DECOMPOSITION OF FINITE GRAPHS INTO OPEN CHAINS

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1. Introduction. If m, n are integers, "$m \equiv n$" will mean "$m \equiv n \pmod 2$)." The cardinal number of a set A will be denoted by $|A|$. The set whose elements are $a_1, a_2, \ldots, a_n$ will be denoted by $\{a_1, a_2, \ldots, a_n\}$. The empty set will be denoted by $\emptyset$. If A, B, C are sets, $A - B$ will denote the set of those elements of A which do not belong to B, and $A - B - C$ will denote $(A - B) - C$. The expression $\sum_{\xi \in A} f(\xi)$ will be denoted by $f.A$. The statements "$f = g$ on A," "$f \equiv g$ on A" will mean that $f(\xi) = g(\xi)$ or $f(\xi) \equiv g(\xi)$ respectively for every $\xi \in A$.

An *unoriented graph* U consists, for the purposes of this paper, of two disjoint finite sets $V(U)$, $E(U)$, together with a relationship whereby with each $\lambda \in E(U)$ is associated an unordered pair of (not necessarily distinct) elements of $V(U)$ which λ is said to *join*. An *oriented graph* is a triple $N = (U, t, h)$, where U is an unoriented graph and t, h are mappings of $E(U)$ into $V(U)$ such that each $\lambda \in E(U)$ joins λt to λh. We write $V(U) = V(N)$, $E(U) = E(N)$ and call λt, λh the *tail* and *head* of λ respectively. Either an unoriented or an oriented graph may be referred to as a *graph*. Throughout this paper, U will denote an unoriented graph, N will denote an oriented graph, and G may denote either. The elements of $V(G)$ and $E(G)$ are called *vertices* and *edges* of G respectively. A *subgraph* of U is an unoriented graph H such that $V(H) \subset V(U)$, $E(H) \subset E(U)$ and each edge of H joins the same vertices in H as in U. A *subgraph* of $N = (U, t, h)$ is an oriented graph (U_1, t_1, h_1) such that U_1 is a subgraph of U and t_1, h_1 are the restrictions of t, h respectively to $E(U_1)$. An *orientation* of U is an oriented graph of the form (U, t, h). A vertex ξ and edge λ of G are *incident* if ξ is one or both of the vertices joined by λ. The *order*, ord G, of G is $|V(G) \cup E(G)|$. G is *empty* if $V(G) = E(G) = \emptyset$. The *degree* $d(\xi)$ of a vertex ξ of a graph is $2a(\xi) + b(\xi)$, where $a(\xi)$ is the number of edges joining ξ to itself and $b(\xi)$ is the number joining ξ to other vertices. A vertex is *even* or *odd* according as its degree is even or odd respectively. G is *Eulerian* if its vertices are all even. A collection of subgraphs of G are *disjoint* (*edge-disjoint*) if no two of them have a vertex (edge) in common. The *union* of the subgraphs $H_1, H_2, \ldots, H_n$ of G is the subgraph H of G such that

$$V(H) = \bigcup_{i=1}^{n} V(H_i), \qquad E(H) = \bigcup_{i=1}^{n} E(H_i).$$

A *decomposition* of G is a set of edge-disjoint subgraphs of G whose union is G. G is *connected* if it is not the union of two disjoint non-empty subgraphs. The

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components of a non-empty graph are its maximal connected subgraphs. (An empty graph is deemed to have 0 components.) A *chain-sequence* of G is a finite sequence

$$\xi_0, \lambda_1, \xi_1, \lambda_2, \xi_2, \lambda_3, \ldots, \lambda_n, \xi_n \qquad (n \geqslant 0)$$

in which the ξ_i are vertices of G, the λ_i are *distinct* edges of G and λ_i joins ξ_{i-1} to ξ_i for $i = 1, 2, \ldots, n$. If G is an oriented graph, this chain-sequence is *forwards-directed* if

