Bull. Aust. Math. Soc. **89** (2014), 415–419
doi:10.1017/S0004972713000658

SPECIAL MAXIMAL SUBGROUPS OF p-GROUPS

JOHN COSSEY

(Received 31 May 2013; accepted 10 June 2013; first published online 7 August 2013)

Abstract

In the 2006 edition of the *Kourovka Notebook*, Berkovich poses the following problem (Problem 16.13): *Let p be a prime and P be a finite p-group. Can P have every maximal subgroup special?* We show that the structure of such groups is very restricted, but for all primes there are groups of arbitrarily large size with this property.

2010 *Mathematics subject classification*: primary 20D15.

Keywords and phrases: p-groups, special maximal subgroups.

1. Introduction

In the 2006 edition of the *Kourovka Notebook* (see [5]), Berkovich poses the following problem (Problem 16.13):

Let p be a prime and P be a finite p-group. Can P have every maximal subgroup special?

Berkovich defines a special p-group S to be one which satisfies $S' = Z(S) = \Phi(S)$ (where $Z(S)$ is the centre of S and $\Phi(S)$ is the Frattini subgroup of S). Other authors allow S to be elementary abelian; we will follow Berkovich here. Clearly, the requirement that all maximal subgroups be special places severe restrictions on the structure of such a group. Our aim here is to establish some of these restrictions and to show that such groups are more common than might be expected. We show that, except for some small values of n, there will be examples on n generators for any prime p.

Throughout this paper, P will be a p-group and n will denote the minimal number of generators of P.

THEOREM 1.1. *Let p be a prime and P a p-group with minimal number of generators n.*

(i) *If $n = 2$ then P contains a maximal subgroup which is not special.*
(ii) *If $n = 3$, then P contains a maximal subgroup which is not special unless $p = 3$, P has exponent at most 9 and P has class 3. There are exactly two 3-groups on three generators with all maximal subgroups special.*

(iii) *For $n \geq 4$ and all p, if P is not special then P contains a maximal subgroup which is not special. There exist n-generator special p-groups P with maximal subgroups special for all $n \geq 4$ and all p.*

If P has all maximal subgroups special then Theorem 1.1 gives a bound of $p^{n(n+1)/2}$ on the order of P, and hence for given n, p there are only finitely many n-generator p-groups with all maximal subgroups normal.

If P is a group in which every maximal subgroup is special then P cannot be abelian. Thus P has at least two generators and any proper subgroup of P has exponent dividing p^2 and is nilpotent of class at most 2. We begin with a general observation. If $x \in P$, then x is contained in a special subgroup and hence has order at most p^2. Further, if x^p is not in P', then $\langle x, P' \rangle$ is contained in a maximal subgroup M such that x^p is not in M' and so M is not special, which is a contradiction. Note also that if P has at least three generators then for any $x, y \in P$ we have $[x, y, y] = 1$ since $\langle x, y \rangle$ is nilpotent of class at most 2. The following lemma is an immediate consequence.

Lemma 1.2. *If P is a group in which every maximal subgroup is special, then P has exponent dividing p^2 and P/P' has exponent p (and hence $P' = \Phi(P)$). Further, if P has at least three generators then P satisfies the second Engel condition ($[x, y, y] = 1$ for all $x, y \in P$).*

Our notation is mostly standard and usually follows Doerk and Hawkes [2]. Note that the ith term of the lower central series of a group G is denoted by $\mathrm{K}_i(G)$.

2. The 2-generator case

Suppose $P = \langle x, y \rangle$. If $u \in P \backslash P'$, then $M = \langle u, P' \rangle$ is maximal and hence special (and so, in particular, nonabelian). Conversely, if M is maximal in P, then $M = \langle u, P' \rangle$ for some $u \in P \backslash P'$. Note that $M' \leq K_3(P)$. If $u^p \notin \mathrm{K}_3(P)$ then $u^p \notin M'$ and so M/M' has exponent p^2, which is a contradiction. We now have $u^p \in \mathrm{K}_3(P)$ for all $u \notin P'$. Thus $P/\mathrm{K}_3(P)$ has exponent p and so is extraspecial of exponent p with p odd. It follows that $M/\mathrm{K}_3(P)$ is elementary and then $[u, v] \notin M'$ and also that P has class at least 3. Since $\langle y, [x, y] \rangle$ has class at most 2, it follows that P satisfies the third Engel condition (that is, $[x, y, y, y] = 1$ for any $x, y \in P$).

