_t - 1} x_t^l ({}^s x_t)^{\mu_t - l - 1} \right) {}^s p_\mu^{\mathrm{end}}(t). \tag{3.8}$$

The above methods for computing $\Delta_c(\rho)$ are implemented in exactly this way in CHAMP. To use the grading of Verma modules, we implemented a new type `ModGr` allowing us to handle graded modules in general. Moreover, we observed that Verma modules are usually very sparse and so we use sparse matrices in our implementation. Even Verma modules of dimension a few thousand can in this way be computed quite fast and with low memory usage.

3.3. *Decomposing Verma modules: the abstract idea*

After this initial problem being solved, we turn to the actual questions in § 2.4, namely: how can we compute the simple modules $\mathrm{L}_c(\lambda)$, that is, the heads of the Verma modules, and how can we compute the constituents of the Verma modules? Over finite fields, this can be achieved using the MEATAXE algorithm (see [**20**, **22**, **27**], [**21**, § 7.4], [**23**, § 1.3], [**13**, § 7.1.1]), which is also implemented in MAGMA. In the generic situations (where the base ring is a rational function field) and in case of base rings of characteristic zero, however, there does not exist any practical algorithm capable of solving our problems. Although there are some recent approaches to a 'characteristic zero MEATAXE', for example the general method developed by Steel [**29**], which is also implemented in MAGMA, no existing algorithm was successful even in smaller examples (see the experiments in § 9.2 proving this). We therefore conceived a method aiming to solve this problem. Although our whole idea is based on necessary conditions so that the resulting algorithm might not produce a result at all, it turned out to be extremely successful and efficient for Verma modules of restricted rational Cherednik algebras and was the key tool of our progress on Gordon's questions for exceptional complex reflection groups (see § 7).

Our approach is very general; so, it has nothing to do with Cherednik algebras and relies on the fact that we can solve the problems over finite fields using the MEATAXE. As we do not have a finite field at hand, we first need a way to transfer the situation to a finite field and then we have to figure out what the situation over the finite field tells us about our original situation. The following proposition, formulated abstractly, is the main ingredient for our approach.

DEFINITION 5. If $\mathscr{A}$ and $\mathscr{B}$ are two essentially small abelian categories, then a group morphism $d : \mathrm{K}_0(\mathscr{A}) \to \mathrm{K}_0(\mathscr{B})$ of the zeroth K-groups is called *positive* if $d(\mathrm{K}_0^+(\mathscr{A})) \subseteq \mathrm{K}_0^+(\mathscr{B})$, where K_0^+ is the submonoid represented by objects, and is called *strongly positive* if it is positive and $d([X]) = 0$ implies $[X] = 0$ for all $[X] \in \mathrm{K}_0^+(\mathscr{A})$.

PROPOSITION 3.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be two abelian categories of finite length and let $d : \mathrm{K}_0(\mathscr{A}) \to \mathrm{K}_0(\mathscr{B})$ be a strongly positive morphism. Let $X \in \mathscr{A}$. The following hold:*
(i) *if $d([X])$ is simple, then X itself is simple;*
(ii) *let $(S_i)_{i\in I}$ be a set of representatives of the simple objects of $\mathscr{A}$ and let $(T_j)_{j\in J}$ be a set of representatives of the simple $\mathscr{B}$-objects. Let $X \in \mathscr{A}$ and let $J_X^d := \{j \in J \mid [d([X]) : T_j] \neq 0\}$. Suppose that there exists a subset $I_X^d \subseteq I$ such that $[X : S_i] = 0$ for all $i \in I \setminus I_X^d$ and such that there exists a bijection $\lambda_X^d : J_X^d \to I_X^d$ with $d([S_{\lambda_X^d(j)}]) = T_j$ for all $j \in J_X^d$. Then*

$$[X] = \sum_{j\in J_X^d} [d([X]) : T_j][S_{\lambda_X^d(j)}]$$

and in this case we say that d is X-generic.

Proof. (i) Suppose that X is not simple. Then we can write $[X] = [X_1] + [X_2]$ with $[X_1], [X_2] \neq 0$ and we get the relation $[T] = d([X]) = d([X_1]) + d([X_2])$ in $\mathrm{K}_0^+(\mathscr{B})$ with $T \in \mathscr{B}$ simple. Since d is strongly positive, we have $d([X_1]), d([X_2]) \neq 0$. But then the above relation in $\mathrm{K}_0^+(\mathscr{B})$ is impossible. Hence, X must be simple.

(ii) The basis representation of $[X]$ is

$$[X] = \sum_{i\in I}[X : S_i][S_i] = \sum_{i\in I_X^d}[X : S_i][S_i].$$

Using the fact that λ_X^d is a bijection, we get

$$d([X]) = \sum_{i\in I_X^d}[X : S_i]d([S_i]) = \sum_{j\in J_X^d}[X : S_{\lambda_X^d(j)}]d([S_{\lambda_X^d(j)}]) = \sum_{j\in J_X^d}[X : S_{\lambda_X^d(j)}][T_j].$$

Since the basis representation of $d([X])$ is

$$d([X]) = \sum_{j\in J}[d([X]) : T_j][T_j] = \sum_{j\in J_X^d}[d([X]) : T_j][T_j],$$

the claim is proven. □

For a finite-dimensional algebra A over a field, we denote by $\mathrm{G}_0(A) := \mathrm{K}_0(A\text{-}\mathsf{mod})$ the Grothendieck group of A, where A-mod is the category of finite-dimensional left A-modules. Applying Proposition 3.1 to the Grothendieck groups of finite-dimensional algebras A and B over fields shows us that if we have a strongly positive morphism $d : \mathrm{G}_0(A) \to \mathrm{G}_0(B)$ and we can compute decompositions in $\mathrm{G}_0(B)$, for example if the base field of B is finite using the MEATAXE, then we can computationally prove that an A-module is simple and we have a chance of computing decompositions of A-modules in $\mathrm{G}_0(A)$. The morphism d is really the link between a computationally manageable ring B and the ring A. Our proposition leads us to the following two strategies.

STRATEGY 1. For computing the head of a finite-dimensional module V with simple head over a finite-dimensional algebra A over a field, we propose the following method.

(i) Find a strongly positive morphism $d : \mathrm{G}_0(A) \to \mathrm{G}_0(B)$ with B a finite-dimensional algebra over a *finite* field.

(ii) Create a submodule J of V, which is to be considered as a candidate for the radical, compute the quotient V/J, and check using the MEATAXE if $d(V/J)$ is irreducible. If it is, then we know that V/J is simple and is therefore the head of V.

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STRATEGY 2. Let A be a finite-dimensional algebra over a field. Suppose that we have a family $(V_\lambda)_{\lambda\in\Lambda}$ of finite-dimensional A-modules with simple heads $(S_\lambda)_{\lambda\in\Lambda}$. Suppose furthermore that this family is *constituent-closed*, meaning that every constituent of a member V_λ of this family is the head S_μ of some V_μ. We then have

$$[V_\lambda] = \sum_{\mu\in\Lambda} m_{\lambda,\mu}[S_\mu] \in \mathrm{G}_0(A)$$

for some $m_{\lambda,\mu} \in \mathbb{N}$ and we propose the following method for computing these decomposition numbers.

(1) Find a strongly positive morphism $d : \mathrm{G}_0(A) \to \mathrm{G}_0(B)$ with B a finite-dimensional algebra over a *finite* field such that $d(S_\lambda)$ is simple for all $\lambda \in \Lambda$.

(2) For each $\lambda \in \Lambda$, compute using the MEATAXE the constituents $(T_{\lambda,\theta})_{\theta\in\Theta_\lambda}$ and their multiplicities $m_{\lambda,\theta}$. Now, check if there exists an injection $\iota_\lambda : \Theta_\lambda \hookrightarrow \Lambda$ such that $d(S_\mu) \cong T_{\lambda,\theta}$ for some $\mu \in \Lambda$ and $\theta \in \Theta_\lambda$ if and only if $\mu = \iota_\lambda(\theta)$. In this case

$$[V_\lambda] = \sum_{\mu\in\Lambda} m_{\lambda,\mu}[S_\mu] \in \mathrm{G}_0(A),$$

where $m_{\lambda,\iota_\lambda(\theta)} := m_{\lambda,\theta}$ for $\theta \in \Theta_\lambda$ and $m_{\lambda,\mu} := 0$ for all $\mu \notin \mathrm{Im}\,\iota_\lambda$.

While decomposition morphisms (more precisely, compositions of decomposition morphisms which do not necessarily have to be decomposition morphisms themselves, whence the formulation using strongly positive morphisms) will certainly play a central role for finding appropriate strongly positive morphisms to algebras over finite fields, it is completely unclear at this stage what we should do in Strategy 1(ii) to produce a candidate for the radical of a module with simple head. In the following two sections, we will discuss methods to solve these two problems. Our final algorithm is presented in § 6.

4. *Finite field specializations*

In this section, we discuss a quite general method to produce for an algebra A (satisfying some assumptions) a strongly positive morphism $d : \mathrm{G}_0(A) \to \mathrm{G}_0(B)$ into the Grothendieck group of a finite-dimensional algebra over a finite field: this is the first step in the strategies outlined in § 3.3. In § 4.2, we discuss when this works for restricted rational Cherednik algebras, and this leads us to the notion of *integral structures* of these algebras.

4.1. *Finite field specializations in general*

Let us fix a Dedekind domain $\mathscr{O}$ with quotient field K, a normal commutative K-algebra R, and an R-algebra A which is free and finitely generated as an R-module.

DEFINITION 6. A *finite field specialization* of A is a pair $(\mathfrak{m}, u)$ such that:
(1) $\mathfrak{m}$ is a maximal ideal of $\mathscr{O}$ with finite residue field;
(2) u is a K-point of the K-scheme $\mathrm{Spec}(R)$ such that the K-algebra $A(u) := u^*A$ splits and has an $\mathscr{O}_\mathfrak{m}$-free $\mathscr{O}_\mathfrak{m}$-structure $\widetilde{A}(u)$, that is, the scalar extension $\widetilde{A}(u)^K$ of $\widetilde{A}(u)$ to K is isomorphic to $A(u)$.

Since $A(u)$ splits, the theory of decomposition morphisms by Geck and Rouquier [**15**] and Geck and Pfeiffer [**14**, § 7] implies that the decomposition morphism

$$\mathrm{d}_A^u : \mathrm{G}_0(A(\mathbf{u})) \to \mathrm{G}_0(A(u))$$

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exists, where $\mathbf{u}$ is the generic point of $\mathrm{Spec}(R)$, that is, $A(\mathbf{u}) = A^{\mathrm{Q}(R)}$, where $\mathrm{Q}(R)$ is the quotient field of R. Now, by assumption $A(u)$ has an $\mathscr{O}_{\mathfrak{m}}$-free $\mathscr{O}_{\mathfrak{m}}$-structure $\widetilde{A}(u)$. Since $\mathscr{O}_{\mathfrak{m}}$ is a valuation ring, the decomposition morphism

$$\mathrm{d}_{\widetilde{A}(u)}^{\mathfrak{m}_{\mathfrak{m}}} : \mathrm{G}_0(A(u)) \to \mathrm{G}_0(\widetilde{A}(u)(\mathfrak{m}_{\mathfrak{m}}))$$

exists, where $\widetilde{A}(u)(\mathfrak{m}_{\mathfrak{m}})$ is the scalar extension of $\widetilde{A}(u)$ to the residue field of $\mathfrak{m}_{\mathfrak{m}}$. As decomposition morphisms are strongly positive, we obtain a strongly positive morphism

$$\begin{tikzcd} \mathrm{G}_0(A(\mathbf{u})) \arrow[r, "\mathrm{d}_A^u"] \arrow[rr, bend right, "\mathrm{d}_A^{\mathfrak{m},u}"'] & \mathrm{G}_0(A(u)) \arrow[r, "\mathrm{d}_{\widetilde{A}(u)}^{\mathfrak{m}}"] & \mathrm{G}_0(\widetilde{A}(u)(\mathfrak{m}_{\mathfrak{m}})) \end{tikzcd} \quad (4.1)$$

We have omitted the choice of the $\mathscr{O}_{\mathfrak{m}}$-free $\mathscr{O}_{\mathfrak{m}}$-structure of $A(u)$ in the notation $\mathrm{d}_A^{\mathfrak{m},u}$ as this will not be important, although the knowledge about the existence of such a structure is of course crucial. We call $\mathrm{d}_A^{\mathfrak{m},u}$ the *decomposition morphism* of A in $(\mathfrak{m}, u)$ but note that this does not have to be a decomposition morphism itself.