$$\lambda_i t = \xi_{i-1}, \lambda_i h = \xi_i \qquad (i = 1, 2, \ldots, n)$$

and *backwards-directed* if

$$\lambda_i h = \xi_{i-1}, \lambda_i t = \xi_i \qquad (i = 1, 2, \ldots, n).$$

A finite sequence is *closed* or *open* according as its first and last terms are the same or different respectively. If s is a chain-sequence of G, the subgraph of G formed by those vertices which appear at least once and those edges which appear exactly once in s will be said to be *derived* from s. A subgraph of G is an *open chain* of G if it is derivable from an open chain-sequence of G. If ξ, η are the first and last terms of an open chain-sequence s of G and C is the open chain derived from s, then clearly ξ, η are odd in C and every other vertex of C is even in C. It follows that an open chain has precisely two odd vertices which are the end-terms of every chain-sequence from which it is derivable; these are called the *end-vertices* of the open chain. If S, T are subsets of $V(G)$, $\bar{S}$ will denote $V(G) - S$, $S \circ T$ will denote the set of those edges of G which join elements of S to elements of T, and $S\delta$ will denote $S \circ \bar{S}$. A subgraph of G is an *ST-chain* if it is derivable from a chain-sequence of G whose first and last terms belong to S, T respectively. A *cincture* of G is a subset of $E(G)$ which is of the form $S\delta$ for some subset S of $V(G)$. If $\xi \in V(N)$, an edge λ is an *exit* of ξ if $\lambda t = \xi$ and an *entry* of ξ if $\lambda h = \xi$. The number of exits [entries] of ξ will be denoted by $x(\xi)$ [$e(\xi)$]. The *flux out of* ξ, denoted by $f(\xi)$, is $x(\xi) - e(\xi)$. N is *quasi-symmetrical* if $x = e$ on $V(N)$. A *route-sequence* of N is a chain-sequence of N which is either forwards- or backwards-directed. A subgraph of N is a *route* (*closed route*, *open route*) of N if it is derivable from a route-sequence (closed route-sequence, open route-sequence) of N.

When, to avoid ambiguity, it is necessary to specify the graph relative to which a graph-theoretical symbol is defined, the letter denoting the graph will be attached to the symbol in some convenient way. For example, if ξ is a common vertex of two oriented graphs M and N, $d_M(\xi)$ will denote the degree of ξ in M. We shall, however, make the convention that, in any context in which an oriented graph denoted by the letter N is under consideration, all graph-theoretical symbols relate to N unless the contrary is indicated; for example, $d(\xi)$ would mean $d_N(\xi)$ in the situation instanced above.

Let s be a forwards-directed route-sequence of N, R be the route derived from s and ξ, η be the first and last terms of s respectively. Then clearly R is quasi-symmetrical if $\xi = \eta$ and $f_R(\xi) = 1, f_R(\eta) = -1$ and $f_R = 0$ on $V(R) - \{\xi, \eta\}$ if $\xi \neq \eta$. It follows that a closed route cannot also be an open route and that an open route R has uniquely determined vertices ξ, η such that $f_R(\xi) = 1$, $f_R(\eta) = -1$ and ξ, η are the first and last terms respectively of every forwards-directed route-sequence from which R is derivable; we call ξ, η the *tail* and *head* respectively of R.

By a *G-function*, we shall mean a non-negative integer-valued function defined on the vertices of G. A G-function g is *congruential* if $g \equiv d$ on $V(G)$. If g is a G-function and $\xi \in S \subset V(G)$, $F_g(\xi; S)$ will denote

$$-g(\xi) + g\,.\,(S - \{\xi\}) + |S\delta|.$$

We shall call g *tolerable* if $F_g(\xi; S) \geqslant 0$ for every pair ξ, S such that $\xi \in S \subset V(G)$. A subset S of $V(G)$ is *g-critical* if $F_g(\xi; S) = 0$ for some $\xi \in S$. A cincture C of G is *g-critical* if $C = S\delta$ for some g-critical subset S of $V(G)$. A *g-chain-factor* of G is a set Φ of edge-disjoint open chains of G such that each vertex ξ of G is an end-vertex of exactly $g(\xi)$ elements of Φ. A *g-decomposition* of G is a g-chain-factor of G which is a decomposition of G.

Let u, v be N-functions. Then a *(u, v)-route-factor* of N is a set Φ of edge-disjoint open routes of N such that each vertex ξ of N is the tail of exactly $u(\xi)$ and head of exactly $v(\xi)$ elements of Φ. A *(u, v)-decomposition* of N is a (u, v)-route-factor of N which is a decomposition of N.

