

MOMENTS AND HYBRID SUBCONVEXITY FOR SYMMETRIC-SQUARE L -FUNCTIONS

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Abstract We establish sharp bounds for the second moment of symmetric-square L -functions attached to Hecke Maass cusp forms u_j with spectral parameter t_j , where the second moment is a sum over t_j in a short interval. At the central point $s = 1/2$ of the L -function, our interval is smaller than previous known results. More specifically, for $|t_j|$ of size T , our interval is of size $T^{1/5}$, whereas the previous best was $T^{1/3}$, from work of Lam. A little higher up on the critical line, our second moment yields a subconvexity bound for the symmetric-square L -function. More specifically, we get subconvexity at $s = 1/2 + it$ provided $|t_j|^{6/7+\delta} \leq |t| \leq (2 - \delta)|t_j|$ for any fixed $\delta > 0$. Since $|t|$ can be taken significantly smaller than $|t_j|$, this may be viewed as an approximation to the notorious subconvexity problem for the symmetric-square L -function in the spectral aspect at $s = 1/2$.

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1. Introduction

1.1. Background

The widely studied subconvexity problem for automorphic L -functions is completely resolved for degree ≤ 2 . For uniform bounds, over arbitrary number fields, this is due to Michel and Venkatesh [25]; for superior-quality bounds in various special cases, this is due to many authors (e.g., [17, 6, 9, 7, 29]). The next frontier is degree 3, but here the subconvexity problem remains a great challenge, save for a few spectacular successes. The first breakthrough is due to Xiaoqing Li [23], who established subconvexity for $L(f, 1/2 + it)$ on the critical line (t -aspect), where f is a fixed self-dual Hecke–Maass cusp form for $SL_3(\mathbb{Z})$. This result was generalized by Munshi [26], by a very different method, to forms f that are not necessarily self-dual. Munshi [27] also established subconvexity for twists $L(f \times \chi, 1/2)$ in the p -aspect, where χ is a primitive Dirichlet character of prime



modulus p . Subconvexity in the spectral aspect of f itself is much harder, and even more so when f is self-dual due to a conductor-dropping phenomenon. Blomer and Buttcane [5], Kumar, Mallesham, and Singh [21], and Sharma [30] have established subconvexity for $L(1/2, f)$ in the spectral aspect of f in many cases, but excluding the self-dual forms.

A self-dual GL_3 Hecke–Maass cusp form is known to be a symmetric-square lift from GL_2 [31]. Let u_j be a Hecke–Maass cusp form for the full modular group $SL_2(\mathbb{Z})$, with Laplace eigenvalue $1/4 + t_j^2$. It is an outstanding open problem to prove subconvexity for the associated symmetric-square L -function $L(\text{sym}^2 u_j, 1/2)$ in the t_j -aspect. Such a bound would represent major progress in the problem of obtaining a power-saving rate of decay in the quantum unique ergodicity problem [15]. A related problem is that of establishing the Lindelöf-on-average bound

$$\sum_{T \leq t_j \leq T+\Delta} |L(\text{sym}^2 u_j, 1/2 + it)|^2 \ll \Delta T^{1+\epsilon}, \quad (1.1)$$

where we assume throughout that $T^\epsilon \leq \Delta \leq T^{1-\epsilon}$, and we generally aim to take Δ as small as possible. Such an estimate is interesting in its own right, and also yields by positivity a bound for each L -value in the sum. At the central point ($t = 0$), if formula (1.1) can be established for $\Delta = T^\epsilon$, it would give the convexity bound for $L(\text{sym}^2 u_j, 1/2)$; the hope would then be to insert an amplifier in order to prove subconvexity. While a second moment bound which implies convexity at the central point is known in the level aspect by the work of Iwaniec and Michel [14], in the spectral aspect the problem is much more difficult. The best known result until now for formula (1.1) was $\Delta = T^{1/3+\epsilon}$ by Lam [22]. (Lam’s work actually involves symmetric-square L -functions attached to holomorphic Hecke eigenforms, but his method should apply equally well to Hecke–Maass forms.) Other works involving moments of symmetric square L -functions include [3, 18, 16, 19, 2, 1, 28].

