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PERIODIC ORBITS FOR GENERALIZED GRADIENT FLOWS

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ABSTRACT. Let M^n be an n-dimensional compact oriented connected Riemannean manifold. It is proved that either of the following conditions is sufficient to insure that the flow defined by a generalized gradient vector field in M^n has either a stationary point or a periodic orbit:

a) M^n is the product of a circle with an $(n-1)$ dimensional manifold of non-zero Euler characteristic.

b) The $(n-1)$ dimensional Stiefel-Whitney class of M^n is different from zero and in addition M^n possesses no one-dimensional 2-torsion.

In what follows M^n will always be a compact oriented connected n-dimensional Riemannean manifold. If V is a smooth vector field in M^n, using the Riemannean metric we get by duality a smooth one-form α. When α is a closed form we say the flow defined by V is a generalized gradient flow. It is known that every generalized gradient flow in M^n has a stationary point exactly when it is impossible to fibre M^n smoothly over the circle [5]. We are going to get two theorems each of which gives topological conditions on M^n that guarantee that every generalized gradient flow on M^n possesses either a stationary point or a periodic orbit.

THEOREM 1. *Suppose that the $(n-1)$ dimensional Stiefel-Whitney class of M^n is different from zero and that M^n possesses no one-dimensional 2-torsion. Then any generalized gradient flow on M^n has either a stationary point or a periodic orbit.*

PROOF. Suppose we have a generalized gradient flow on M^n that has no stationary point. Then the tangent bundle of M^n is the direct sum of the normal bundle to the vector field V defining the flow and the trivial one-dimensional bundle. Therefore the normal bundle to V has its $(n-1)$ dimensional Stiefel-Whitney class different from zero. Thus the Euler class with integer coefficients of the normal bundle to V is different from zero. Since M^n has no one-dimensional 2-torsion, the $(n-1)$ dimensional cohomology group of M^n with integer coefficients has no 2-torsion. From these considerations it follows that the Euler class of the normal bundle to V with real coefficients is different from zero. In the note at the end of [4] it is pointed out that if the normal bundle to a nowhere vanishing vector field V on a compact oriented manifold has the property that its Euler class with real coefficients is different from zero and the flow carries a closed one form, then there is a periodic orbit. In our present situation, if α is the closed one-form dual to V then the interior product of α and V is nowhere zero. Our theorem follows, and in fact it is clear

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that all that was needed for the proof was the assumption that our flow carried a closed one-form α together with the topological assumptions about M^n.

THEOREM 2. *Suppose that M^n is diffeomorphic to the product of the circle with a manifold N of Euler characteristic different from zero. Then every generalized gradient flow on M^n possesses either a stationary point or a periodic orbit.*

PROOF. Using the representation of M^n as the product of the circle with N, let the real line act on M^n by acting periodically on the circle with period one and acting via the identity map on N. We will refer to this action of the real line as the canonical periodic flow on M^n.

Next let α be the closed one-form on M^n dual to the vector field V defining our generalized gradient flow, and let λ be the one-dimensional homology class on M^n corresponding to an orbit under the canonical periodic flow. Assume that V and therefore α is never zero. We can approximate α as closely as we want to in the C^1 topology by a differential form β such that the cap product of λ with the one-dimensional real cohomology class determined by β is not zero and so that this one-dimensional real cohomology class is the image under the coefficient homomorphism of a one-dimensional rational cohomology class. Since the interior product of α and V is never zero, one may therefore chose β so that the interior product of β and V is never zero, and the cap product of λ with the one-dimensional cohomology class determined by β is not zero.

Next we can choose a positive rational number r so that $r\beta$ determines a real one-dimensional cohomology class that is the image under the coefficient homomorphism of an integral one-dimensional cohomology class, and if we wish we may assume that this integral cohomology class is not a positive integral multiple of any other integral cohomology class. Then there will exist a complex valued C^1 function ψ on M^n of absolute value one such that in any simply connected open set O, if we write ψ on O in the form $e^{2\pi i f}$ with f real valued, then on O $r\beta$ equals df.

Since the interior product of $r\beta$ and V is not zero and the integral cohomology class corresponding to $r\beta$ is not a positive integral multiple of any other integral cohomology class, the set on which ψ equals one is a connected cross section to our generalized gradient flow. With any cross section to a flow there is associated a continuous function of absolute value one and hence an integral one-dimensional cohomology class; in our case the cohomology class is that corresponding to $r\beta$.

Next we turn to the canonical periodic flow on M^n. The asymptotic cycle [2] determined by each orbit is λ. Therefore for any invariant measure μ the μ-*asymptotic* cycle is λ. Either the cap product of λ with the cohomology class determined by $r\beta$ or the cap product of λ with the negative of this cohomology class is positive. Suppose the canonical flow does not have a cross section associated with the cohomology class of ψ. If the canonical flow takes the pair (t,x) into $F(t,x)$, define the reverse canonical flow to be the flow that sends (t,x) into $F(-t,x)$. Then by the necessary and sufficient condition given for the existence of a cross section in [2], the reverse canonical flow will have a cross section associated with the cohomology class of ψ.

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Thus the cohomology class of ψ will have a cross section associated with our generalized gradient flow and also one associated either with the canonical periodic flow or its reverse.

However, by Theorem 3 of [3] when K and K' are cross sections of two different flows in a compact manifold M^n associated with the same cohomology class, then $K \times R$ is homeomorphic to $K' \times R$, so K and K' have the same homotopy type and therefore the same Euler characteristic. Now a theorem of Fuller [1] asserts that a homeomorphism of a compact polyhedron of Euler characteristic different from zero onto itself possesses a periodic point. If we can show that the cross section of our generalized gradient flow has Euler characteristic different from zero it will follow that there is a periodic orbit. By Fuller's theorem it is only necessary to show that the cross section of the canonical periodic flow or its reverse associated with the cohomology class of ψ has Euler characteristic different from zero.

However by the way β was chosen, the cap product of the one-dimensional cohomology class determined by β with λ is not zero. Let ρ be the Euler characteristic of N; by assumption this is not zero. Therefore the cap product of the one-dimensional cohomology class determined by $r\beta$ with $\rho\lambda$ is not zero.

Next we note that the cross section of the canonical periodic flow (or its reverse) associated with the integral cohomology class determined by ψ determines an $(n-1)$ dimensional real homology class on M^n that corresponds via Poincaré duality to $r\beta$. Moreover the $(n-1)$ dimensional real cohomology class which is the Euler class of the normal bundle to the canonical periodic flow corresponds via Poincaré duality to $\rho\lambda$. Therefore the cap product of the $(n-1)$ dimensional homology and cohomology classes we are considering on M^n is not zero. But this cap product is essentially nothing but the Euler characteristic of our cross section to the canonical periodic flow (or its reverse). Thus our theorem is proved.

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