The object of this paper is to prove the following two parallel results:

THEOREM 1. *Let g be a U-function. Then U has a g-decomposition if and only if g is tolerable and congruential and $g\,.\,V(H) > 0$ for each component H of U.*

THEOREM 2. *Let u, v be N-functions. Then N has a (u, v)-decomposition if and only if $u + v$ is tolerable, $u - v = f$ on $V(N)$ and $(u + v)\,.\,V(H) > 0$ for each component H of N.*

Our procedure will be to prove Theorem 2 and deduce Theorem 1 from it. Certain generalizations of the theorems will be mentioned at the end of the paper.

2. Proof of Theorem 2.

LEMMA 1. *If G has a g-chain-factor, g is tolerable.*

Proof. Let Φ be a g-chain-factor of G. For any pair of disjoint subsets S, T of $V(G)$, let $S*T$ denote the number of ST-chains in Φ. Then, if $\xi \in S \subset V(G)$,

$$g(\xi) = (\{\xi\}*\bar{S}) + \sum_{\eta \in S - \{\xi\}} (\{\xi\}*\{\eta\}).$$

But $\{\xi\}*\{\eta\} \leqslant g(\eta)$ for every $\eta \in S - \{\xi\}$; and $\{\xi\}*\bar{S} \leqslant |S\delta|$ since $\xi \in S$ and so each $\{\xi\}\bar{S}$-chain must include an element of $S\delta$. Hence $g(\xi) \leqslant g\,.\,(S - \{\xi\}) + |S\delta|$; and the lemma is proved.

LEMMA 2. *If A, B are disjoint subsets of $V(G)$, $|(A \cup B)\delta| + |A\delta| \geqslant |B\delta|$.*

Proof. If $V(G) - (A \cup B) = C$, the above inequality follows from the relations

$$|A\delta| = |A \circ B| + |C \circ A|, |B\delta| = |B \circ C| + |A \circ B|, |(A \cup B)\delta| = |C \circ A| + |B \circ C|.$$

LEMMA 3. *If* $S \subset V(G)$, $|S\delta| \equiv d \, . \, S$.

Proof. An edge contributes 2, 1, or 0 to $d \, . \, S$ according as it belongs to $S \circ S$, $S\delta$ or $\bar{S} \circ \bar{S}$ respectively.

COROLLARY 3A. *If g is a congruential G-function and $\xi \in S \subset V(G)$, $F_g(\xi; S)$ is even.*

COROLLARY 3B. (= (**1**, chapter II, Theorem 3)). *The number of odd vertices of a graph is even.*

Proof. Take $S = V(G)$ in Lemma 3.

Definition. Let λ, μ be distinct edges of N such that $\lambda h = \mu t = \xi$. Then the oriented graph M obtained from N by *fusion of* λ *and* μ *at* ξ is defined by the rules:

(i) $V(M) = V(N)$, $E(M) = [E(N) - \{\lambda, \mu\}] \cup \{\nu\}$, where ν is a newly added edge and is not an element of the set $V(N) \cup E(N)$;

(ii) $\nu t_M = \lambda t$, $\nu h_M = \mu h$;

(iii) $\kappa t_M = \kappa t$, $\kappa h_M = \kappa h$ for every $\kappa \in E(N) - \{\lambda, \mu\}$.

LEMMA 4. *If, in the circumstances of the above definition, g is a tolerable congruential N-function and no g-critical cincture of N includes both λ and μ, then g is tolerable in M.*

Proof. Let $\xi \in S \subset V(M)$ $(= V(N))$. If λ, μ do not both belong to $S\delta$, then $|S\delta_M| = |S\delta|$ and so ${}_M F_g(\xi; S) = F_g(\xi; S) \geqslant 0$. If λ, μ both belong to $S\delta$, then (i) $|S\delta_M| = |S\delta| - 2$, whence ${}_M F_g(\xi; S) = F_g(\xi; S) - 2$, and (ii) $S\delta$ must not be g-critical, whence, by the tolerability of g and Corollary 3A, $F_g(\xi; S) \geqslant 2$. Hence ${}_M F_g(\xi; S) \geqslant 0$.

Definitions. If $S \subset V(N)$, S^* will denote the subgraph of N defined by $V(S^*) = S$, $E(S^*) = S \circ S$, and N_S will denote the oriented graph M defined as follows.

(i) $V(M) = \bar{S} \cup \{S'\}$, $E(M) = \bar{S} \circ V(N)$, where S' $[\notin V(N) \cup E(N)]$ is a newly introduced vertex.