1.2. Main results

One of the main results of this paper is an approximate version of the subconvexity bound for $L(\text{sym}^2 u_j, 1/2)$. Namely, we establish subconvexity for $L(\text{sym}^2 u_j, 1/2 + it)$ for t small, but not too small, compared to $2t_j$. This hybrid bound (stated precisely later) seems to be the first subconvexity bound for symmetric-square L -functions in which the dominant aspect is the spectral parameter t_j . For comparison, note that bookkeeping the proofs of Li [23] or Munshi [26] would yield hybrid subconvexity bounds for t_j (very) small compared to t . Our method also yields a hybrid subconvexity bound for $L(\text{sym}^2 u_j, 1/2 + it)$ when t is larger (but not too much larger) than $2t_j$, but for simplicity we refrain from making precise statements. We do not prove anything when t is close to $2t_j$, for in this case the analytic conductor of the L -function drops. In fact, it is then the same size as the analytic conductor at $t = 0$, where the subconvexity problem is the hardest.

Our approach is to establish a sharp estimate for the second moment as in formula (1.1), which is strong enough to yield subconvexity in certain ranges.

Theorem 1.1. *Let $0 < \delta < 2$ be fixed, and let $U, T, \Delta > 1$ be such that*

$$\frac{T^{3/2+\delta}}{\Delta^{3/2}} \leq U \leq (2 - \delta)T. \quad (1.2)$$

We have

$$\sum_{T < t_j < T + \Delta} |L(\text{sym}^2 u_j, 1/2 + iU)|^2 \ll \Delta T^{1+\epsilon}. \tag{1.3}$$

Corollary 1.2. *Let $0 < \delta < 2$ be fixed. For $|t_j|^{6/7+\delta} \leq U \leq (2 - \delta)|t_j|$, we have the hybrid subconvexity bound*

$$L(\text{sym}^2 u_j, 1/2 + iU) \ll |t_j|^{1+\epsilon} U^{-1/3}. \tag{1.4}$$

Proof. The bound follows by taking $\Delta = T^{1+\delta} U^{-2/3}$ in Theorem 1.1 with δ chosen small enough. When $U \geq T^{6/7+\delta}$, this bound is subconvex. \square

Note that in Theorem 1.1, we are able to take Δ as small as $T^{1/3}$ at best. This requires $T \ll U \leq (2 - \delta)T$ and for instance yields the subconvexity bound $L(\text{sym}^2 u_j, 1/2 + it_j) \ll |t_j|^{2/3+\epsilon}$.

We might also speculate that the lower bound in formula (1.2) could plausibly be relaxed to $\Delta U \gg T^{1+\delta}$ (possibly with an additional term on the right-hand side of formula (1.3), as in formula (12.10)) which would give subconvexity in the wider range $T^{2/3+\delta} \leq U \leq (2 - \delta)T$. For some reasoning on this, see the remarks following formula (9.16).

For the central values we do not get subconvexity, but we are able to improve the state of the art for the second moment. This is the other main result of this paper: We establish a Lindelöf-on-average estimate for the second moment with Δ as small as $T^{1/5+\epsilon}$:

Theorem 1.3. *For $\Delta \geq T^{1/5+\epsilon}$ and $0 \leq U \ll T^\epsilon$, we have*

$$\sum_{T < t_j < T + \Delta} |L(\text{sym}^2 u_j, 1/2 + iU)|^2 \ll \Delta T^{1+\epsilon}. \tag{1.5}$$

It is a standing challenge to prove a Lindelöf-on-average bound in formula (1.5) with $\Delta = 1$.

Theorem 1.3 also has implications for the quantum variance problem. To explain this, recall that quantum unique ergodicity [24, 32] says that for any smooth, bounded function ψ on $\Gamma \backslash \mathbb{H}$, we have that $\langle |u_j|^2, \psi \rangle \rightarrow \frac{3}{\pi} \langle 1, \psi \rangle$ as $t_j \rightarrow \infty$. By spectrally decomposing ψ , this is equivalent to demonstrating the decay of $\langle |u_j|^2, \varphi \rangle$ and $\langle |u_j|^2, E_U \rangle$, where φ is a fixed Hecke–Maass cusp form and $E_U = E(\cdot, \frac{1}{2} + iU)$ is the standard Eisenstein series with U fixed. The quantum variance problem is the problem of understanding the variance of these crucial quantities. More precisely, the quantum variance problem asks for nontrivial bounds on

$$\sum_{T < t_j < T + \Delta} |\langle u_j^2, \varphi \rangle|^2, \tag{1.6}$$

as well as the Eisenstein contribution

$$\sum_{T < t_j < T + \Delta} |\langle u_j^2, E_U \rangle|^2. \tag{1.7}$$

Our Theorem 1.3 gives, by classical Rankin–Selberg theory, a sharp bound on expression (1.7) for $\Delta \geq T^{1/5+\epsilon}$. In turn, by Watson’s formula [33], a sharp estimate for expression