REMARK 4. The notion of finite field specializations can of course be generalized to arbitrary finite chains of decomposition morphisms ending in the Grothendieck group of an algebra over a finite field. One only has to make sure in each step that the decomposition morphism exists with the main problem being the existence of integral structures.

REMARK 5. In [**31**], it is proven that decomposition morphisms are generically trivial for finite free algebras with split generic fiber over noetherian normal rings. Hence, assuming that R is noetherian and that A has split fibers, the morphism $\mathrm{d}_A^{\mathfrak{m},u}$ is trivial for generic u and for generic $\mathfrak{m}$, meaning that it induces a bijection between simple modules. Hence, finite field specializations can be used to employ Proposition 3.1(3.1) generically. This already indicates that it makes sense to choose finite field specializations randomly, as the probability is quite high to stay in the generic region.

REMARK 6. If $(\mathfrak{m}, u)$ is a finite field specialization of A and V is a finite-dimensional $A(\mathbf{u})$-module, it will be important to explicitly compute a representative of $\mathrm{d}_A^{\mathfrak{m},u}([V])$. To this end, suppose that the image of u is contained in $\mathscr{O}_{\mathfrak{m}}$ and that we have an $R_{\mathfrak{p}}$-free $A_{\mathfrak{p}}$-structure $\widetilde{V}$ of V for some $\mathfrak{p} \in \mathrm{Spec}(R)$. Let $\widetilde{\mathscr{V}}$ be an $R_{\mathfrak{p}}$-basis of $\widetilde{V}$ and let $\mathscr{A}$ be an R-algebra generating system of A. If we apply the map u to the entries of the matrices describing the action of $a \in \mathscr{A}$ on V in terms of the basis $\mathscr{V}$, we obtain a representative of $\mathrm{d}_A^u([V])$. As the image of u is contained in $\mathscr{O}_{\mathfrak{m}}$ by assumption, the entries of the matrices just obtained are contained in $\mathscr{O}_{\mathfrak{m}}$ and so we can reduce them modulo $\mathfrak{m}_{\mathfrak{m}}$, and this a representative of $\mathrm{d}_{\widetilde{A}(u)}^{\mathfrak{m}} \circ \mathrm{d}_A^u([V]) = \mathrm{d}_A^{\mathfrak{m},u}([V])$. In this situation, we do not even see the chosen $\mathscr{O}$-free $\mathscr{O}$-structure $\widetilde{A}(u)$ of $A(u)$.

Although formally a bit complicated, this whole process is actually quite straightforward in explicit situations and is automatically performed by the command `Specialize` in CHAMP. That a pair $(\mathfrak{m}, u)$ is indeed a finite field specialization has to be checked manually, however.

4.2. *Integral structures of restricted rational Cherednik algebras*

Let us now turn to the problem of finding finite field specializations of restricted rational Cherednik algebras.

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ASSUMPTION 1. By $\Gamma := (G, V)$, we denote a finite reflection group over a field K containing a Dedekind domain $\mathscr{O}$ with quotient field K. We assume as usual that all reflections are diagonalizable. Furthermore, we assume that the action of G on V and on V^* has no non-zero fixed points, that is, the G-modules V and V^* are *essential*. This certainly holds if Γ is irreducible.

DEFINITION 7. We say that a Cherednik parameter $c \in \mathfrak{R}_\Gamma(K)$ is *$\mathscr{O}$-integral* if the K-algebra $\overline{\mathrm{H}}_c$ has an $\mathscr{O}$-free $\mathscr{O}$-structure. We call any such structure an *$\mathscr{O}$-integral structure*.

It seems that the existence of integral structures of restricted rational Cherednik algebras has never been considered before. Before we give a sufficient condition for their existence, note that for any $s \in \mathrm{Ref}_\Gamma$ the set

$$\mathrm{Che}_\Gamma(s) := \{(y_j, x_i)_s \mid i, j \in [1, n]\} \subseteq K$$

for a K-basis $(y_i)_{i=1}^n$ of V with dual basis $(x_i)_{i=1}^n$ is independent of the chosen basis.

DEFINITION 8. We say that $c \in \mathfrak{R}_\Gamma(K)$ is *potentially $\mathscr{O}$-integral* if $c(s)\mathrm{Che}_\Gamma(s) \subseteq \mathscr{O}$ for all $s \in \mathrm{Ref}_\Gamma$.

THEOREM 4.1. *If there exists a datum $(\mathbf{y}, \mathscr{A}, \mathscr{B}, \mathscr{G})$ consisting of a basis $\mathbf{y}$ of V with dual basis $\mathbf{x}$, a basis $\mathscr{A}$ of $K[V]_G$, a basis $\mathscr{B}$ of $K[V^*]_G$, and a generating system $\mathscr{G}$ of G satisfying all of the following properties, then any potentially $\mathscr{O}$-integral parameter $c \in \mathfrak{R}_\Gamma(K)$ is already $\mathscr{O}$-integral.*

(i) *$\mathscr{A}$ contains the images of the elements of $\mathbf{x}$ in $K[V]_G$ and every element of $\mathscr{A}$ is an $\mathscr{O}$-linear polynomial in these images. The basis $\mathscr{B}$ satisfies the analogous conditions.*

(ii) *The structure constants of $K[V]_G$ with respect to $\mathscr{A}$ are contained in $\mathscr{O}$. The structure constants of $K[V^*]_G$ with respect to $\mathscr{B}$ satisfy the analogous conditions.*

(iii) *For all $g \in \mathscr{G}$, the action of g on V in the basis $\mathbf{y}$ and the action of g on V^* in the basis $\mathbf{x}$ are described by matrices with entries in $\mathscr{O} \subseteq K$.*

Proof. Let $\mathbf{x} = (x_i)_{i=1}^n$ and $\mathbf{y} = (y_i)_{i=1}^n$. Let $\overline{x}_i$ and $\overline{y}_i$ denote the images of x_i and y_i in $K[V]_G$ and $K[V^*]_G$, respectively. A K-basis of $\overline{\mathrm{H}}_c$ is given by $(abg)_{a\in\mathscr{A}, b\in\mathscr{B}, g\in G}$ and it suffices to show that the structure constants of $\overline{\mathrm{H}}_c$ with respect to this basis are contained in $\mathscr{O}$. Due to (ii), products of the form aa' and bb' with $a, a' \in \mathscr{A}$ and $b, b' \in \mathscr{B}$ are $\mathscr{O}$-linear combinations of elements of $\mathscr{A}$ and $\mathscr{B}$, respectively. Let $g \in \mathscr{G}$. Then, by (iii), we have ${}^g x_i = \sum_{j=1}^n \alpha_{ij} x_j$ with $\alpha_{ij} \in \mathscr{O}$. Since the $\overline{x}_i$ are contained in $\mathscr{A}$ by (i), it follows that ${}^g \overline{x}_i = \sum_{j=1}^n \alpha_{ij} \overline{x}_j$ is the basis expansion of ${}^g \overline{x}_i$ in the basis $\mathscr{A}$. Hence, the structure constants of the action of G on the elements of $\overline{\mathbf{x}} := (\overline{x}_i)_{i=1}^n$ are contained in $\mathscr{O}$. If $\lambda \in \mathbb{N}^n$, then

$${}^g\overline{\mathbf{x}}^\lambda = {}^g\left(\prod_{i=1}^n \overline{x}_i^{\lambda_i}\right) = \prod_{i=1}^n {}^g\overline{x}_i^{\lambda_i}.$$

By what we have just said, the elements ${}^g\overline{x}_i$ are $\mathscr{O}$-linear combinations of the elements of $\mathbf{x}$. It now follows from (ii) that ${}^g\overline{\mathbf{x}}^\lambda$ is an $\mathscr{O}$-linear combination of the elements of $\mathscr{A}$. This extends to the action of G on all elements of $K[V]_G$ and therefore the structure constants of the multiplication of elements of $K[V]_G \subseteq \overline{\mathrm{H}}_c$ with group elements are also contained in $\mathscr{O}$. Analogously, this holds for the action of G on $K[V^*]_G$. The only products of basis elements not already covered are those of the form ba for $b \in \mathscr{B}$ and $a \in \mathscr{A}$. We have

$$\overline{y}_j \overline{x}_i = \overline{x}_i \overline{y}_j + \sum_{s \in \mathrm{Ref}_\Gamma} (y_j, x_i) c(s) s$$

and this is an $\mathscr{O}$-linear combination of basis elements. By a recursive application of this and the fact that all other basis elements of $\mathscr{A}$ and $\mathscr{B}$ are polynomials in the $\overline{x}_i$ and the $\overline{y}_i$, respectively, we see that all the products ba are $\mathscr{O}$-linear combinations of basis elements. This shows that $\overline{\mathrm{H}}_c$ has an $\mathscr{O}$-free $\mathscr{O}$-structure. $\square$

Proposition 4.2. *For any basis* $\mathbf{y}$ *of* V, *there are a basis* $\mathscr{A}$ *of* $K[V]_G$ *and a basis* $\mathscr{B}$ *of* $K[V^*]_G$ *satisfying Theorem 4.1(i).*

Proof. Let $\mathbf{x} = (x_i)_{i=1}^n$. We can then write $K[V] = K[x_1, \dots, x_n]$. Let $\mathbf{f}$ be a system of fundamental invariants of Γ. Note that the degrees of the elements of $\mathbf{f}$ are strictly greater than 1, since if $f \in \mathbf{f}$ were of degree equal to 1, then it would be an element of V^* fixed by G and thus equal to zero as Γ^* is essential by assumption. Since the Hilbert ideal $\mathfrak{h}_\Gamma$ of Γ is the homogeneous ideal generated by $\mathbf{f}$, it follows that $\mathfrak{h}_\Gamma$ does not contain linear polynomials. Now, extend $\mathbf{f}$ to a Gröbner basis $\widetilde{\mathbf{f}}$ of the Hilbert ideal $\mathfrak{h}_\Gamma$ of Γ with respect to the lexicographical order. A monomial basis $\mathscr{A}$ of $K[V]_G$ is then given by the images of the elements

$$\{\mathbf{x}^\alpha \mid \alpha \in \mathbb{N}^n \text{ and } \mathbf{x}^\alpha \text{ is not divisible by some } \mathrm{LT}(f) \text{ for } f \in \mathscr{F}\}$$

in $K[V]_G$ (see Example 1). Now, suppose that the image of x_i in $K[V]_G$ were not contained in $\mathscr{A}$. Then by definition there exists $f \in \widetilde{\mathbf{f}}$ such that $\mathrm{LT}(f)$ divides x_i. But this means that f is a linear polynomial and we just argued that no linear polynomial is contained in the Hilbert ideal, so this is not possible. We can apply the same arguments to $K[V^*]_G$ and this proves the claim. $\square$

Proposition 4.3. *For all but finitely many maximal ideals* $\mathfrak{m}$ *of* $\mathscr{O}$, *any potentially* $\mathscr{O}_\mathfrak{m}$*-integral parameter* $c \in \mathfrak{R}_\Gamma(K)$ *is* $\mathscr{O}_\mathfrak{m}$*-integral. We call those* $\mathfrak{m}$ *for which this is true good for the restricted rational Cherednik algebras of* Γ.

Proof. Let $\mathbf{y}$ be a basis of V. We know from Proposition 4.2 that we can find $\mathscr{A}$ and $\mathscr{B}$ satisfying Theorem 4.1(i). Since everything is finite dimensional, the set S of the structure constants occurring in Theorems 4.1(ii) and (iii) is finite. Since $\mathscr{O}$ is a Dedekind domain, we have $S \subseteq \mathscr{O}_\mathfrak{m}$ for all but finitely many maximal ideals $\mathfrak{m}$ of $\mathscr{O}$ and so the assumptions in Theorem 4.1 are satisfied for the bases $\mathscr{A}$ and $\mathscr{B}$, and the ring $\mathscr{O}_\mathfrak{m}$. $\square$

The proof of Propositions 4.2 and 4.3 gives us an explicit way to find good maximal ideals of $\mathscr{O}$. This is summarized in Algorithm 2.

Algorithm 2: Find good maximal ideals.

Choose an explicit realization $\mathbf{G} \subseteq \mathrm{GL}_n(K)$ of G. This amounts to choosing a basis $\mathbf{y}$ of V. Let $\mathbf{x}$ be the dual basis.
Compute fundamental invariants $\mathbf{f}$ of $\mathbf{G}$ and $\mathbf{f}^*$ of the dual group $\mathbf{G}^*$.
Compute Gröbner bases of $\mathfrak{h}_\mathbf{G} = \langle \mathbf{f} \rangle$ and $\mathfrak{h}_{\mathbf{G}^*} = \langle \mathbf{f}^* \rangle$.
Compute monomial bases $\mathscr{A}$ of the coinvariant algebra $K[\mathbf{x}]/\mathfrak{h}_\mathbf{G}$ and $\mathscr{B}$ of $K[\mathbf{y}]/\mathfrak{h}_{\mathbf{G}^*}$ using the Gröbner bases.
Compute using the Gröbner bases the structure constants of the coinvariant algebras with respect to the bases $\mathscr{A}$ and $\mathscr{B}$, respectively.
Let S be the set of all denominators occurring in these structure constants.
Choose a generating system $\mathscr{G}$ of G and extend S by the denominators occurring in the corresponding matrices and their inverses.
Then all $\mathfrak{m}$ not containing any element of S are good.