(ii) Write $\phi(\xi) = \xi$ if $\xi \in \bar{S}$ and $\phi(\xi) = S'$ if $\xi \in S$. Then $\lambda t_M = \phi(\lambda t)$, $\lambda h_M = \phi(\lambda h)$ for every $\lambda \in E(M)$.

Thus N_S is obtained from N by contracting the subgraph S^* to a single vertex S'.

LEMMA 5. *Let g be a tolerable N-function and C be a g-critical subset of $V(N)$. If $g(C')$, $g(\bar{C}')$ are both defined to be $|C\delta|$, then g is tolerable in N_C and $N_{\bar{C}}$.*

Proof. Write $N_{\bar{C}} = H$, $N_C = K$. Since C is critical,

$$g(\xi) = g \, . \, (C - \{\xi\}) + |C\delta| \tag{1}$$

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for some $\xi \in C$. Since $g(C') = g(\bar{C}') = |C\delta|$, (1) can be rewritten in each of the forms

$$(1') \qquad g(\xi) = g\,.\,[V(H) - \{\xi\}],$$

$$(1'') \qquad g(C') = g(\xi) - g\,.\,(C - \{\xi\}).$$

LEMMA 5A[1]. *If* $S \subset V(H) - \{\xi\}$, $g\,.\,S \leqslant |S\delta_H|$.

Proof. Since $F_g(\xi; C - S) \geqslant 0$,

$$(2) \qquad g(\xi) - g\,.\,(C - S - \{\xi\}) \leqslant |(C - S)\delta|.$$

If $\bar{C}' \notin S$,

$$|S\delta_H| = |S\delta| \geqslant |(C-S)\delta| - |C\delta| \geqslant g(\xi) - g\,.\,(C - S - \{\xi\}) - |C\delta| = g\,.\,S$$

by Lemma 2, (2) and (1). If $\bar{C}' \in S$,

$$|S\delta_H| = |(C - S)\delta| \geqslant g(\xi) - g\,.\,(C - S - \{\xi\}) = g\,.\,S$$

by (2) and (1′).

Suppose that $Y \subset V(H)$. Let $V(H) - Y = W$. If $\xi \notin Y$, then, for every $\eta \in Y$,

$${}_HF_g(\eta;\, Y) \geqslant |Y\delta_H| - g(\eta) \geqslant |Y\delta_H| - g\,.\,Y \geqslant 0$$

by Lemma 5A. If $\xi \in Y$, then by (1′),

$${}_HF_g(\xi;\, Y) = |Y\delta_H| - g\,.\,W = |W\delta_H| - g\,.\,W \geqslant 0$$

by Lemma 5A, and, for every $\eta \in Y - \{\xi\}$,

$${}_HF_g(\eta;\, Y) \geqslant g\,.\,(Y - \{\eta\}) - g(\eta) \geqslant 0$$

by (1′). Hence g is tolerable in H.

Suppose that $Z \subset V(K)$. If $C' \notin Z$, then $Z\delta_K = Z\delta$ and so ${}_KF_g(\eta; Z) = F_g(\eta; Z) \geqslant 0$ for every $\eta \in Z$. If $C' \in Z$, then

$$(3) \qquad Z\delta_K = \tilde{Z}\delta$$

where $\tilde{Z} = (Z - \{C'\}) \cup C$. By (1″) and (3), ${}_KF_g(C'; Z) = F_g(\xi; \tilde{Z}) \geqslant 0$; and, by (3) and Lemma 2,

$$g(C') + |Z\delta_K| = |C\delta| + |\tilde{Z}\delta| \geqslant |(Z - \{C'\})\delta|,$$

whence ${}_KF_g(\eta; Z) \geqslant F_g(\eta; Z - \{C'\}) \geqslant 0$ for every $\eta \in Z - \{C'\}$. Hence g is tolerable in K.

