(negative). We have $B_n \sim \frac{1}{\varepsilon_n}$. Set r_n to be a positive sequence with $r_n \to 1$. Now, Theorem 2 is equivalent to

$$\lim_{n \to \infty} (1 + \varepsilon_n)^{\frac{r_n}{\varepsilon_n}} = e. \tag{1}$$

The sign of ε_n does not matter for this limit, so we can generalise the left-hand side of (1). For any constant k and δ_n a sequence with $|\delta_n|$ monotone decreasing to 0, we have

$$\lim_{n \to \infty} (1 + \varepsilon_n)^{\delta_n + k} = 1. \tag{2}$$

Multiplying (1) by (2) we obtain

$$\lim_{n \to \infty} (1 + \varepsilon_n)^{\frac{r_n}{\varepsilon_n} + \delta_n + k} = e. \tag{3}$$

This allows the reader to choose parameters to optimise convergence.

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108.28 π is a mean of 2 and 4

A series of *Mathematical Gazette* contributions, [1, 2, 3, 4], deals with limits of infinite sequences where the first n entries are specified and where latter entries correspond to a specified type of average of the n preceding entries. To the list of recursively defined averages may be added also the more well-known arithmetic-geometric mean, the arithmetic-harmonic mean and the geometric-harmonic mean. We are not aware of studies of recursions where some property of the index k dictates what average to consider when calculating a_k from preceding entries. There is virtually no limit to the possibilities we hereby face, but we will in this short Note restrict ourselves to cases where only a_0 and a_1 are specified and where the parity of k dictates which one of two specified averages to consider in order to generate a_k from a_{k-1} and a_{k-2}.

We shall define six functions $\alpha(x, y)$, $\beta(x, y)$ and $\gamma(x, y)$ as well as $\alpha^*(x, y)$, $\beta^*(x, y)$ and $\gamma^*(x, y)$. The functions are, as will be seen, defined in analogous ways and we provide a definition of $\gamma(x, y)$ to start with.

Definition: For non-negative real numbers x and y, set $a_0 = x$ and $a_1 = y$. When for integers $k \geqslant 2$ we set a_k equal to the **geometric** mean of a_{k-1} and a_{k-2} if k is even, and the **harmonic** mean of a_{k-1} and a_{k-2} if k is odd, then we refer to $\lim_{n\to\infty} a_k$ as $\boldsymbol{\gamma(x, y)}$.

Using the same non-bold text while changing the bold parts allows to define the other functions in a compact way. If replacing the non-bold parts with triple dots we get as compact definitions

"… arithmetic… geometric… $\alpha(x, y)$" "… geometric … arithmetic … $\alpha^*(x, y)$"
"… arithmetic… harmonic… $\beta(x, y)$" "… harmonic… arithmetic … $\beta^*(x, y)$"
"… geometric… harmonic… $\gamma(x, y)$" "…harmonic … geometric… $\gamma^*(x, y)$".

Among trivial rules may be mentioned that

$$\alpha(x, y) = \alpha^*\left(y, \tfrac{1}{2}(x + y)\right) \qquad \alpha^*(x, y) = \alpha\left(y, \sqrt{xy}\right)$$
$$\beta(x, y) = \beta^*\left(y, \tfrac{1}{2}(x + y)\right) \qquad \beta^*(x, y) = \beta\left(y, \tfrac{2xy}{x+y}\right)$$
$$\gamma(x, y) = \gamma^*\left(y, \sqrt{xy}\right) \qquad \gamma^*(x, y) = \gamma\left(y, \tfrac{2xy}{x+y}\right)$$

which not only (implicitly) remind us about the definitions of the involved averages, but also show us that whenever a general closed form solution is established for one of the functions we get a closed form solution for one of the other functions essentially for free (specifically the one with the same letter, with or without the star).

Computing to high precision the values of any of the described functions for arbitrary x and y is a straightforward exercise. Linking the arrived-at numbers to known constants or expressions present a tougher challenge, but the work is simplified by the rich content and handy search functions on the Online Encyclopedia of Integer Sequences (OEIS) [5]. The following results limited to $x = 1$ and $y = 2$ were established (validated to machine precision) after direct or more tedious searching (references to specific sequences on OEIS are given in brackets)

$\alpha(1, 2) = \frac{3\sqrt{3}}{\pi}$ [A306712] $\alpha^*(1, 2) = \frac{\sqrt{2}}{\ln(1+\sqrt{2})}$ [divide A169800 by 2)]

$\beta(1, 2) = \sum_{j=0}^{\infty} 2^{-T_j}$ [A299998] $\beta^*(1, 2) = \dfrac{\sum_{j=0}^{\infty} 2^{1-2T_j}}{\sum_{j=0}^{\infty} 2^{-T_j}}$ [via A299998]

$\gamma(1, 2) = \frac{\pi}{2}$ [A019669] $\gamma^*(1, 2) = \frac{2\ln(2+\sqrt{3})}{\sqrt{3}}$ [multiply A196530 by 2]

where the j-th triangular number $T_j = \frac{1}{2}j(j + 1)$.