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Precisely this method is performed by the command `BadPrimesForRRCA` in CHAMP. In [**32**, § 22], we computed sets of primes which contain all bad primes (for explicit choices of the bases) for the exceptional complex reflection groups G_4 up to G_{28} to ensure correctness of our computations. We remark that some of these primes are surprisingly large and we do not yet have a theoretical explanation for them.

4.3. *The generic situation for restricted rational Cherednik algebras*

The primary case we are considering is the following. Let K be a number field with ring of integers $\mathscr{O}$ and let R be the polynomial ring over K with indeterminates $(c_s)_{s \in \mathscr{C}_\Gamma}$. Let $c : \mathscr{C}_\Gamma \to R$ be the obvious map and let $\mathbf{c}$ be the composition of this map with the embedding into the quotient field of R. Let $\overline{\mathbf{H}} := \overline{\mathrm{H}}_c$ be the *generic restricted rational Cherednik algebra* for Γ. Let $\mathfrak{m} \in \mathrm{Max}(\mathscr{O})$ be a good maximal ideal. Then, for any $u \in \mathrm{Che}_\Gamma^{-1} \mathscr{O}^{\mathscr{C}_\Gamma}$, the pair $(\mathfrak{m}, u)$ is a finite field specialization and we have the morphism

$$\begin{tikzcd} \mathrm{G}_0(\overline{\mathrm{H}}_{\mathbf{c}}) \arrow[r, "\mathrm{d}_{\overline{\mathbf{H}}}^{u}"] \arrow[rr, bend right, "\mathrm{d}_{\overline{\mathbf{H}}}^{\mathfrak{m},u}"'] & \mathrm{G}_0(\overline{\mathrm{H}}_u) \arrow[r, "\mathrm{d}_{\widetilde{\mathrm{H}}_u}^{\mathfrak{m}}"] & \mathrm{G}_0(\widetilde{\mathrm{H}}_u(\mathfrak{m}_{\mathfrak{m}})) \end{tikzcd}$$

where $\widetilde{\mathrm{H}}_u$ is some $\mathscr{O}_{\mathfrak{m}}$-integral structure of $\overline{\mathrm{H}}_u$. As explained in Remark 5, the probability of this morphism being trivial in the sense that it induces a bijection between the simple modules is quite high. Thus, a random choice of u will bring us in position of employing Proposition 3.1. It remains to understand how we can lift back the results from the right to the left in this diagram and this is the topic of the next section.

Before we go there, we point out that the same idea works of course if instead of a parameter c yielding the generic point of the whole parameter space $\mathfrak{R}_\Gamma$ as above we take a parameter yielding the generic point of some closed subscheme of $\mathfrak{R}_\Gamma$, for example some hyperplane. To have this possibility at hand was one of the reasons why we chose a general commutative K-algebra as base ring everywhere and why we put emphasis on CHAMP being able to handle general base rings. In exactly this way, starting with the generic situation and then considering restrictions to hyperplanes, we approach the cases G_4, G_{13}, and G_{20}.

5. *Reconstructing submodules from abstract structures*

Now that we have found a way of transporting modules to an algebra over a finite field, we have to figure out how we can lift back the results obtained there to the initial setting. The idea is the following: if the morphism d induced by a finite field specialization as in (4.1) satisfies the condition in Proposition 3.1(3.1), then we can think of it as not destroying the structure of modules. Hence, the 'abstract structure' of the radical of the image of a module V with simple head under this morphism should be the same as the one of V itself. From this 'abstract structure' we might be able to compute a candidate for the radical of V and, using the morphism d, we can check if this candidate was the correct one. Let us now make precise what we mean by 'abstract structure' and how the candidate production works.

5.1. *Abstract structures*

Let V be an n-dimensional vector space over a field K with basis $\mathbf{v}$ and let U be an m-dimensional subspace. For a basis $\mathbf{u}$ of U, let $\mathrm{M}_{\mathbf{u}}^{\mathbf{v}} \in \mathrm{Mat}_{n \times m}(K)$ be the matrix of the embedding $U \hookrightarrow V$ with respect to the chosen bases. The class $\mathscr{M}_U^{\mathbf{v}}$ of $\mathrm{M}_{\mathbf{u}}^{\mathbf{v}}$ in $\mathrm{Mat}_{n \times m}(K)/\mathrm{GL}_m(K)$ consists precisely of the matrices $\mathrm{M}_{\mathbf{u}'}^{\mathbf{v}}$ for bases $\mathbf{u}'$ of U. It is an elementary fact that inside $\mathscr{M}_U^{\mathbf{v}}$ there exists precisely one matrix in reduced column echelon form, which we denote by $\mathrm{M}_U^{\mathbf{v}}$. Hence, once we fixed a basis of V, the subspaces of V are in bijection with $n \times m$ matrices in reduced column echelon form. We will now define the notion of the abstract structure of U with respect to $\mathbf{v}$ by using the matrix $\mathrm{M}_U^{\mathbf{v}}$.

Let $M \in \mathrm{Mat}_{n\times m}(K)$. If $\mathscr{E}(M)$ denotes the set of entries of M and if $\theta : \mathscr{E}(M) \to S$ is a map into a set S, then we denote by $\theta^*(M) \in \mathrm{Mat}_{n\times m}(S)$ the matrix defined by $(\theta^*(M))_{ij} := \theta(M_{ij})$. We denote by $M_{i,\bullet}$ the ith row of M and by $M_{\bullet,j}$ the jth column of M. We define $\mathrm{Supp}(M_{i,\bullet}) := \{j \in [1,m] \mid M_{ij} \neq 0\}$. Analogously, we define $\mathrm{Supp}(M_{\bullet,j})$ and $\mathrm{Supp}(M)$.

Now, suppose that M is in reduced column echelon form. We define two matrices ${}_{\mathrm{c}}M, {}_{\mathrm{f}}M \in \mathrm{Mat}_{n\times m}(\mathbb{N}_{>0})$ as follows. First, decompose M as $M = {}_{\mathrm{c}}M + {}_{\mathrm{f}'}M$, where each column of ${}_{\mathrm{c}}M$ just consists of the leading entry 1 of the corresponding column of M (if there is one) and ${}_{\mathrm{f}'}M$ is the matrix $M - {}_{\mathrm{c}}M$. We call ${}_{\mathrm{c}}M$ the *coarse structure* of M. Let $\mathscr{E}$ be the set of entries of ${}_{\mathrm{f}'}M$ and, for $x \in \mathscr{E}$, let $\mathscr{E}_x := \{(i,j) \in [1,n] \times [1,m] \mid M_{ij} = x\}$. We equip each $\mathscr{E}_x$ with the lexicographical order, which is a total order so that $\mathscr{E}_x$ has a unique minimum, and define an order $\leqslant$ on $\mathscr{E}$ by $x \leqslant y$ if and only if $\min \mathscr{E}_x \leqslant \min \mathscr{E}_y$. This is a total order on the finite set $\mathscr{E}$, so that assigning to each $x \in \mathscr{E}$ its position in $\mathscr{E}$ relative to $\leqslant$ defines a function $e : \mathscr{E} \to \mathbb{N}_{>0}$. We now define ${}_{\mathrm{f}}M := e^*({}_{\mathrm{f}'}M)$ and call this the *fine structure* of M. We call the pair $\mathrm{Abs}(M) := ({}_{\mathrm{c}}M, {}_{\mathrm{f}}M)$, which we also write as ${}_{\mathrm{c}}M + {}_{\mathrm{f}}M$, the *abstract structure* of M and call $\#\mathscr{E}$ the *complexity* of M. By $\mathrm{Abs}_{n\times m}$, we denote the set of abstract structures of $n \times m$ matrices in reduced column echelon form.

EXAMPLE 2. Let

$$M := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \\ 1 & 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{{}_{\mathrm{c}}M} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 1 \\ 1 & 4 \end{pmatrix}}_{{}_{\mathrm{f}'}M} \in \mathrm{Mat}_{3\times 2}(\mathbb{Q}).$$

Then

$$\mathrm{Abs}(M) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{{}_{\mathrm{c}}M} + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}}_{{}_{\mathrm{f}}M} \in \mathrm{Mat}_{3\times 2}(\mathbb{N}_{>0}).$$

In this example, we have $\mathscr{E} = \{2, 1, 4\}$ and $e : \mathscr{E} \to [1, 3]$ is defined by $e(2) = 1$, $e(1) = 2$, $e(4) = 3$. The complexity of M is equal to 3.

DEFINITION 9. If V is a finite-dimensional vector space over a field K with basis $\mathbf{v}$, then the *abstract structure* $\mathrm{Abs}_U^{\mathbf{v}}$ with respect to $\mathbf{v}$ of a subspace U of V is the abstract structure of the matrix $\mathrm{M}_U^{\mathbf{v}}$.

DEFINITION 10. If an abstract structure $M := ({}_{\mathrm{c}}M, {}_{\mathrm{f}}M) \in \mathrm{Abs}_{n\times m}$ with $m \leqslant n$ is given, then, for any map $\theta : \mathscr{E}({}_{\mathrm{f}}M) \to K^\times$ with $\theta(i) \neq \theta(j)$ for $i \neq j$, we get a matrix ${}_{\mathrm{c}}M + \theta^*{}_{\mathrm{f}}M \in \mathrm{Mat}_{n\times m}(K)$ in reduced column echelon form describing a unique subspace $\mathrm{U}_{M,\theta}^{\mathbf{v}}$ of V with respect to the basis $\mathbf{v}$. We call this subspace the *concretization* of M with respect to θ and $\mathbf{v}$.

Note that an abstract structure itself is *independent* of a base field: this is precisely the point of abstract structures.

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5.2. *Existence of submodules with prescribed abstract structure*

We can now formulate the primary aim of this section and we do this in a graded setting as the efficiency of CHAMP also relies on the fact that we make use of gradings throughout.

QUESTION 1. Let A be a finite-dimensional $\mathbb{Z}$-graded algebra over a field K, let V be a $\mathbb{Z}$-graded n-dimensional A-module, and let $\mathbf{v} := (v_i)_{i=1}^n$ be a homogeneous basis of V. The question this whole section is about is formulated as follows.

> Given an abstract structure $M := ({}_{\mathrm{c}}M, {}_{\mathrm{f}}M) \in \mathrm{Abs}_{n\times m}$ with $m \leqslant n$, is there a graded submodule U of V with $\mathrm{Abs}_U^{\mathbf{v}} = M$? In other words, is there a map $\theta : \mathscr{E}({}_{\mathrm{f}}M) \to K^\times$ with $\theta(i) \neq \theta(j)$ for $i \neq j$ such that the concretization $\mathrm{U}_{M,\theta}^{\mathbf{v}}$ is a graded submodule of V?