Definitions. An edge λ of N is a *loop* if $\lambda t = \lambda h$. If g is an N-function, a vertex ξ is *g-critical* if the set $\{\xi\}$ is g-critical, that is, if $g(\xi) = |\{\xi\}\delta|$, and is *g-safe* if $F_g(\xi; \{\xi\}) > 0$, that is, if $g(\xi) < |\{\xi\}\delta|$. A *one-edge-route* is a route which has exactly one edge. If $S \subset V(N)$, an edge λ is an *exit* of S if $\lambda t \in S$, $\lambda h \in \tilde{S}$, and is an *entry* of S if $\lambda h \in S$, $\lambda t \in \tilde{S}$. If $A \subset E(N)$, $N - A$ will denote the

[1]We give the names Lemma nA, Lemma nB to lemmas which themselves form part of the proof of Lemma n.

subgraph of N defined by the relations $V(N - A) = V(N)$, $E(N - A) = E(N) - A$.

LEMMA 6. *If u and v are N-functions such that $u - v = f$ on $V(N)$ and $u + v$ is tolerable, then N has a (u, v)-route-factor.*

Proof. Since Lemma 6 is trivially true for an oriented graph of order 0, it may be proved by induction on ord N. We shall therefore make the inductive hypothesis that Lemma 6 is true for all oriented graphs of lower order than N. Let $u + v = g$. If N has a loop λ, then λ belongs to no cincture. Therefore g, being tolerable in N, is tolerable in $N - \{\lambda\}$. It is also clear that $f_{N-\{\lambda\}} = f = u - v$ on $V(N)$. Therefore, by the inductive hypothesis, $N - \{\lambda\}$ has a (u, v)-route-factor, and hence so has N. We shall therefore henceforward assume that N is loopless. We shall consider separately the following two cases: (I) $V(N)$ has a g-critical subset C such that $|C| \geqslant 2$ and $|\bar{C}| \geqslant 2$; (II) $V(N)$ has no such subset.

Proof for Case I. Let the exits of C be $\lambda_1, \lambda_2, \ldots, \lambda_p$ and its entries be $\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_r$. If we write $N_C = K$, $u(C') = p$, $v(C') = r - p$ and $g(C') = |C\delta|$, then u, v, and g are defined on all vertices of K and $g = u + v$ on $V(K)$. By Lemma 5, g is tolerable in K. It is clear that $u(C') - v(C') = f_K(C')$ and that $f_K = f = u - v$ on $\bar{C}$; hence $u - v = f_K$ on $V(K)$. Since $|C| \geqslant 2$, ord $K <$ ord N. Therefore, by the inductive hypothesis, K has a (u, v)-route-factor Φ. Since $u(C') + v(C') = r$ and $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the only edges incident with C' in K, it is clear that $\lambda_1, \lambda_2, \ldots, \lambda_r$ must be distributed in a one-to-one fashion amongst the r elements of Φ which have C' as an end-vertex; let R_i be that element of Φ which includes λ_i among its edges. Then clearly R_i is derivable from a route-sequence of the form C', λ_i, s_i, where s_i is a route-sequence of $\bar{C}^*$. Clearly C', λ_i, s_i and hence also s_i must be forwards- or backwards-directed according as C' is the tail or head respectively of λ_i in K, that is, according as $i \leqslant p$ or $i > p$ respectively. Moreover, if $\Phi - \{R_1, R_2, \ldots, R_r\} = \Delta$, then, since the λ_i are the only edges incident with C' in K and $\lambda_i \in E(R_i)$ $(i = 1, 2, \ldots, r)$, it follows that each element of Δ is a route of $\bar{C}^*$.

If we write $u(\bar{C}') = r - p$, $v(\bar{C}') = p$, an argument similar to that of the preceding paragraph, but using the hypothesis that $|\bar{C}| \geqslant 2$ and the assertion concerning $N_{\bar{C}}$ in Lemma 5, shows that $N_{\bar{C}}$ has a (u, v)-route-factor $\bar{\Delta} \cup \{\bar{R}_1, \bar{R}_2, \ldots, \bar{R}_r\}$ such that the elements of $\bar{\Delta}$ are routes of C^* and, for $i = 1, 2, \ldots, r$, $\bar{R}_i$ is derivable from a route-sequence of the form $\bar{s}_i, \lambda_i, \bar{C}'$, where $\bar{s}_i$ is a route-sequence of C^* and is forwards- or backwards-directed according as $i \leqslant p$ or $i > p$ respectively. It is now not difficult to see that, if S_i is the route derived from the route-sequence $\bar{s}_i, \lambda_i, s_i$, then $\Delta \cup \bar{\Delta} \cup \{S_1, S_2, \ldots, S_r\}$ is a (u, v)-route-factor of N.

Proof for Case II.