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A range of other special identities were established (not presented here), but based on these we have thus far – among $\alpha(x, y)$, $\beta(x, y)$ and $\gamma(x, y)$ – only managed to crack the pattern for $\gamma(x, y)$, for which we propose any of the following relations

$$\gamma(x, y) = \frac{y}{\sqrt{\frac{y}{x} - 1}} \tan^{-1}\sqrt{\frac{y}{x} - 1}, \tag{1}$$

$$\gamma(x, y) = \frac{y}{\sqrt{1 - \frac{y}{x}}} \tanh^{-1}\sqrt{1 - \frac{y}{x}}. \tag{2}$$

The equations are equivalent but the implicit use of complex arguments can be avoided when using (1) for cases when $y > x$ and (2) for cases when $y < x$. For completeness may be added the trivial result that $\gamma(x, x) = x$. A key step in approaching the equation(s) was the discovery that the numerically determined value of $\gamma(2, 1)$, following division by $\sqrt{2}$, equalled 0.8813735 …, a decimal expansion present on OEIS [sequence A091648] including the remark that it equals $\tanh^{-1}\frac{1}{\sqrt{2}}$. Note that while (1) (and (2)) has been validated numerically it lacks a rigorous proof. It seems that (1) and (2) apply also when extending the definition of $\gamma(x, y)$ to the domain of complex numbers.

A suitable exercise (for various levels) is to show that (2), in combination with Euler's identity $\left(e^{i\pi} = -1\right)$, basic rules of logarithms, and the well-known identity $\tanh^{-1} z = \frac{1}{2}\ln\frac{1+z}{1-z}$, aligns with the result that justifies the title of this work

$$\gamma(2, 4) = \pi. \tag{3}$$

Another exercise of similar complexity is to arrive at the corollary result that for real $t \geqslant 1$

$$\ln t = \frac{t^2 - 1}{2t}\gamma\left(1, \frac{4t}{(t + 1)^2}\right). \tag{4}$$

We end this Note with some further encouragements to the wider community:

- Prove (1) rigorously.
- Find closed-form solutions for $\alpha(x, y)$ and $\beta(x, y)$ (and prove these).
- Extend to the cases when generating the next element from a k-dependent type of average of the n most recent elements.

I declare no conflict of interest and apologise in advance should it turn out that the key proposal (1) is merely a rediscovery. I am thankful to Andreas Dieckmann, Roberto Tauraso and an anonymous reviewer for help with high-precision numerical validation, guidance to relevant literature and help with improving the original manuscript, respectively.

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108.29 A geometric mean–arithmetic mean ratio limit

One of the truly delightful results related to the natural numbers is the following limit of the ratio of the geometric and arithmetic means of the first n natural numbers:

$$\lim_{n \to \infty} \frac{\sqrt[n]{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n}}{\frac{1}{n}(1 + 2 + 3 + \ldots + n)} = \frac{2}{e}. \tag{1}$$

Obviously, the ratio in (1) approaches its limit really slowly. In fact, the relative difference between the ratio and its limiting value is of order $(n + 1)^{(2n)^{-1}}$, as $n \to \infty$. For example, this is about 2% when $n = 100$.

Some generalisations of the limit can be found in [1], [2] and [3].

In this Note, we offer a short proof and generalisation of limit (1). Our result is narrower here, but the techniques are wholly different from [1], [2] and [3], and rely solely, in theory, on algebraic limit properties. Our proof relies on the following well-known result.

Lemma [See e.g. [4, p. 81]]: Let a_n be a sequence of positive reals with $\lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n} = L$. Then $\lim\limits_{n \to \infty} \sqrt[n]{a_n} = L$.

We now establish a generalisation of (1) in the following theorem.

Theorem: Let $\{b_n\}$ be a sequence of positive reals with $\lim\limits_{n \to \infty} b_n - n = 0$. Then

$$\lim_{n \to \infty} \frac{\sqrt[n]{b_1 b_2 b_3 \ldots b_n}}{\frac{1}{n}(b_1 + b_2 + b_3 + \ldots + b_n)} = \frac{2}{e}.$$

Proof: We apply the Lemma to

$$a_n = \frac{\prod_{i=1}^{n} b_i}{\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right)^n}.$$