To analyze this question, we choose a set $\mathbf{a} := (a_k)_{k=1}^r$ of homogeneous K-algebra generators of A and denote for each $k \in [1, r]$ by $X^{(k)} \in \mathrm{Mat}_n(K)$ the matrix describing the action of a_k on V in the basis $\mathbf{v}$, that is,

$$a_k v_i = \sum_{l=1}^{n} X_{li}^{(k)} v_l = \sum_{l\in D_{ki}^{\mathrm{r}}} X_{li}^{(k)} v_l \tag{5.1}$$

for all $j \in [1, n]$, where

$$D_{ki}^{\mathrm{r}} := \{l \in [1, n] \mid \deg(a_k) + \deg(v_i) = \deg(v_l)\}.$$

THEOREM 5.1. *The answer to Question 1 is positive if and only if the following conditions are satisfied:*

(i) *for each $j \in [1, m]$, the degree of v_i is constant for all $i \in \mathrm{Supp}(M_{\bullet,j})$. We define $d_M^{\mathrm{c}}(j)$ to be this degree;*

(ii) *there exist pairwise different $\theta_1, \ldots, \theta_s \in K^\times$, where s is the complexity of M, and a family*

$$(Y_l^{(k,j)})_{\substack{k\in[1,r], j\in[1,m]\\ l\in D_{kj}^{\mathrm{c}}}} \subseteq K,$$

where

$$D_{kj}^{\mathrm{c}} := (d_M^{\mathrm{c}})^{-1}(\deg(a_k) + d_M^{\mathrm{c}}(j))$$

such that the equations

$$E_{i,j,k}^1 : \sum_{l\in I_{ijk}} {}_{\mathrm{c}}M_{lj} X_{li}^{(k)} + \sum_{l\in I_{ijk}} \theta_{{}_{\mathrm{f}}M_{lj}} X_{il}^{(k)} = 0 \tag{5.2}$$

hold for all $j \in [1, m]$, $k \in [1, r]$, and $i \in \mathrm{Supp}(M_{\bullet,j})$ and such that the equations

$$E_{ijk}^2 : \sum_{l\in I_{ijk}} {}_{\mathrm{c}}M_{lj} X_{il}^{(k)} + \sum_{l\in I_{ijk}} \theta_{{}_{\mathrm{f}}M_{lj}} X_{il}^{(k)} = \sum_{l\in D_{kj}^{\mathrm{c}}} Y_l^{(k,j)} {}_{\mathrm{c}}M_{il} + \sum_{l\in D_{kj}^{\mathrm{c}}} Y_l^{(k,j)} \theta_{{}_{\mathrm{f}}M_{il}} \tag{5.3}$$

hold for all $j \in [1, m]$, $k \in [1, r]$, and $i \in [1, n] \setminus \mathrm{Supp}(M_{\bullet,j})$, where

$$I_{ijk} := \{l \in \mathrm{Supp}(M_{\bullet,j}) \mid i \in D_{kl}^{\mathrm{r}}\}.$$

Proof. Suppose that the conditions are satisfied. Let $\theta : [1, s] \to K^\times$ be the map with $\theta(i) := \theta_i$. Then the concretization $U := \mathrm{U}_{M,\theta}^{\mathbf{v}}$ defines a unique subspace of V. Let

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$N_{\bullet,j} := {}_{\mathrm{c}}M_{\bullet,j} + (\theta^*{}_{\mathrm{f}}M)_{\bullet,j}$ be the 'specialization' of the jth column of M in θ. Define

$$u_j := \sum_{i=1}^{n} N_{i,j} v_i = \sum_{i \in \mathrm{Supp}(M_{\bullet,j})} N_{i,j} v_i = \sum_{i \in \mathrm{Supp}(M_{\bullet,j})} {}_{\mathrm{c}}M_{ij} v_i + \theta_{{}_{\mathrm{f}}M_{ij}} v_i. \tag{5.4}$$

Then $(u_j)_{j=1}^m$ is a basis of U and because of (i) this is a graded subspace. It remains to show that U is A-invariant. This holds if and only if $a_k U \subseteq U$ for all $k \in [1,r]$, and this in turn holds if and only if $a_k u_j \in U$ for all k and j, so $a_k u_j \in \langle u_1, \ldots, u_m \rangle_K$. As u_j is homogeneous of degree $\deg(a_k) + d_M^{\mathrm{c}}(j)$, this is equivalent to $a_k u_j \in \langle u_l \mid l \in D_{kj}^{\mathrm{c}} \rangle$. This is equivalent to the existence of elements $Y_l^{(k,j)} \in K$ such that

$$a_k u_j = \sum_{l \in D_{kj}^{\mathrm{c}}} Y_l^{(k,j)} u_l. \tag{5.5}$$

Combining equations (5.1), (5.4), and (5.5) implies that this is equivalent to the following equality for each $j \in [1,m]$ and $k \in [1,r]$:

$$\begin{aligned}
& a_k \Bigg(\sum_{i \in \mathrm{Supp}(M_{\bullet,j})} {}_{\mathrm{c}}M_{ij} v_i + \theta_{{}_{\mathrm{f}}M_{ij}} v_i \Bigg) \\
&\quad = \sum_{l \in D_{kj}^{\mathrm{c}}} Y_l^{(k,j)} \Bigg(\sum_{i \in \mathrm{Supp}(M_{\bullet,j})} {}_{\mathrm{c}}M_{il} v_i + \theta_{{}_{\mathrm{f}}M_{il}} v_i \Bigg) \\
&\quad \Leftrightarrow \sum_{i \in \mathrm{Supp}(M_{\bullet,j})} \sum_{l \in D_{ki}^{\mathrm{r}}} {}_{\mathrm{c}}M_{ij} X_{li}^{(k)} v_l + \sum_{i \in \mathrm{Supp}(M_{\bullet,j})} \sum_{l \in D_{ki}^{\mathrm{r}}} \theta_{{}_{\mathrm{f}}M_{ij}} X_{li}^{(k)} v_l \\
&\quad = \sum_{i \in \mathrm{Supp}(M_{\bullet,j})} \sum_{l \in D_{kj}^{\mathrm{c}}} Y_l^{(k,j)} {}_{\mathrm{c}}M_{il} v_i + \sum_{i \in \mathrm{Supp}(M_{\bullet,j})} \sum_{l \in D_{kj}^{\mathrm{c}}} Y_l^{(k,j)} \theta_{{}_{\mathrm{f}}M_{il}} v_i \\
&\quad \Leftrightarrow \sum_{i=1}^{n} \sum_{l \in I_{ijk}} {}_{\mathrm{c}}M_{lj} X_{il}^{(k)} v_i + \sum_{i=1}^{n} \sum_{l \in I_{ijk}} \theta_{{}_{\mathrm{f}}M_{lj}} X_{il}^{(k)} v_i \\
&\quad = \sum_{i \in \mathrm{Supp}(M_{\bullet,j})} \sum_{l \in D_{kj}^{\mathrm{c}}} Y_l^{(k,j)} {}_{\mathrm{c}}M_{il} v_i + \sum_{i \in \mathrm{Supp}(M_{\bullet,j})} \sum_{l \in D_{kj}^{\mathrm{c}}} Y_l^{(k,j)} \theta_{{}_{\mathrm{f}}M_{il}} v_i.
\end{aligned}$$

As $\mathbf{v}$ is a basis of V, each of these equations holds if and only if the coefficients of v_i for each $i \in [1,n]$ are the same. If $i \notin \mathrm{Supp}(M_{\bullet,j})$, the coefficient equation is

$$\sum_{l \in I_{ijk}} {}_{\mathrm{c}}M_{lj} X_{il}^{(k)} + \sum_{l \in I_{ijk}} \theta_{{}_{\mathrm{f}}M_{lj}} X_{il}^{(k)} = 0.$$

If $i \in [1,n] \setminus \mathrm{Supp}(M_{\bullet,j})$, the coefficient equation is

$$\sum_{l \in I_{ijk}} {}_{\mathrm{c}}M_{lj} X_{il}^{(k)} + \sum_{l \in I_{ijk}} \theta_{{}_{\mathrm{f}}M_{lj}} X_{il}^{(k)} = \sum_{l \in D_{kj}^{\mathrm{c}}} Y_l^{(k,j)} {}_{\mathrm{c}}M_{il} + \sum_{l \in D_{kj}^{\mathrm{c}}} Y_l^{(k,j)} \theta_{{}_{\mathrm{f}}M_{il}}.$$

These are the two asserted types of equations. It is evident from the discussion that these equations are also necessary for the existence of a graded submodule. $\square$

5.3. *Finding submodules with prescribed abstract structure (ModFinder)*

Let $E_{M,\mathbf{v}}^1 := (E_{i,j,k}^1)$ be the system of equations defined by (5.2), let $E_{M,\mathbf{v}}^2 := (E_{i,j,k}^2)$ be the system of equations defined by (5.3), and let $E_{M,\mathbf{v}}$ be the whole system. For finding a graded

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submodule of V with abstract structure M, we have to solve the system $E_{M,\mathbf{v}}$ for the *θ-variables* $\theta_1, \ldots, \theta_s$ and the *auxiliary variables* $Y_l^{(k,j)}$. If there is a unique submodule with this abstract structure, for example if M is the abstract structure of the unique maximal submodule when V has simple head, this system will have a unique solution we are searching for.

While $E^1_{M,\mathbf{v}}$ is an inhomogeneous linear system for the θ-variables, the system $E^2_{M,\mathbf{v}}$ is quadratic because of the products $Y_l^{(k,j)}\theta_{\mathrm{f}M_{il}}$ occurring in the equations. Hence, it will be very difficult in general to solve this system. But we can still try to consecutively solve *linear parts* of this system. Namely, we can start solving $E^1_{M,\mathbf{v}}$, which is easy as it is a linear system. The point is now that this system might already pin down one of the θ-variables. When inserting the determined θ-variables into the system $E^2_{M,\mathbf{v}}$, we might get further *linear* equations just involving the auxiliary variables. If we can determine some of the auxiliary variables, then inserting them into $E^2_{M,\mathbf{v}}$ might yield new *linear* equations for the θ-variables, which might pin down further θ-variables etc. This means that we consecutively solve the 'specialized systems' $L_{M,\mathbf{v}}(\theta', Y')$ given by the linear equations of the system $E_{M,\mathbf{v}}$ when inserting a family θ' of θ-variables and a family Y' of auxiliary variables. If this process leads to a (unique) solution of $E_{M,\mathbf{v}}$, we say that this system is (uniquely) *linearly solvable*. It might happen, however, that at some stage we cannot determine any new variables: then the system is not linearly solvable.

As we will work with modules of dimension up to 3000, we need a very efficient strategy for determining the θ-variables by linear equations of $E_{M,\mathbf{v}}$ (if this is possible at all). To this end, we define for any $q \in [1,s]$ a subsystem of $L^q_{M,\mathbf{v}}(\theta', Y')$ just consisting of the linear equations of $E_{M,\mathbf{v}}(\theta', Y')$ involving θ_q and all *dependent variables*. To make this precise, denote for a subsystem E of $E_{M,\mathbf{v}}$ by $\Theta(E)$ the set of non-determined θ-variables occurring in these equations. For $q \in [1,s]$, let $\widetilde{L}^q_{M,\mathbf{v}}(\theta', Y')$ just consist of the equations of $L_{M,\mathbf{v}}(\theta', Y')$ involving the variable θ_q, that is,

$$\widetilde{L}^q_{M,\mathbf{v}}(\theta', Y') := \{L \in L_{M,\mathbf{v}}(\theta', Y') \mid \theta_q \in \Theta(L)\}.$$

Now, define $L^q_{M,\mathbf{v}}(\theta', Y')$ inductively as follows. First, $L^q_{M,\mathbf{v}}(\theta', Y') := \widetilde{L}^q_{M,\mathbf{v}}(\theta', Y')$. For each $\theta_p \in \Theta(L^q_{M,\mathbf{v}}(\theta', Y'))$, we add to $L^q_{M,\mathbf{v}}(\theta', Y')$ the equations of $\widetilde{L}^p_{M,\mathbf{v}}(\theta', Y')$. We repeat this process until $L^q_{M,\mathbf{v}}(\theta', Y')$ stabilizes.

We will split the system $E_{M,\mathbf{v}}$ once more by defining $L^{q,\mathbf{g}}_{M,\mathbf{v}}(\theta', Y')$ for a subset $\mathbf{g} \subseteq [1,r]$ as the subsystem of $L^q_{M,\mathbf{v}}(\theta', Y')$ just involving equations E^1_{ijk} or E^2_{ijk} with $k \in \mathbf{g}$. The idea behind this is that we do not want to consider all algebra generators at once: perhaps a few algebra generators will be sufficient to determine all θ-variables and this means that we have to consider fewer equations. This idea turned out to be very efficient in experiments (see § 9).

Our idea of solving $E_{M,\mathbf{v}}$ is now summarized in Algorithm 3. This algorithm, which we call the ModFinder algorithm, has been implemented in this way (and with several additional ideas we cannot discuss here) in Champ in the subpackage `ModFinder`. In line 22, we have to check whether the concretization $\mathrm{U}^{\mathbf{v}}_{M,\theta}$ is indeed a submodule as we are just solving subsystems of $E_{M,\mathbf{v}}$ and just verifying necessary conditions up to this point. This can efficiently be checked using the *graded spinning algorithm*: a graded adaptation of the standard spinning algorithm explained for example in [**23**, § 1.3]. All this is provided by the new type `ModGr` for graded modules we have implemented in Champ.

Remark 7. Obviously, there is no reason why we can solve $E_{M,\mathbf{v}}$ just by consecutively solving specialized linear subsystems. For the radicals of Verma modules for restricted rational Cherednik algebras, however, this surprisingly turned out to be almost always the case and our algorithm was amazingly efficient; we cannot yet give theoretical arguments in favor of this.

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Algorithm 3: Finding submodules with prescribed abstract structure (ModFinder).

Data: Data as in Question 1 and Theorem 5.1 satisfying Theorem 5.1(i), and a subset $\mathbf{g} \subseteq [1, r]$.