LEMMA 6A. *A vertex ξ of N is g-critical if $V(N) - \{\xi\}$ is g-critical.*

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Proof. If $V(N) - \{\xi\}$ is g-critical,

$$-g(\eta) + g \,.\, (V(N) - \{\xi, \eta\}) + |\{\xi\}\delta| = 0$$

for some $\eta \in V(N) - \{\xi\}$. But

$$-g(\eta) + g \,.\, (V(N) - \{\xi, \eta\}) + g(\xi) = F_g(\eta; V(N)) \geqslant 0.$$

Therefore $g(\xi) \geqslant |\{\xi\}\delta|$, that is, $F_g(\xi; \{\xi\}) \leqslant 0$. Hence, since g is tolerable, $F_g(\xi; \{\xi\}) = 0$ and so ξ is g-critical.

COROLLARY 6AA. *In Case* II, *every non-empty g-critical cincture is of the form $\{\xi\}\delta$ for some g-critical vertex ξ.*

If ξ is a g-critical vertex, $g(\xi) = |\{\xi\}\delta|$, that is, since N is loopless, $u(\xi) + v(\xi) = x(\xi) + e(\xi)$. But, by hypothesis, $u(\xi) - v(\xi) = f(\xi) = x(\xi) - e(\xi)$. Hence $u(\xi) = x(\xi)$ and $v(\xi) = e(\xi)$. Hence, since N is loopless, the one-edge-routes in N constitute a (u, v)-route-factor of N if every vertex of N is g-critical. We may therefore assume that N has a g-safe vertex σ. Since σ is g-safe,

$$|\{\sigma\}\delta| > g(\sigma) \geqslant |u(\sigma) - v(\sigma)| = |f(\sigma)|$$

by hypothesis. Therefore

$$x(\sigma) > 0, \qquad e(\sigma) > 0. \tag{4}$$

LEMMA 6B. *The vertex σ has an entry λ and an exit μ such that no g-critical cincture includes both λ and μ.*

Proof. (Throughout this proof, the reader should bear in mind that N is assumed to be loopless.) If σ is adjacent to two or more other vertices, it is easily seen from (4) that σ has an entry λ and an exit μ which join it to different vertices; since σ is g-safe and is the only vertex incident with both λ and μ, Corollary 6AA shows that no g-critical cincture includes both λ and μ. We may therefore assume that σ is adjacent to at most one, and hence, by (4), to exactly one other vertex; let this vertex be τ. Since σ is adjacent only to τ, $|\{\sigma, \tau\}\delta| = |\{\tau\}\delta| - |\{\sigma\}\delta|$. Therefore

$$-\, g(\tau) + g(\sigma) + |\{\tau\}\delta| - |\{\sigma\}\delta| = F_g(\tau; \{\sigma, \tau\}) \geqslant 0.$$

But $|\{\sigma\}\delta| > g(\sigma)$ since σ is g-safe. Therefore $|\{\tau\}\delta| > g(\tau)$. Hence τ is also g-safe. But, by (4), we can select an entry λ and an exit μ of σ. Since λ, μ must both join σ, τ, which are both g-safe, Corollary 6AA again implies the required result.

Since

$$g = u + v \equiv u - v = f = x - e \equiv x + e = d$$

on $V(N)$, g is congruential in N. Therefore, by Lemmas 6B and 4, g is tolerable in the oriented graph (M, say) obtained from N by fusion of λ and μ at σ. It is also clear that $f_M = f = u - v$ on $V(N) = V(M)$ and that ord M = ord N $-$ 1. Therefore, by the inductive hypothesis, M has a (u, v)-route-factor, and it is easily seen that this is converted into a (u, v)-route-factor of N when we reverse the fusion of λ and μ at σ.

LEMMA 7. *If N has a decomposition of the form $\Phi \cup \Theta$, where Φ is a (u, v)-route-factor of N and Θ is a set of closed routes each of which has a vertex in common with some element of Φ, then N has a (u, v)-decomposition.*

Proof. Let $\Phi = \{R_1, R_2, \ldots, R_r\}$, and let $\Theta = \Theta_1 \cup \Theta_2 \cup \ldots \cup \Theta_r$, where the Θ_i are disjoint and each element of Θ_i has a vertex in common with R_i. If S_i is the union of R_i and the elements of Θ_i, it is easily seen that S_i is an open route with the same head and tail as R_i. Hence $\{S_1, S_2, \ldots, S_r\}$ is a (u, v)-decomposition of N.