Suppose P has class 3. We have that $M = \langle u, P' \rangle$ is maximal. If $[u, v, v] \notin \langle u, [u, v] \rangle'$ then $[u, v, v] \in Z(M) \backslash M'$ and M is not special, which is a contradiction. Thus we must have $[u, v, v] \in \langle [u, v, u] \rangle$. If $[u, v, v] = [u, v, u]^t$, then $[u, v, vu^{-t}] = 1$. But now $\langle vu^{-t}, [u, v], [u, v, u] \rangle$ is an abelian maximal subgroup, which is again a contradiction.

Hence P must have class at least 4 and so P must be a 2-group of class 4 with $\mathrm{K}_3(P)$ of order two (Heineken [3, Satz 1 and Zusatz 2]), contradicting p odd.

3. The 3-generator case

Since every maximal subgroup of P is special, every 2-generator subgroup of P has class at most 2 and hence by a theorem of Levi (Neumann [6, Theorem 34.31]) P is a

https://doi.org/10.1017/S0004972713000658 Published online by Cambridge University Press

group of class 2 unless $p = 3$. If $p = 3$, then P has class at most 3 and if it has class 3, $\mathrm{K}_3(P)$ has exponent 3.

Suppose P has class 2 and let M be a maximal subgroup of P. Then for any $x, y \in P$, $M = \langle x, y, P' \rangle$ is special and hence $\langle [x, y] \rangle = M'$ has order p. Moreover, if $M' < P'$ and $c \in P' \backslash M'$ then $c \in P' \cap M \leq Z(P) \cap M \leq Z(M)$ and so $c \in Z(M) \backslash M'$, which is a contradiction. Thus P' is cyclic of order p. If $Z(P) = P'$ we would have P special and thus extraspecial. But $|P/Z(P)| = p^3$, contradicting Huppert [4, Satz III.13.7.c]. It follows that $Z(P) > P'$. If $z \in Z(P) \backslash P'$ and $x \notin \langle z, P' \rangle$ then $\langle x, z, P' \rangle$ is an abelian maximal subgroup, which is a contradiction. Hence P cannot be of class 2.

Now suppose that P has class 3 (so that P is a 3-group and $\mathrm{K}_3(P)$ has exponent 3). Suppose also that P has exponent 3. Since the free group B of exponent 3 on three generators has order 3^7 with $Z(B) = \mathrm{K}_3(B)$ of order three, B has all proper quotients of class 2 and hence we must have $P \cong B$. Suppose that M is maximal in P, so that $M = \langle x, y, P' \rangle$. Let $u \in P \backslash M$. Then it is easy to calculate that $M' = \langle [x, y], [x, u, y] \rangle = Z(M) = \Phi(M)$ and so M is special.

A small variation on the above group gives another example. Let B be the free group of exponent 3 on generators x, y, z and $C = \langle c \rangle$ be a cyclic group of order nine. In $B \times C$ let $d = cx$ and set $Q = \langle d, y, z \rangle$. Then $c^3 = d^3 \leq Z(Q)$. If we set $Z = \langle d^3 \rangle$ and then $P = Q/Z$, P is a group of order 3^7 with all maximal subgroups special.

Now suppose that P has exponent 9. Then $P'/\mathrm{K}_3(P)$ is an abelian group of exponent 3 and so has order at most 27. It follows from the second Engel condition and Witt's identity [4, Satz III.1.4] that $|P| \leq 3^7$. A computer search shows that the group of the previous paragraph is the only such group with all maximal subgroups special. (I am grateful to Mike Newman for doing the search.)

4. The n-generator case, $n \geq 4$

Note that as above P will have P/P' of exponent p and P' elementary abelian. Since $n \geq 4$ any three elements will generate a proper subgroup and hence a subgroup of class at most 2. This gives that any commutator of weight 3 is trivial. Hence P has class 2 and so $P' \leq Z(P)$. Clearly, if $z \in Z(P) \backslash \Phi(P)$ there is a maximal subgroup M of P with $z \in M \backslash \Phi(P)$. We then have M not special, which is a contradiction. Thus $P' \leq Z(P) \leq \Phi(P)$. On the other hand, by Lemma 1.2 we have $\Phi(P) = P'$. We now have $P' = Z(P) = \Phi(P)$ and so P is special. As a consequence, $|P| \leq p^{\frac{1}{2}n(n+1)}$ and so for any prime p there are only a finite number of n-generator p-groups with every maximal subgroup special.