Result: Decides if the system $E_{M,\mathbf{v}}$ is uniquely linearly solvable. If so, returns a graded submodule U of V with $\mathrm{Abs}_U^{\mathbf{v}} = M$.

1 $\theta' := \emptyset$; $Y := \emptyset$;
2 **while** $\#\theta' \neq s$ **do**
3   progress := false;
4   **for** $q \in \Theta(E_{M,\mathbf{v}}(\theta', Y'))$ **do**
5   **if** $L_{M,\mathbf{v}}^{q,\mathbf{g}}(\theta', Y')$ is not consistent **then**
6   **return** *There is no graded submodule with abstract structure* M;
7   **end**
8   Let θ'' and Y'' be the θ-variables and auxiliary variables, respectively, determined by $L_{M,\mathbf{v}}^{q,\mathbf{g}}(\theta', Y')$;
9   **if** θ'' *or* Y'' contains a variable not in θ' or Y', respectively, **then**
10             $\theta' := \theta' \cup \theta''$; $Y' := Y' \cup Y''$;
11   progress := true;
12   **end**
13   **end**
14   **if** progress = false **then**
15   **if** $\mathbf{g} = [1, r]$ **then**
16   **return** $E_{M,\mathbf{v}}$ *is not uniquely linearly solvable*;
17   **else**
18   Repeat the above algorithm with $\mathbf{g} = [1, r]$;
19   **end**
20   **end**
21 **end**
22 Check if $\mathrm{U}_{M,\theta}^{\mathbf{v}}$ is indeed a submodule of V;
23 **if** this is true **then**
24   **return** $\mathrm{U}_{M,\theta}^{\mathbf{v}}$;
25 **else**
26   **return** $E_{M,\mathbf{v}}$ *is not uniquely linearly solvable*;
27 **end**

Remark 8. In experiments, we observed that the choice of $\mathbf{g}$ and the order in which we try to determine θ-variables (line 4 in Algorithm 3) can have a serious impact on the run time of the algorithm (see § 9). We do not know yet how to determine an optimal choice of $\mathbf{g}$ and the sequence of θ-variables to solve for. The interaction between the subsystems $L_{M,\mathbf{v}}^{q,\mathbf{g}}(\theta', Y')$ seems to be very hard to understand. In Champ we have implemented a selection process for the systems, which performs quite well in experiments.

6. *A Las Vegas algorithm for computing heads and constituents*

With the theory of finite field specializations and the ModFinder algorithm, we can now turn our idea explained abstractly in Strategy 2 into an algorithm. The result is Algorithm 4. Remember that we are considering a finite-dimensional algebra A over a field and a family $(V_\lambda)_{\lambda \in \Lambda}$ of finite-dimensional A-modules with simple heads $(S_\lambda)_{\lambda \in \Lambda}$ such that this family is constituent-closed, meaning that every constituent of a member V_λ of this family is the head

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Algorithm 4: Computing heads and decomposition matrices.

Data: Data as explained in § 6

Result: If successful, returns the simple modules S_λ and the multiplicity $m_{\lambda,\mu}$ of S_μ in V_λ.

Randomly choose a strongly positive morphism $d : \mathrm{G}_0(A) \to \mathrm{G}_0(B)$ with B a finite-dimensional algebra over a finite field;

for $\lambda \in \Lambda$ **do**

  Compute a representative $\overline{V}_\lambda$ of $d([V])$;

  Compute using the MeatAxe the radical $\overline{J}_\lambda$ of $\overline{V}_\lambda$;

  Determine the abstract structure $\overline{J}_\lambda^{\mathrm{abs}}$ of J in $\overline{V}_\lambda$;

  Using Algorithm 3, try to find a submodule J_λ of V_λ with abstract structure $\overline{J}_\lambda^{\mathrm{abs}}$;

  if J_λ could not be determined **then**

    return *No success*;

  else

    $Q_\lambda := V_\lambda / J_\lambda$;

    Compute a representative $\overline{Q}_\lambda$ of $d([Q_\lambda])$;

    Check using the MeatAxe if $\overline{Q}_\lambda$ is irreducible;

    if this is not true **then**

      return *No success*;

    end

  end

end

for $\lambda \in \Lambda$ **do**

  Compute using the MeatAxe the constituents $(\overline{U}_{\lambda,\theta})_{\theta \in \Theta_\lambda}$ and their multiplicities $m_{\lambda,\theta}$ of $\overline{V}_\lambda$;

  Find using the MeatAxe an injection $\iota_\lambda : \Theta_\lambda \hookrightarrow \Lambda$ such that $\overline{U}_{\lambda,\theta} \cong \overline{Q}_\mu$ for $\mu \in \Lambda$ and $\theta \in \Theta_\lambda$ if and only if $\mu = \iota_\lambda(\theta)$;

  if no such injection exists **then**

    return *No success*;

  end

  $m_\lambda, \iota_\lambda(\theta) := m_{\lambda,\theta}$ for all $\theta \in \Theta_\lambda$ and $m_{\lambda,\mu} := 0$ for all $\mu \notin \operatorname{Im} \iota_\lambda$;

end

return $(Q_\lambda)_{\lambda \in \Lambda}$, $(m_{\lambda,\mu})_{\lambda,\mu \in \Lambda}$;

S_μ of some V_μ. Algorithm 4 attempts to compute the simple modules S_λ and the multiplicities of S_μ in V_λ.

We see that there are three branches in our algorithm whose result will be that the algorithm is not successful. On the other hand, if the algorithm is successful, it follows from our discussion that the result returned is the correct result. This means that our algorithm is a so-called Las Vegas algorithm, like the MeatAxe itself. Because of this, it is not easy to provide a complexity analysis of our approach. Note that whenever the algorithm is unsuccessful, it makes sense to run it again with a different finite field specialization.

Remark 9. In Champ we have implemented an extension of the above algorithm motivated by the few cases where it was not successful. Namely, in this case, we randomly pick a vector $v \in V_\lambda$ and compute (using the graded spinning algorithm) the submodule U of V_λ it generates. In case it is a proper submodule, we compute the quotient $Q := V_\lambda / U$ and apply our algorithm to Q. If it is again not successful, we repeat this process. With this simple extension we could indeed obtain all results for restricted rational Cherednik algebras we could compute so far.

6.1. *Application to Gordon's questions*

Let us discuss how we apply our algorithm to Gordon's questions § 2.4 in case of generic restricted rational Cherednik algebras (see § 4.3) for irreducible complex reflection groups. First, we choose a realization Γ of the reflection group over a number field K with ring of integers $\mathscr{O}$ (this is always possible). Then we compute which maximal ideals of $\mathscr{O}$ are certainly good using Algorithm 2. Next, we compute the generic Euler families $\mathrm{Eu}_{\mathbf{c}}$ (see § 2.7). For each Euler family Λ, the Verma modules $(\Delta_{\mathbf{c}}(\lambda))_{\lambda\in\Lambda}$ form a constituent-closed family of modules with simple head to which we apply our algorithm.

The random finite field specialization (line 1 of Algorithm 4) is chosen as $\mathrm{d}_{\overline{\mathbf{H}}}^{\mathfrak{m},u}$ by randomly choosing a good maximal ideal $\mathfrak{m}$ and a point $u \in \mathrm{Che}_{\Gamma}^{-1}\mathscr{O}^{\mathscr{C}_{\Gamma}}$ as explained in § 4.3. All this is automatically performed in CHAMP by the commands `HeadOfLocalModule` and `HeadsOfLocalModules` contained in the subpackage `RadicalLift`. This command is in general applicable to any constituent-closed family of modules with simple head over an algebra over a rational function field over a number field. Note, however, that one has to ensure by theory that the chosen data $(\mathfrak{m}, u)$ is indeed a finite field specialization in the sense of § 4.

If successful, our algorithm computes the generic Verma families (see § 2.8) and, due to the result by Bonnafé and Rouquier (see § 2.8), we also know the Calogero–Moser families. Note that in case of success we have also explicitly computed the simple modules so that we know their dimension, their Poincaré series, and using character theory we can also compute their structure as graded G-modules. In this way we can answer all of Gordon's questions.

The same idea is of course applicable if we do not start with the generic algebra $\overline{\mathbf{H}}$ but with its restriction to a hyperplane, say. This is exactly what we did to get the results for G_4, G_{13}, and G_{22}.

REMARK 10. If we work with a generic restricted rational Cherednik algebra $\overline{\mathbf{H}}$ for a reflection group Γ over a *finite* field K which splits over K, the choice of the morphism d in line 1 of the algorithm is actually simpler. As restricted rational Cherednik algebras split, we have a decomposition morphism $\mathrm{d}_{\overline{\mathbf{H}}}^{\mathfrak{p}} : \mathrm{G}_0(\overline{\mathbf{H}}(0)) \to \mathrm{G}_0(\overline{\mathbf{H}}(\mathfrak{p}))$ for any prime ideal $\mathfrak{p}$ of the base ring of $\overline{\mathbf{H}}$ and we can choose for $\mathfrak{p}$ any K-point of $\mathfrak{R}_{\Gamma}$. This approach is also covered by CHAMP and it applies in particular to Verma modules for rational Cherednik algebras at $t = 1$ in positive characteristic (see [**2**]).

7. *Summary of the results*

We summarize here as theorems the results we have obtained so far using CHAMP. All results are listed explicitly in tabular form in the online supplementary material available from the publisher's website. We also comment on some observations in the hope that some general theorem lies behind them. The reader should check the web site http://thielul.github.io/CHAMP/ and [**33**] for further results obtained after publication of this article.

THEOREM 7.1. *For generic parameters for the groups*

$$\mathrm{G}_4, \mathrm{G}_5, \mathrm{G}_6, \mathrm{G}_7, \mathrm{G}_8, \mathrm{G}_9, \mathrm{G}_{10}, \mathrm{G}_{12}, \mathrm{G}_{13}, \mathrm{G}_{14}, \mathrm{G}_{15}, \mathrm{G}_{16}, \mathrm{G}_{20}, \mathrm{G}_{22}, \mathrm{G}_{23} = \mathrm{H}_3, \mathrm{G}_{24}$$

the following hold:

(i) *we have the explicit answers to all of Gordon's questions;*
(ii) *Martino's generic parameter conjecture holds;*

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(iii) *the Calogero–Moser families are equal to the Euler families. This implies that the locus of 'exceptional' parameters, that is, those parameters for which the Calogero–Moser families become coarser than the generic Calogero–Moser families, is contained in the Euler variety and is thus a union of hyperplanes;*
(iv) *the Poincaré series of the simple modules are palindromic, that is, their list of coefficients can be reversed without changing the polynomial.*

THEOREM 7.2. *For all parameters for the groups*

$$\mathrm{G}_4, \mathrm{G}_{12}, \mathrm{G}_{13}, \mathrm{G}_{20}, \mathrm{G}_{22}, \mathrm{G}_{23} = \mathrm{H}_3, \mathrm{G}_{24}$$

the following hold†*:*
(i) *we have the explicit answers to all of Gordon's questions;*
(ii) *Martino's conjecture holds in its complete form, that is, the Rouquier $k^\sharp$-families refine the Calogero–Moser k-families for all parameters k.*

THEOREM 7.3. *In all the cases covered by Theorems 7.1 and 7.2, the following property holds: if λ is a character of minimal degree d in a Calogero–Moser family $\mathscr{F}$, then the multiplicity of $\mathrm{L}(\mu)$ in $\Delta(\lambda)$ for $\mu \in \mathscr{F}$ is a positive multiple of d.*

THEOREM 7.4. *For the groups*

$$\mathrm{G}_4, \mathrm{G}_6, \mathrm{G}_8, \mathrm{G}_{12}, \mathrm{G}_{13}, \mathrm{G}_{14}, \mathrm{G}_{20}, \mathrm{G}_{22}, \mathrm{G}_{23} = \mathrm{H}_3, \mathrm{G}_{24}$$

we explicitly know the locus of 'exceptional' parameters. Except for the group G_8, it coincides precisely with the union of Chlouveraki's essential hyperplanes for cyclotomic Hecke algebras [8]. For G_8, however, the Euler hyperplane $k_{1,1} - k_{1,2} + k_{1,3}$ is one additional 'exceptional' non-essential hyperplane‡*.*

THEOREM 7.5. *Also for G_6, G_8, and G_{14} we have the answers to all of Gordon's questions for the generic points of all Euler hyperplanes.*

REMARK 11. Theorem 7.5 does not yet imply that we know the results for *all* parameters for G_6, G_8, and G_{14}, as the parameter space for these groups is three dimensional and there is no theory of 'semi-continuity' of the representation theory of restricted rational Cherednik algebras so far. To this end, we would also have to consider all intersections of the Euler hyperplanes, but this would be way too much to compute and document. So, to solve these cases we need new theory.