Proof of Theorem 2. The necessity of the first condition follows from Lemma 1, and the necessity of the other two is obvious. Conversely, suppose that these three conditions are satisfied. Then, by Lemma 6, N has a (u, v)-route-factor Φ. If T is the union of the elements of Φ, then clearly $f_T = u - v$ on $V(T)$ and $u = v = 0$ on $V(N) - V(T)$. But $f = u - v$ on $V(N)$ by hypothesis. Therefore $N - E(T)$ is quasi-symmetrical. Therefore, by **(1**, chapter II, Theorem **7)**, every component of $N - E(T)$ is a closed route. Moreover, since $(u + v) \,.\, V(H) > 0$ for each component H of N, each component of N contains an element of Φ and hence each component of $N - E(T)$ has a vertex in common with an element of Φ. Therefore, by Lemma 7 (with Θ taken to be the set of components of $N - E(T)$), N has a (u, v)-decomposition.

3. Proof of Theorem 1.

LEMMA 8. *Every unoriented graph has an orientation in which $f(\xi) = 0$ for each even vertex ξ and $f(\xi) = \pm 1$ for each odd vertex ξ.*

Proof. Let U be a given unoriented graph. By Corollary 3B, the number of odd vertices of U is even; let it be $2r$. Then U can be converted into an Eulerian unoriented graph H by the addition of r new edges joining its odd vertices in pairs.[2] H, being Eulerian, has by **(1**, p. 30, ll. 4-9**)**, a quasi-symmetrical orientation, and this clearly induces in U an orientation of the required type.

Proof of Theorem 1. The necessity of the condition that g be tolerable follows from Lemma 1, and the necessity of the remaining conditions is obvious. Conversely, let the conditions of Theorem 1 be satisfied, and let N be an orientation of U satisfying the condition of Lemma 8. Write $u = \frac{1}{2}(g + f)$, $v = \frac{1}{2}(g - f)$, where f denotes flux in N. Then, by Theorem 2, N has a (u, v)-decomposition, and hence U has a g-decomposition.

4. Generalizations.

Definitions. A *semi-oriented graph* is a quintuple $S = (U, \mathfrak{x}, \mathfrak{e}, p, q)$ such that U is an unoriented graph, $\mathfrak{x}$, $\mathfrak{e}$ are disjoint sets and p, q are mappings of $\mathfrak{x} \cup \mathfrak{e}$ into $V(U)$, $E(U)$ respectively, subject to the condition that each edge λ of U is the image under q of exactly two elements of $\mathfrak{x} \cup \mathfrak{e}$ and that, if these elements are ϵ, ϵ', then λ joins ϵp to $\epsilon' p$ in U. Vertices and edges of U are

[2]This procedure is suggested by the proof of **(1**, chapter II, Theorem **4)**.

called *vertices* and *edges* of S respectively, and elements of $\mathfrak{x} \cup \mathfrak{e}$ are called *hinges* of S. A vertex ξ (edge λ) of U is *incident* with a hinge ϵ if $\epsilon p = \xi$ ($\epsilon q = \lambda$). Two hinges are *opposed* if one of them belongs to $\mathfrak{x}$ and the other to $\mathfrak{e}$. If $\xi \in V(U)$, $f(\xi)$ will denote $|\mathfrak{s} \cap \mathfrak{x}| - |\mathfrak{s} \cap \mathfrak{e}|$, where $\mathfrak{s}$ is the set of those hinges of S which are incident with ξ. An *open route-sequence* of S is a finite sequence

$$\xi_0,\ \epsilon_1,\ \lambda_1,\ \bar{\epsilon}_1,\ \xi_1,\ \epsilon_2,\ \lambda_2,\ \bar{\epsilon}_2,\ \xi_2,\ \epsilon_3,\ \ldots,\ \lambda_n,\ \bar{\epsilon}_n,\ \xi_n \tag{5}$$

such that $\xi_0, \lambda_1, \xi_1, \lambda_2, \ldots, \lambda_n, \xi_n$ is an open chain-sequence of U, the ϵ_i and $\bar{\epsilon}_i$ are hinges of S, the relations