Observe that if P is special and a maximal subgroup M is also special we have $P' = Z(P) = \Phi(P) \leq M$ and $M' = Z(M) = \Phi(M)$ and thus $Z(P) \leq Z(M) = M' \leq P'$. It now follows that $Z(P) = Z(M)$, $P' = M'$ and $\Phi(P) = \Phi(M)$.

We have seen that if P has two or three generators and is special it cannot have all maximal subgroups special. We now show that for all $n \geq 4$ and all primes p there are special p-groups on n generators with all maximal subgroups special.

Our examples start with the following groups. Let F be a field of characteristic p and order p^m, $m \geq 2$, and set $P = \{(a, b, c) \ : \ a, b, c \in F\}$, with multiplication defined

https://doi.org/10.1017/S0004972713000658 Published online by Cambridge University Press

by $(a, b, c)(d, e, f) = (a + d, b + e, c + f + ae)$. That P is a special p-group is proved by Beisiegel [1]. (These groups have been called ultraspecial by Beisiegel [1]; note also that P is the Sylow p-subgroup of $\mathrm{SL}_3(F)$.) Set $H = \{(a, 0, 0) : a \in F\}$ and $K = \{(0, a, 0) : a \in F\}$ and note that $P' = \{(0, 0, a) : a \in F\}$. We will also need some information about centralisers of elements in P. We have $P' = Z(P)$. Suppose that $u = (a, b, 0) \in P \backslash P'$. If $a \neq 0$ then $C_K(u) = 1$. To see this, if $v = (0, b, 0) \in K$ then $[u, v] = (0, 0, ab) = 1$ if and only if $b = 0$. The maximal subgroups of P are of three types, which we consider separately. It is easy to see that M is a maximal subgroup of P if and only if $M = \langle A, K, P' \rangle$, $M = \langle H, B, P' \rangle$ or $M = \langle xy, A, B, P'' \rangle$, with A and B maximal subgroups of H and K respectively and $x \in H \backslash A$, $y \in K \backslash B$.

Suppose now that $N = \langle A, K, P' \rangle$ with $1 \neq A \leq H$. If $1 \neq x \in A$ then $[x, K] = P'$. To see this, if $x = (a, 0, 0)$ $(a \neq 0)$ and $u = (0, 0, c) \in P'$, we set $y = (0, a^{-1}c, 0)$ and then we have $[x, y] = u$. Thus $N' \leq \Phi(N) \leq \Phi(P) = P' \leq N'$, giving $P' = N' = \Phi(N)$. Moreover, if $y = (0, b, 0)$ then $[x, y] = (0, 0, ab)$ and so $C_K(x) = 1$. It now follows that $Z(N) = P'$ and so any subgroup of P containing K and a nontrivial subgroup of H is special. Similarly, any subgroup of P containing H and a nontrivial subgroup of K is special. In particular, maximal subgroups of the first two types are special.

If M is a maximal subgroup of the third type, with $x = (a, 0, 0)$ and $y = (0, b, 0)$, then $xy = (a, b, ab)$. Take $1 \neq v = (d, 0, 0) \in A$ and $u = (0, c, 0) \in B$. We have $[v, u] = (0, 0, dc)$ and since $[v, u] = 1$ if and only if $c = 0$ we have $|[v, B]| = p^{m-1}$. Note that $[v, xy] = (0, 0, db) \notin [v, B]$. Similarly, we have $[xy, A] = \{(0, 0, be) : e \in A\}$. Since $(0, 0, bd) \in [xy, A] \backslash [v, B]$, we have $\langle [xy, A], [v, B] \rangle = P'$. Thus $M' = P'$. Now let $u = (e, d, c) \in Z(M)$. Since $B \neq 1$ we must have $e = 0$ by the remark above. Similarly, since $A \neq 1$ we must have $d = 0$ and so $u \in P'$. Since $P' \leq Z(M)$ we have $P' = Z(M)$ and then M special.

This establishes the existence of n-generator special groups with all maximal subgroups special for even $n \geq 4$ and any prime p. For odd $n \geq 5$ we use subgroups of the examples above. Let P be the ultraspecial p-group on $2m$ generators $(m \geq 3)$ with subgroups H, K as above. Let f be a generator of the multiplicative group of F and then let E be a maximal subgroup of the additive group of F containing both f and the multiplicative identity ι of F. Then $A = \{(a, 0, 0) : a \in E\}$ is a maximal subgroup of H. Set $Q = \langle A, K, P' \rangle$ and note that the argument above shows $Q' = P'$. If M is a maximal subgroup of Q, then $M = \langle C, K, P' \rangle$, $M = \langle A, B, P' \rangle$ or $M = \langle x, S, T, P' \rangle$, where C and S are maximal subgroups of A, B and T are maximal subgroups of K and $x = (a, b, 0)$ with $(a, 0, 0) \notin S$, $(0, b, 0) \notin T$. Let $S_0 = \{a : (a, 0, 0) \in S\}$ and $T_0 = \{b : (0, b, 0) \in T\}$ (note S_0 and T_0 are subgroups of F).