QUESTION 2. Our results suggest the following questions.
(i) Are the Poincaré series of simple modules for generic parameters always palindromic? If not, what lies behind this property?
(ii) Is the property about the decomposition matrices of the Verma modules in Theorem 7.3 always true?
(iii) Is the locus of 'exceptional parameters' always a union of hyperplanes (this was already asked by Bonnafé and Rouquier [5])? Does it always contain the union of Chlouveraki's essential hyperplanes?

†Note that there is just one parameter for G_{12}, G_{22}, G_{23}, and G_{24}, so these results are just the generic ones. But for G_4, G_{13}, and G_{20} there are two parameters and here much more work has to be done.

‡This was first discovered by Bonnafé using different methods.

Remark 12. For special parameters it is no longer true that the Poincaré series of simple modules is palindromic. For G_4 on the hyperplane $k_{1,1} - 2k_{1,2} = 0$ we find a simple module with Poincaré series $1 + 2t$, which is not palindromic. There are many more counter-examples.

Remark 13. The first examples we found where the Rouquier families are strictly finer than the Calogero–Moser families are for G_{20} and the hyperplanes $k_{1,1} = 0$, $k_{1,2} = 0$, and $k_{1,1} - k_{1,2} = 0$.

Remark 14. So far we have no idea about general properties of the (graded) G-module structures of the simple modules. We hope that our explicit results help to reveal them.

Remark 15. We discussed rational Cherednik algebras for reflection groups over arbitrary fields as long as all reflections are diagonalizable and designed Champ to work in this generality. In [**32**], we computed for example the representation theory of the restricted rational Cherednik algebra attached to the general orthogonal group $GO_3(3)$ and to modular reflection representations of some symmetric groups. These cases are not yet understood theoretically and we hope that such examples will help to develop a general theory.

8. Champ

Now, we pass to the experimental part of this article. Everything we discussed so far has been implemented in Champ. The source code and documentation (including a Wiki) of Champ are freely available at http://thielul.github.io/CHAMP/. All parts are licensed under the GPL. Due to some operating system functions used in Champ, it will not work on Windows systems, just on Linux and Mac OS X systems. Moreover, a Magma version of at least 2.19 (released in December 2012) is necessary as we make use of user-defined types which did not exist in earlier versions.

8.1. *Running* Champ

Once the downloaded package is unpacked, one has to configure Champ by running

```
$ ./configure
```

in a terminal and inside the directory of Champ. This sets several variables to the absolute path of Champ and is necessary for working with it. Champ is now started by running

```
$ ./champ
Loading file "/CHAMP/CHAMP.m"

CHAMP (CHerednik Algebra Magma Package)
Version v1.5
Copyright (C) 2013, 2014 Ulrich Thiel
Licensed under GNU GPLv3, see LICENSE.txt
thiel@mathematik.uni-stuttgart.de
http://thielul.github.io/CHAMP/

>
```

Before we give a rough description of the capabilities of Champ, we point out the following important aspect.

> All actions in Magma are *right* actions. This means that whenever we start with a reflection group acting from the left and we consider left modules over rational Cherednik algebras, we have to transpose all matrices in Magma. Moreover, the

rational Cherednik algebra implemented in CHAMP is the *opposite* algebra of the one we are describing here theoretically. Hence, we have to reverse all products when passing between theory and CHAMP.

This reversion process between theory and CHAMP might be confusing at first, but we found it much more confusing when artificially working with left actions in MAGMA.

8.2. *Reflection groups*

As one aim of CHAMP was to verify Martino's conjecture, we had to make sure that we use the same labelings of irreducible characters of complex reflection groups as the one used by Chlouveraki [8] for the computation of Rouquier families. This is why we imported all relevant data from CHEVIE (see [12]) and implemented basic database support in CHAMP to deal with this data. This is illustrated by the following example.

```
> G:=ExceptionalComplexReflectionGroup(4);
> CharacterTable(~G);
> G`CharacterNames;
[ \phi_{1,0}, \phi_{1,4}, \phi_{1,8}, \phi_{2,5}, \phi_{2,3}, \phi_{2,1},
  \phi_{3,2} ]
```

In this example, we loaded the exceptional complex reflection group G_4. The realization is the same as in CHEVIE, but note that all matrices are transposed. Then we attached the character table to this group. When doing this, the names of the characters used in CHEVIE are automatically loaded and stored in the attribute `CharacterNames` of the group. We see in this example that one philosophy of CHAMP is to work with procedures taking a reference to an object as input and storing their result in the corresponding attribute of the object. The reason for this is that we want to have easy access to all data already computed and to handle the large amount of data necessary to work with rational Cherednik algebras. The absolutely irreducible characteristic zero representations are now attached using the procedure `Representations(~G,0)` and can be accessed via `G`Representations[0]`. Again we use the exact same realizations of these representations as in CHEVIE. Absolutely irreducible representations in characteristic p can be attached by calling the above command with p instead of 0.

Next to the characters and representations, the reflections are important. A structured collection of the reflections is attached by the command `ReflectionLibrary`, which gathers all the reflections of a reflection group Γ in a nested list of the form

$$(((s)_{\mathrm{H}_s=H})_{H\in\Omega})_{\Omega\in\mathscr{A}_\Gamma}.$$

Hence, for each orbit Ω of reflection hyperplanes of Γ, we have for each $H\in\Omega$ a list consisting of the reflections with hyperplane H. This allows us to label a reflection of Γ by a triple (i,j,k), where i refers to the ith reflection hyperplane orbit, j refers to the jth hyperplane in the orbit labeled by i, and k refers to the kth reflection with hyperplane j. This is precisely the triple we get when passing a reflection to the function `ReflectionID`. From the reflection library we automatically store representatives of the conjugacy classes of reflections in the attribute `ReflectionClasses`.

8.3. *Cherednik algebras*

A generic Cherednik parameter can be obtained as follows.

```
> G:=ExceptionalComplexReflectionGroup(4);
> c:=CherednikParameter(G : Type:="GGOR"); c;
Mapping from: { 1 .. 2 } to Multivariate rational function field of rank 2
over Cyclotomic Field of order 3 and degree 2
    <1, (-zeta_3 + 1)*k_{1,1} + (2*zeta_3 + 1)*k_{1,2}>
    <2, (zeta_3 + 2)*k_{1,1} + (-2*zeta_3 - 1)*k_{1,2}>
```

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This will be a map $c : [1, N] \to L$, where N is the number of conjugacy classes of reflections and L is the appropriate rational function field (the residue field in the generic point of $\mathfrak{R}_\Gamma$). The numbers 1 to N of the domain of c refer to the numbers in `ReflectionClasses`. So, if s is a reflection of Γ and i is its reflection class number, then $c(i) = c(s)$.

The command `CherednikParameter` has the additional option `Type`, which allows specification of different types of parameters. In the above, we selected the GGOR type (see § 2.6). We can instead also pass `EG` as type, which are the parameters used in [**9**], or we can pass `BR`, which are the parameters used in [**5**]. There is a further option `Rational`, which, when set to false, returns the parameter with values in the polynomial ring instead of the rational function field. Instead of using generic parameters, the user can define any map $c : [1, N] \to L$ as above, which can be used for a Cherednik parameter.

Rational Cherednik algebras can be created as follows.

```
> G:=ExceptionalComplexReflectionGroup(4);
> c:=CherednikParameter(G : Type:="EG");
> H:=RationalCherednikAlgebra(G,<1,c>); H;
Rational Cherednik algebra
Generators:
    g1, g2, y1, y2, x1, x2
Generator degrees:
    0, 0, -1, -1, 1, 1
Base ring:
    Multivariate rational function field of rank 2 over Cyclotomic Field of
    order 3 and degree 2
    Variables: k_{1,1}, k_{1,2}
Group:
    MatrixGroup(2, Cyclotomic Field of order 3 and degree 2) of order 2^3 * 3
    Generators:
    [      1       0]
    [      0  zeta_3]

    [1/3*(2*zeta_3 + 1) 1/3*(2*zeta_3 - 2)]
    [  1/3*(zeta_3 - 1)   1/3*(zeta_3 + 2)]
t-parameter:
    1
c-parameter:
    Mapping from: { 1 .. 2 } to Multivariate rational function field of rank 2
    over Cyclotomic Field of order 3 and degree 2
    <1, (-zeta_3 + 1)*k_{1,1} + (2*zeta_3 + 1)*k_{1,2}>
    <2, (zeta_3 + 2)*k_{1,1} + (-2*zeta_3 - 1)*k_{1,2}>
> H.3*H.5;
[1 0]
[0 1]*(y1*x1)
> H.5*H.3;
[1/3*(-2*zeta_3 - 1) 1/3*(-2*zeta_3 - 4)]
[  1/3*(-zeta_3 - 2)   1/3*(-zeta_3 + 1)]*(1/3*(2*zeta_3 + 4)*k_{1,1} +
1/3*(-4*zeta_3 - 2)*k_{1,2})
+
[1/3*(-2*zeta_3 - 1) 1/3*(-2*zeta_3 + 2)]
[ 1/3*(2*zeta_3 + 1)   1/3*(-zeta_3 + 1)]*(1/3*(2*zeta_3 + 4)*k_{1,1} +
1/3*(-4*zeta_3 - 2)*k_{1,2})
+
[1/3*(-2*zeta_3 - 1)  1/3*(4*zeta_3 + 2)]
[  1/3*(-zeta_3 + 1)   1/3*(-zeta_3 + 1)]*(1/3*(2*zeta_3 + 4)*k_{1,1} +
1/3*(-4*zeta_3 - 2)*k_{1,2})
+
[1 0]
[0 1]*(y1*x1 + 1)
+
[ 1/3*(2*zeta_3 + 1)  1/3*(2*zeta_3 + 4)]
```

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```
[1/3*(-2*zeta_3 - 1)    1/3*(zeta_3 + 2)]*(1/3*(-2*zeta_3 + 2)*k_{1,1} +
1/3*(4*zeta_3 + 2)*k_{1,2})
+
[ 1/3*(2*zeta_3 + 1) 1/3*(-4*zeta_3 - 2)]
[   1/3*(zeta_3 + 2)    1/3*(zeta_3 + 2)]*(1/3*(-2*zeta_3 + 2)*k_{1,1} +
1/3*(4*zeta_3 + 2)*k_{1,2})
+
[1/3*(2*zeta_3 + 1) 1/3*(2*zeta_3 - 2)]
[  1/3*(zeta_3 - 1)   1/3*(zeta_3 + 2)]*(1/3*(-2*zeta_3 + 2)*k_{1,1} +
1/3*(4*zeta_3 + 2)*k_{1,2})
```

In the above example, we created the opposite rational Cherednik algebra $H^{\mathrm{op}} := \mathrm{H}_{1,\mathbf{c}}^{\mathrm{op}}$ for G_4 and the rational point $\mathbf{c}$ of $\mathfrak{R}_\Gamma$. The generators of H can be accessed via `H.i`, where i lies between 1 and $2d+e$, where d is the dimension of Γ and e is the number of generators of Γ. We see in the above output that the generators are ordered as $g_1, g_2, y_1, y_2, x_1, x_2$. In PBW basis expressions, group algebra elements are always on the left and in matrix form. In the example, we computed the products $y_1 x_1$ and $x_1 y_1$. Keep in mind that H as created is the *opposite* algebra to what we treated theoretically before. This is why $x_1 y_1$ is *not* in PBW form: it is actually the product $y_1 x_1$, and this has to be rewritten.

EXAMPLE 3. The following very elaborate example from Bonnafé and Rouquier [5, § 19] can be treated easily in CHAMP. The Weyl group of type B_2 can be realized as the matrix group Γ in $\mathrm{GL}_2(\mathbb{Q})$ generated by the reflections

$$s := g_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t := g_2 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let y_1, y_2 be the standard basis of $V := \mathbb{Q}^2$ and let x_1, x_2 be the dual basis. Let $\{A, B\}$ be algebraically independent over $\mathbb{Q}$ and define $\mathbf{c}_s := -2A$, $\mathbf{c}_t := -2B$. As s and t are representatives of the conjugacy classes of reflections of Γ, this yields a map $\mathbf{c} : \mathscr{C}_\Gamma \to \mathbb{Q}(A, B)$ giving the generic point of $\mathfrak{R}_\Gamma$. Now, define the following elements of $\mathrm{H}_{0,\mathbf{c}}$:

$$\sigma := y_1^2 + y_2^2, \quad \pi := y_1^2 y_2^2, \quad \Sigma := x_1^2 + x_2^2, \quad \Pi := x_1^2 x_2^2.$$

In [5, 19.4.5], it is now proven that the Euler element $\mathrm{eu}_{\mathbf{c}} \in \mathrm{H}_{0,\mathbf{c}}$ is a zero of the polynomial

$$\begin{aligned} t^8 - 2(\sigma\Sigma + 4A^2 + 4B^2)t^6 + (\sigma^2\Sigma^2 + 2(\sigma^2\Pi + \Sigma^2\pi - 8\pi\Pi) + 8(A^2 + B^2)\sigma\Sigma + 16(A^2 - B^2)^2)t^4 \\ - 2((\sigma\Sigma + 4A^2 - 4B^2)(\sigma^2\Pi + \Sigma^2\pi) - 8\sigma\Sigma\pi\Pi + 2B^2\sigma^2\Sigma^2)t^2 + (\sigma^2\Pi - \Sigma^2\pi)^2. \end{aligned}$$

This fact was one essential part in determining the Calogero–Moser cells and to prove the Calogero–Moser cell conjecture for B_2. In [5], this is proven by an argument based on the undeformed situation in $\mathrm{H}_{0,0}$. As the computation is quite elaborate and one does not want to write down all its details, let us see if we can verify this fact with CHAMP.