$$\epsilon_i p = \xi_{i-1},\ \bar{\epsilon}_i p = \xi_i,\ \epsilon_i q = \bar{\epsilon}_i q = \lambda_i,\ \epsilon_i \neq \bar{\epsilon}_i$$

hold for $i = 1, 2, \ldots, n$ and $\bar{\epsilon}_i$, ϵ_{i+1} are opposed for $i = 1, 2, \ldots, n-1$. (The last condition is vacuous if $n = 1$.) The vertex ξ_0 [ξ_n] is a *tail* or *head* of (5) according as ϵ_1 [$\bar{\epsilon}_n$] belongs to $\mathfrak{x}$ or $\mathfrak{e}$ respectively. (Thus an open route-sequence of S may have two tails, two heads, or one tail and one head.) An *open route* of S is a subgraph of S derivable from an open route-sequence of S. (We shall leave the reader to guess the definitions of *subgraph of S*, *derivable* and certain other terms relating to semi-oriented graphs from corresponding definitions given for unoriented and oriented graphs.) If R is an open route of S, ξ is a vertex of R, and s is any open route-sequence from which R is derivable, then clearly $f_R(\xi) = 1$ if and only if ξ is a tail of s and $f_R(\xi) = -1$ if and only if ξ is a head of s; we shall therefore call ξ a *tail* of R if $f_R(\xi) = 1$ and a *head* of R if $f_R(\xi) = -1$. A *decomposition* of S is a set of edge-disjoint subgraphs of S whose union is S. If u, v are U-functions, a *(u, v)-decomposition* of S is a decomposition D of S into open routes such that each vertex ξ is a tail of exactly $u(\xi)$ and head of exactly $v(\xi)$ elements of D. Semi-oriented graphs are virtually a generalization of oriented graphs, since an oriented graph may be regarded as a semi-oriented graph in which each edge is incident with two opposed hinges. A *semi-orientation* of an unoriented graph U_1 is a semi-oriented graph having U_1 as its first constituent element.

Theorem 2 admits the following generalization:

THEOREM 3. *Let $S = (U, \mathfrak{x}, \mathfrak{e}, p, q)$ be a semi-oriented graph and u, v be U-functions. Then S has a (u, v)-decomposition if and only if $u + v$ is tolerable, $u - v = f$ on $V(U)$ and $(u + v) \,.\, V(H) > 0$ for each component H of U.*

The proof of Theorem 3 is a fairly easy adaptation of that of Theorem 2; but we refrained from giving the argument in this more general form to avoid obscurity. It may be remarked, however, that Theorem 1 is more readily deducible from Theorem 3 than from Theorem 2, since Lemma 8 becomes trivial if, in its statement, "an orientation" be replaced by "a semi-orientation."

Definitions. A *partition* of a set A is a set of disjoint subsets of A whose union is A. If P is a partition of $V(N)$, an N-function g is *P-tolerable* if

$$g \,.\, (S \cap T) \leqslant g \,.\, (S - T) + |S\delta|$$

for every pair S, T of subsets of $V(N)$ such that $T \in P$. A set Φ of open routes of N is *P-restricted* if no element of Φ has both its end-vertices in the same element of P.

THEOREM 2′. *Let P be a partition of $V(N)$ and u, v be N-functions. Then N has a P-restricted (u, v)-decomposition if and only if $u + v$ is P-tolerable, $u - v = f$ on $V(N)$, and $(u + v) \,.\, V(H) > 0$ for each component H of N.*

Theorem 2′ is a generalization of Theorem 2, since it clearly reduces to Theorem 2 when P is taken to be the partition of $V(N)$ into subsets of order 1. The proof of Theorem 2′, which we shall not give in detail, consists in applying Theorem 2 to an oriented graph N_1 and N_1-functions u_1, v_1 defined as follows. N_1 is obtained from N by adding, for each $T \in P$, a new vertex α_T and, for each pair ξ, T such that $\xi \in T \in P$, $u(\xi)$ new edges with tail α_T and head ξ and $v(\xi)$ new edges with tail ξ and head α_T. (Thus $|P|$ new vertices and $(u + v) \,.\, V(N)$ new edges are added altogether.) We write $u_1(\alpha_T) = u \,.\, T$, $v_1(\alpha_T) = v \,.\, T$ and $u_1 = v_1 = 0$ on $V(N)$.

Theorems 1 and 3 admit corresponding generalizations to "P-restricted" decompositions.

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