If $M = \langle C, K, P' \rangle$ then as above we have M special.

If $M = \langle A, B, P' \rangle$ we suppose first that $\{a : (a, 0, 0) \in A\} \neq \{b : (0, b, 0) \in B\}$. If $u = (a, 0, 0)$ with $a \notin \{b : (0, b, 0) \in B\}$ then $[u, (0, b, 0)] = (0, 0, ab)$ and $[(\iota, 0, 0), (0, b, 0)] = (0, 0, b)$. Since $\{(0, 0, b) : (0, b, 0) \in B\}$ is a maximal subgroup of P' and $[u, (0, b, 0)] \notin \{(0, 0, b) : (0, b, 0) \in B\}$, we have $M' = P'$. Since nontrivial elements of A do not commute with nontrivial elements of B we have $Z(M) \leq P' = M' \leq \Phi(M) \leq \Phi(Q) = Q' = M' = Z(Q) \leq Z(M)$, and hence M is special. Now suppose

https://doi.org/10.1017/S0004972713000658 Published online by Cambridge University Press

that $E = \{a : (a, 0, 0) \in A\} = \{b : (0, b, 0) \in B\}$. Since $\iota \in E$ we have $[(\iota, 0, 0), (0, b, 0)] = (0, 0, b) \in M'$ and so $\{b : (0, 0, b) \in M'\} \geq E$. If $\{b : (0, 0, b) \in M'\} = E$ we have $[(f, 0, 0), (0, b, 0)] = (0, 0, fb)$, and hence if $f^i \in E$ then $f^{i+1} \in E$ and so $F \leq E$, which is a contradiction. Since E is maximal in F we get $\{b : (0, 0, b) \in M'\} = F$ and so $M' = P'$. As above we now get M special.

Finally, suppose $M = \langle x, S, T, P' \rangle$ with $x = (a, b, 0)$ and $a \notin S_0$, $b \notin T_0$. Since $n \geq 6$ we have $S_0 \cap T_0 \neq 0$. If $u = (s, 0, 0) \in S$ and $0 \neq s \in S_0 \cap T_0$ then as above $[u, T] = \{(0, 0, st) : t \in T_0\} = \{(0, 0, t) : t \in T_0\}$ and so is a maximal subgroup of P'. If S_0 is not contained in T_0 let $s \in S_0 \setminus T_0$. We then have $[(s, 0, 0), T] = \{(0, 0, st) : t \in T_0\}$. Since $st \notin T_0$ for any t it follows that $\langle \{(0, 0, v) : v \in T_0 \cup sT_0\} \rangle = P'$. If $S_0 \leq T_0$, then $[x, S] = \{(0, 0, sb) : s \in S\}$. Since $S_0 \leq T_0$ and $b \notin T_0$ it follows that $[x, S]$ is not contained in $[u, T]$ and then $\langle [x, S], [u, T] \rangle = P'$. Thus $M' = P'$ and as above we have M special. This completes the proof of Theorem 1.1.

References

[1] B. Beisiegel, 'Semi-extraspezielle p-Gruppen', *Math. Z.* **156** (1977), 247–254.

[2] K. Doerk and T. Hawkes, *Finite Soluble Groups*, De Gruyter Expositions in Mathematics, 4 (De Gruyter, Berlin–New York, 1992).

[3] H. Heineken, 'Engelsche Elemente der Länge drei', *Illinois J. Math.* **5** (1961), 681–707.

[4] B. Huppert, *Endliche Gruppen I* (Springer, Berlin, Heidelberg, New York, 1967).

[5] V. D. Mazurov and E. I. Khukhro (eds.) *Unsolved Problems in Group Theory: The Kourovka Notebook*, 16th edn (Russian Academy of Sciences Siberian Division, Institute of Mathematics, Novosibirsk, 2006).

[6] H. Neumann, *Varieties of Groups* (Springer, Berlin, Heidelberg, New York, 1967).

JOHN COSSEY, Department of Mathematics, Mathematical Sciences Institute,
Australian National University, Canberra ACT 0200, Australia
e-mail: John.Cossey@anu.edu.au

https://doi.org/10.1017/S0004972713000658 Published online by Cambridge University Press