```
> G:=CHAMP_GetFromDB("GrpMat/B2_BR","GrpMat"); //loads B2 as above
> C:=CherednikParameter(G:Type:="BR");
> H:=RationalCherednikAlgebra(G,C);
> eu:=EulerElement(H); eu;
[1 0]
[0 1]*(y1*x1 + y2*x2)
+
[0 1]
[1 0]*(-C1)
+
[-1  0]
[ 0  1]*(-C2)
+
[ 1  0]
```

```
[ 0 -1]*(-C2)
+
[ 0 -1]
[-1  0]*(-C1)
> A:=C(1)*(-1/2); B:=C(2)*(-1/2);
> y2:=H.4; y1:=H.3; g2:=H.2; g1:=H.1; x2:=H.6; x1:=H.5;
> sigma:=y1^2+y2^2; pi:=y2^2*y1^2; Sigma:=x1^2+x2^2; Pi:=x2^2*x1^2;
> time eu^8 - 2*eu^6*(Sigma*sigma + 4*A^2 + 4*B^2) +
eu^4*(Sigma^2*sigma^2 + 2*(Pi*sigma^2 + pi*Sigma^2 - 8*Pi*pi) +
8*Sigma*sigma*(A^2+B^2) + 16*(A^2-B^2)^2) -
2*eu^2*( (Pi*sigma^2 + pi*Sigma^2)*(Sigma*sigma +
4*A^2 - 4*B^2) - 8*Pi*pi*Sigma*sigma + Sigma^2*sigma^2*B^2*2) +
(Pi*sigma^2 - pi*Sigma^2)^2;
0
Time: 2.360
```

Hence, we could indeed verify (within only 2 s) that the Euler element is a zero of the polynomial above. Note again that we reversed all products, as CHAMP works in the opposite algebra.

8.4. *Verma modules*

Let us now see how we can compute in CHAMP with Verma modules for restricted rational Cherednik algebras and how we can answer Gordon's questions.

```
> G:=ExceptionalComplexReflectionGroup(4); Representations(~G,0);
> c:=CherednikParameter(G:Rational:=false); c;
Mapping from: { 1 .. 2 } to Polynomial ring of rank 2 over Cyclotomic
Field of order 3 and degree 2
    <1, (-zeta_3 + 1)*k_{1,1} + (2*zeta_3 + 1)*k_{1,2}>
    <2, (zeta_3 + 2)*k_{1,1} + (-2*zeta_3 - 1)*k_{1,2}>
> R:=Codomain(c); R;
Polynomial ring of rank 2 over Cyclotomic Field of order 3 and degree 2
Order: Lexicographical
Variables: k_{1,1}, k_{1,2}
> cH:=SpecializeCherednikParameterInHyperplane(c, R.1-R.2); c;
Mapping from: { 1 .. 2 } to Multivariate rational function field of
rank 1 over Cyclotomic Field of order 3 and degree 2
    <1, (zeta_3 + 2)*k_{1,2}>
    <2, (-zeta_3 + 1)*k_{1,2}>
> EulerFamilies(G,cH);
{@
    <{@ 5, 6 @}, 2*k_{1,2}>,
    <{@ 7 @}, 0>,
    <{@ 2, 3, 4 @}, -4*k_{1,2}>,
    <{@ 1 @}, 8*k_{1,2}>
@}
> V:=VermaModule(G,cH,G`Representations[0][2]); V;
Graded module of dimension 24 over an algebra with generator degrees
[ -1, -1, 0, 0, 1, 1 ] over Multivariate rational function field of
rank 1 over Cyclotomic Field of order 3 and degree 2.
> res, P, dims, Pseries, D, Gstruct, L := Gordon(G,cH, [ 2,3,4 ] :
GeneratorSets:=[{1,2,3}], pExclude:={2,3,5});
> P;
<[ 735 ], Prime Ideal
Two element generators:
    [1873, 0]
    [115, 1]>
> dims;
[ 9, 1, 7 ]
> Pseries;
[
    1 + 2*t + 3*t^2 + 2*t^3 + t^4,
```

```
    1,
    2 + 3*t + 2*t^2
]
> D;
[1 1 2]
[1 1 2]
[2 2 4]
> Gstruct;
[*
    (    t^4       1       0       0       0 t^3 + t     t^2),
    (0 0 1 0 0 0 0),
    (  0   0   0   1 t^2   0   t)
*]
> L;
[*
    Graded module of dimension 9 over an algebra with generator degrees
    [ -1, -1, 0, 0, 1, 1 ] over Multivariate rational function field of
    rank 1 over Cyclotomic Field of order 3 and degree 2.,
    Graded module of dimension 1 over an algebra with generator degrees
    [ -1, -1, 0, 0, 1, 1 ] over Multivariate rational function field of
    rank 1 over Cyclotomic Field of order 3 and degree 2.,
    Graded module of dimension 7 over an algebra with generator degrees
    [ -1, -1, 0, 0, 1, 1 ] over Multivariate rational function field of
    rank 1 over Cyclotomic Field of order 3 and degree 2.
*]
> IsModuleForRRCA(G,cH,L[1]);
true
```

In this example, we are considering the group G_4. At the beginning we create the (non-rational) generic Cherednik parameter of GGOR type. We specialize this parameter in the hyperplane H defined by $k_{1,1} - k_{1,2}$ of $\mathfrak{R}_\Gamma$ in GGOR parameters and get in this way the generic point $\mathbf{c}_H$ of this hyperplane. We then compute the Verma module $\Delta_{\mathbf{c}_H}(\phi_{1,4})$, which is a graded module of type `ModGr`. The central command is now `Gordon`, which takes as input a reflection group G, a Cherednik parameter, and a list of integers referring to the irreducible representations of G as in the attribute `` G`Representations ``. In the above example, we apply it to the Euler $\mathbf{c}_H$-family $\{\phi_{1,4}, \phi_{1,8}, \phi_{2,5}\}$. This command computes the corresponding Verma modules and applies our algorithms (encapsulated in the command `HeadsOfLocalModules`) to compute their heads and their decompositions. The additional option `GeneratorSets` controls which generators are used for the ModFinder algorithm (we chose in this case y_1, y_2, g_2 as generators) and the option `pExclude` describes the primes to be excluded when picking a finite field specialization (in this case we chose 2, 5, and 7, as they are bad). We remark that many additional techniques on which we cannot comment here are 'secretly' applied while running this command (see also § 9). If successful, the output consists of the parameters and the prime ideal chosen for the finite field specialization, the dimensions of the simple modules, their Poincaré series, the decomposition matrix of the Verma modules (the entry (i, j) in this matrix is the multiplicity of the head of the jth Verma module in the ith Verma module in the list passed to `Gordon`), the graded G-module structure of the simple modules, and the simple modules themselves. Using the command `IsModuleForRRCA`, we can check if a family of matrices indeed defines a module for the restricted rational Cherednik algebra: all necessary relations are checked.

This example is the prototype showing how we can answer all of Gordon's questions by simply applying the command `Gordon` to Euler families.

8.5. *Database*

All results we could compute so far are contained in an easily accessible database, as illustrated by the following example.

```
> G:=ExceptionalComplexReflectionGroup(4);
> answers:=Gordon(G);
> answers;
Associative Array with index universe Polynomial ring of rank 2 over
Cyclotomic Field of order 3 and degree 2
> Keys(answers);
{
    k_{1,2},
    k_{1,1} - 2*k_{1,2},
    k_{1,1},
    2*k_{1,1} - k_{1,2},
    1,
    k_{1,1} + k_{1,2},
    k_{1,1} - k_{1,2}
}
> P:=Universe(Keys(answers)); answers[P.1-P.2];
rec<recformat<Hyperplane, EulerFamilies, SimpleDims, SimplePSeries,
SimpleGModStruct, SimpleGradedGModStruct, VermaDecomposition,
CMFamilies> |
    Hyperplane := k_{1,1} - k_{1,2},
    EulerFamilies := {
        { 1 },
        { 2, 3, 4 },
        { 7 },
        { 5, 6 }
},
    SimpleDims := [ 24, 9, 1, 7, 8, 16, 24 ],
    SimplePSeries := [
        1 + 2*t + 3*t^2 + 4*t^3 + 4*t^4 + 4*t^5 + 3*t^6 + 2*t^7 + t^8,
        1 + 2*t + 3*t^2 + 2*t^3 + t^4,
        1,
        2 + 3*t + 2*t^2,
        2 + 4*t + 2*t^2,
        2 + 4*t + 4*t^2 + 4*t^3 + 2*t^4,
        3 + 6*t + 6*t^2 + 6*t^3 + 3*t^4
    ],
    SimpleGModStruct := [
        (1 1 1 2 2 2 3),
        (1 1 0 0 0 2 1),
        (0 0 1 0 0 0 0),
        (0 0 0 1 1 0 1),
        (0 0 1 1 1 0 1),
        (1 1 0 1 1 2 2),
        (1 1 1 2 2 2 3)
    ],
    SimpleGradedGModStruct := [
        (              1              t^8              t^4
        t^5 + t^7          t + t^3        t^3 + t^5 t^2 + t^4 + t^6),
        (    t^4       1       0       0       0 t + t^3      t^2),
        (0 0 1 0 0 0 0),
        (  0   0   0   1 t^2   0   t),
        (  0   0   t t^2   1   0   t),
        (      t     t^3       0     t^2      t^2 1 + t^4 t + t^3),
        (          t^2           t^2           t^2        t + t^3
        t + t^3        t + t^3 1 + t^2 + t^4)
    ],
    VermaDecomposition := [
        (1 0 0 0 0 0 0),
        (0 1 1 2 0 0 0),
        (0 1 1 2 0 0 0),
        (0 2 2 4 0 0 0),
        (0 0 0 0 2 2 0),
        (0 0 0 0 2 2 0),
        (0 0 0 0 0 0 3)
```

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```
    ],
    CMFamilies := {
        { 1 },
        { 2, 3, 4 },
        { 7 },
        { 5, 6 }
}>
> RouquierFamilies(G)[P.1-P.2];
{
    { 1 },
    { 2, 3, 4 },
    { 7 },
    { 5, 6 }
}
```

In this example, we fetched all the results for the example discussed in § 8.4 from the database. This is done by calling the command `Gordon` for an exceptional complex reflection group created with `ExceptionalComplexReflectionGroup`. The result is an associative array indexed by *normalized* equations for the hyperplanes of the Euler variety, and by 1 signifying generic parameters. It is now easy to test conjectures on these results without performing any additional computations.

REMARK 16. As we want to ensure verifiability of our results, we have included in the directory `Experiments/GordonQuestions` in CHAMP scripts which allow a recomputation from scratch of all our results and show how exactly we computed them.

9. *Experimental aspects*

The run time and success of the MODFINDER algorithm can depend heavily on the input data and on the choices made. We therefore point out some issues we observed in experiments with the hope that future developments will clarify these aspects and lead to further improvements.

9.1. *The effect of the choice of generators and realizations*

In Table 1, we list some data concerning the computation of the Verma modules and the heads of Verma modules using our algorithm. All computations and time measurements have been performed on an Intel Core i7-3930K at 3.2 GHz running the AVX version of MAGMA 2.19-8. We always work with generic GGOR parameters and use the realizations of the exceptional complex reflection groups and their representations as obtained from CHEVIE (these are also the ones used in CHAMP by default).

The columns denoted by t_Δ give the time needed for computing the X-table explained in § 3.2 and the time it then takes to compute the corresponding Verma module. The column Vars lists the number of variables in the abstract structure of the Jacobson radical of the Verma module (note that our algorithm has to be successful to determine this number). The column $\mathbf{g}$ lists the generators we have selected for the MODFINDER algorithm. In the last columns, denoted by $t_{\mathrm{Hd}\,\Delta}$, we list the time the MEATAXE needed to determine the Jacobson radical of the finite field specialization of the Verma module, the time the MODFINDER needed, and the total time (this includes for example the graded spinning algorithm to ensure that we found a submodule). This table shows us immediately how sensitive our approach is to the choices we make throughout. Let us discuss this in more detail.

First of all, we can see that we usually work with very small $\mathbf{g}$. We almost never had to consider all algebra generators for the MODFINDER algorithm. For the computation of the head of the Verma module $\Delta_{\mathbf{k}}(\phi_{3,6})$ for G_5, however, we see that the selection of $\mathbf{g}$ can be important. It this situation, the choice $\mathbf{g} = \{y_2, x_2\}$ is more than twice as fast as $\{y_2\}$. Unfortunately, we

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cannot say yet what makes one choice better than the other; we just found efficient choices by experimenting and it seems best to start with the basis $(y_i)_{i=1}^n$ of V.

Next, we observed that when modifying our explicit realizations of the group and the irreducible representations of the group in such a way that one generator of the group is diagonal and acts diagonally on all representations the ModFinder algorithm usually performs much faster. We denote in this table by $\lambda^{(i)}$ the representation obtained from λ by changing the basis so that the generator i of the chosen realization of the group acts diagonally. Comparing the computations for $\phi_{3,6}$ and $\phi_{3,6}^{(1)}$ for G_5, we see that we obtained the solution for $\phi_{3,6}^{(1)}$ around 20 times faster than for $\phi_{3,6}$. We see that the number of variables in the Jacobson radical drops from 70 to only 24, which is probably the reason for the speedup. Because of this, the command `Gordon` always automatically performs such a diagonalization, respecting the fact that the realizations of the exceptional complex reflection groups in Chevie are usually chosen such that one generator is already diagonal.

In the example $\phi_{3,8}^{(1)}$ for G_{24}, we see that even a very large number of variables (3888 in this case) do not necessarily have to be a problem. We are able to compute the head of the corresponding 1008-dimensional Verma module in just around 15 min. Even more fascinating is the example $\phi_{3,10}^{(1)}$ for G_{24}. Here, we finish the determination of the 1002-dimensional Jacobson radical in just 50 s (the ModFinder algorithm just needs 0.13 s).

We see from these examples that our algorithm can be surprisingly powerful but that it is very hard to control theoretically.

Table 1. *Experimental data about the computation of the heads of Verma modules.*

G	λ	$\dim \Delta$	t_Δ		$\dim \mathrm{Hd}\,\Delta$	Vars	$\mathbf{g}$		$t_{\mathrm{Hd}\,\Delta}$	
G_4	$\phi_{3,2}$	72	0.19	0.21	24	52	$\{y_2\}$	0.01	0.73	1.98
G_5	$\phi_{3,6}$	216	2.19	1.78	24	70	$\{y_2\}$	0.12	52.61	74.58
G_5	$\phi_{3,6}$	.	.	.	.	.	$\{y_2, x_2\}$	.	12.06	33.94
G_5	$\phi_{3,6}^{(1)}$	.	.	2.4	.	24	$\{y_2\}$	0.12	0.67	1.83
G_7	$\phi_{2,15}$	288	10.39	5.73	72	208	$\{y_2, y_1\}$	0.22	735.70	860.56
G_7	$\phi_{2,15}$	.	.	.	.	.	$\{y_2, g_1\}$	.	205.01	329.81
G_{23}	$\phi_{4,4}$	480	12.15	10.27	60	759	$\{y_3, y_2\}$	3.74	40.49	72.04
G_9	$\phi_{3,4}$	576	23.56	11.16	192	491	$\{y_2\}$	?	?	?
G_9	$\phi_{3,4}^{(2)}$	.	.	36.72	.	90	$\{y_2\}$	1.13	4.65	11.83
G_9	$\phi_{3,4}^{(1)}$	.	.	12.66	.	630	$\{y_2\}$	?	?	?
G_{24}	$\phi_{3,8}^{(1)}$	1008	206.43	91.85	156	3888	$\{y_3, y_2\}$	24.22	595.72	849.51
G_{24}	$\phi_{3,10}^{(1)}$	1008	.	99.50	6	14	$\{y_3, y_2\}$	28.47	0.13	50.17

Table 2. *Comparison of* Magma*'s algorithm (left) with ours (right).*

G	λ	Tests	Magma Avg.	Magma α	Champ Avg.	Champ α
S_4	$(2,1,1)$	82	0.65	0.13	0.23	1.0
G_4	$\phi_{3,7}^{(1)}$	84	0.76	0.15	0.7	1.0
G_4	$\phi_{3,7}$	82	–	0.0	5.2	1.0
G_{12}	$\phi_{4,3}^{(3)}$	84	3.29	0.14	0.38	1.0
G_6	$(\phi'_{2,5})^{(2)}$	77	–	0.0	0.25	1.0
G_6	$(\phi''_{2,3})^{(2)}$	79	–	0.0	0.25	1.0
G_5	$\phi_{3,6}^{(1)}$	81	–	0.0	5.1	1.0
G_7	$(\phi'_{2,11})^{(1)}$	78	–	0.0	42.0	1.0

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9.2. *Comparison with the algorithm in* MAGMA

So far, we did not comment on other already existing algorithms to compute the heads of the Verma modules in characteristic zero. The MEATAXE might actually solve this problem in special situations. In his PhD thesis, Steel [**29**] has developed a general characteristic zero MEATAXE which is in theory capable of computing the radical of a module over an algebra over a field of characteristic zero. This algorithm is implemented in MAGMA since 2012 and it is (to our knowledge) the only algorithm which could also be used to compute the head of Verma modules for restricted rational Cherednik algebras[†]. We therefore have to compare our methods with this algorithm. As it is also a Las Vegas algorithm, we cannot simply test it once for a specific problem and record the run time because it might always be the case that the randomly chosen parameters were bad. We thus have to run several tests and determine the average run time. We run each attempt with a time out τ of 900 s (15 min) for each attempt, as the run time of our algorithm is always much lower. We then record the average run time of all successful approaches, and record the success rate α within the time window τ for specific problems. The results are listed in Table 2.

We see that our success rate is always 100% while MAGMA's success rate is below 15% (if there is success at all). For all problems where MAGMA's algorithm did not return a result within the time window τ, we also did not get a result in sporadic attempts after a couple of days. Although this does not mean that MAGMA's algorithm would not eventually solve the problem, it should be quite clear from the table that without our algorithm we would not have been able to obtain most results in § 7; in particular, since the modules we have to work with are much bigger than those listed in the table.

REMARK 17. As our algorithm for determining the head of a module with simple head in characteristic zero is completely general (despite non-trivial theoretical assumptions, which have to be checked in each case), it is in principle applicable to many more situations. We hope that future developments and improvements to this method will enable us to solve problems in other contexts.

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References

1. G. BELLAMY, 'On singular Calogero–Moser spaces', *Bull. Lond. Math. Soc.* 41 (2009) no. 2, 315–326.
2. G. BELLAMY and M. MARTINO, 'On the smoothness of centres of rational Cherednik algebras in positive characteristic', *Glasg. Math. J.* 55 (2011) no. A, 27–54.
3. M. BENARD, 'Schur indices and splitting fields of the unitary reflection groups', *J. Algebra* 38 (1976) no. 2, 318–342.
4. M. BEZEM, J. W. KLOP and R. DE VRIJER, *Term rewriting systems* (Cambridge University Press, 2003).
5. C. BONNAFÉ and R. ROUQUIER, 'Cellules de Calogero–Moser', Preprint, 2013, arXiv:1302.2720.
6. W. BOSMA, J. CANNON and C. PLAYOUST, 'The Magma algebra system. I. The user language', *J. Symbolic Comput.* 24 (1997) no. 3–4, 235–265.
7. M. BROUÉ and S. KIM, 'Familles de caractères des algèbres de Hecke cyclotomiques', *Adv. Math.* 172 (2002) no. 1, 53–136.
8. M. CHLOUVERAKI, *Blocks and families for cyclotomic Hecke algebras*, Lecture Notes in Mathematics 1981 (Springer, 2009).
9. P. ETINGOF and V. GINZBURG, 'Symplectic reflection algebras, Calogero–Moser space, and deformed Harish-Chandra homomorphism', *Invent. Math.* 147 (2002) no. 2, 243–348.

[†]Unfortunately, it seems that there is no publication describing these methods in detail. Nevertheless, we argue here that *in our case at least* we have more significant reasons for not using it.

10. M. Finkelberg and V. Ginzburg, 'Calogero–Moser space and Kostka polynomials', *Adv. Math.* 172 (2002) no. 1, 137–150.

11. M. Geck, *An introduction to algebraic geometry and algebraic groups*, Oxford Graduate Texts in Mathematics 10 (Oxford University Press, 2003).

12. M. Geck, G. Hiss, F. Lübeck, G. Malle and G. Pfeiffer, 'CHEVIE — a system for computing and processing generic character tables', *Appl. Algebra Engrg. Comm. Comput.* 7 (1996) no. 3, 175–210; Pre-packaged GAP3 version by J. Michel (version from March 2012).

13. M. Geck and N. Jacon, *Representations of Hecke algebras at roots of unity*, Algebra and Applications 15 (Springer, London, 2011).

14. M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori–Hecke algebras*, London Mathematical Society Monographs. New Series 21 (Oxford University Press, 2000).

15. M. Geck and R. Rouquier, 'Centers and simple modules for Iwahori–Hecke algebras', *Finite reductive groups (Luminy, 1994)*, Progress in Mathematics 141 (Birkhäuser, Boston, MA, 1997) 251–272.

16. V. Ginzburg, N. Guay, E. Opdam and R. Rouquier, 'On the category $\mathscr{O}$ for rational Cherednik algebras', *Invent. Math.* 154 (2003) no. 3, 617–651.

17. I. Gordon, 'Baby Verma modules for rational Cherednik algebras', *Bull. Lond. Math. Soc.* 35 (2003) 321–336.

18. U. Görtz and T. Wedhorn, *Algebraic geometry I: schemes with examples and exercises*, Advanced Lectures in Mathematics (Vieweg-Teubner, Wiesbaden, 2010).

19. R. R. Holmes and D. K. Nakano, 'Brauer-type reciprocity for a class of graded associative algebras', *J. Algebra* 144 (1991) no. 1, 117–126.

20. D. F. Holt, 'The Meataxe as a tool in computational group theory', *The atlas of finite groups: ten years on (Birmingham, 1995)*, London Mathematical Society Lecture Note Series 249 (Cambridge University Press, 1998) 74–81.

21. D. F. Holt, B. Eick and E. A. O'Brien, *Handbook of computational group theory*, Discrete Mathematics and its Applications (Chapman & Hall/CRC, Boca Raton, FL, 2005).

22. D. F. Holt and S. Rees, 'Testing modules for irreducibility', *J. Aust. Math. Soc. Ser. A* 57 (1994) no. 1, 1–16.

23. K. Lux and H. Pahlings, *Representations of groups: a computational approach*, Cambridge Studies in Advanced Mathematics 124 (Cambridge University Press, Cambridge, 2010).

24. G. Malle and R. Rouquier, 'Familles de caractères de groupes de réflexions complexes', *Represent. Theory* 7 (2003) 610–640 (electronic).

25. M. Martino, 'The Calogero–Moser partition and Rouquier families for complex reflection groups', *J. Algebra* 323 (2010) no. 1, 193–205.

26. M. Martino, 'Blocks of restricted rational Cherednik algebras for $\mathrm{G}(m, d, n)$', *J. Algebra* 397 (2014) 209–224.

27. R. A. Parker, 'The computer calculation of modular characters (the Meat-Axe)', *Computational group theory (Durham, 1982)* (Academic Press, 1984) 267–274.

28. A. Ram and A. Shepler, 'Classification of graded Hecke algebras for complex reflection groups', *Comment. Math. Helv.* 78 (2003) no. 2, 308–334.

29. A. Steel, 'Construction of ordinary irreducible representations of finite groups', PhD Thesis, University of Sydney, 2012.

30. U. Thiel, 'A counter-example to Martino's conjecture about generic Calogero–Moser families', *Algebr. Represent. Theory* 17 (2014) no. 5, 1323–1348.

31. U. Thiel, 'Decomposition matrices are generically trivial', Preprint, 2014, arXiv:1402.5122.

32. U. Thiel, 'On restricted rational Cherednik algebras', Dissertation, TU Kaiserslautern, 2014.

33. U. Thiel, 'CHAMP: a Cherednik algebra Magma package', Preprint, 2015, arXiv:1403.6686.

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