(1.6) boils down to establishing

$$\sum_{T < t_j < T + \Delta} L(\text{sym}^2 u_j \otimes \varphi, 1/2) \ll \Delta T^{1+\epsilon}. \tag{1.8}$$

It is plausible that the methods used to prove Theorem 1.3 should also generalize to show formula (1.8) for $\Delta \geq T^{1/5+\epsilon}$, which would improve on [16], but this requires a rigorous proof. For quantum variance in the level aspect, see [28].

1.3. Overview

We now give a rough sketch of our ideas for Theorems 1.1 and 1.3, both of which consider the second moment of the symmetric-square L -function. Let $h(t)$ be a smooth function supported essentially on $T < |t| < T + \Delta$, such as the one given in equation (6.2). For $0 \leq U \leq (2 - \delta)T$, the analytic conductor of $L(\text{sym}^2 u_j, 1/2 + iU)$ is of size $T^2(U + 1)$, so using an approximate functional equation, we have roughly

$$\sum_{j \geq 1} |L(\text{sym}^2 u_j, 1/2 + iU)|^2 h(t_j) = \sum_{j \geq 1} \sum_{m, n \leq T^{1+\epsilon}(U+1)^{1/2}} \frac{\lambda_j(m^2) \lambda_j(n^2)}{m^{1/2+iU} n^{1/2-iU}} h(t_j),$$

which we need to show is bounded by $T^{1+\epsilon} \Delta$. Applying the Kuznetsov formula, the diagonal contribution is of size $O(T^{1+\epsilon} \Delta)$, while the off-diagonal contribution is roughly

$$\sum_{m, n \leq T^{1+\epsilon}(U+1)^{1/2}} \frac{1}{m^{1/2+iU} n^{1/2-iU}} \sum_{c \geq 1} \frac{S(m^2, n^2, c)}{c} H\left(\frac{4\pi mn}{c}\right)$$

for some transform H of h , given in equation (6.6). By developing equation (6.12), we have that $H(x)$ is essentially supported on $x \geq T^{1-\epsilon} \Delta$ and roughly has the shape $H(x) = \frac{T\Delta}{x^{1/2}} e^{i(x - T^2/x)}$. Thus in the generic ranges $m, n \sim T(U + 1)^{1/2}$ and $c \sim \frac{mn}{T\Delta}$, writing $(n/m)^{iU} = e(U \log(n/m)/2\pi)$ and not being very careful about factors of π and such, the off-diagonal is

$$\frac{\Delta^{3/2}}{U^{3/2} T^{3/2}} \sum_{m, n \sim T(U+1)^{1/2}} \sum_{c \sim T(U+1)/\Delta} S(m^2, n^2, c) e\left(\frac{2mn}{c}\right) e\left(-\frac{T^2 c}{mn} + U \log(n/m)\right). \tag{1.9}$$

The oscillatory factor $e\left(-\frac{T^2 c}{mn} + U \log(n/m)\right)$ behaves differently according to whether U is large or small. When U is large, the dominant phase is $U \log(n/m)$, whereas when U is small, the dominant phase is $-\frac{T^2 c}{mn} \sim -\frac{T}{\Delta}$.

Consider one extreme end of our problem: the case $U = T$ (covered by Theorem 1.1), so that the convexity bound is $T^{3/4+\epsilon}$. Since the diagonal after Kuznetsov is $O(T^{1+\epsilon} \Delta)$, the largest we can take Δ to establish subconvexity is $\Delta = T^{1/2-\delta}$ for some $\delta > 0$.

Thus for the off-diagonal, what we need to prove is roughly (specializing expression (1.9) to $U = T, \Delta = T^{1/2}$, and retaining only the dominant phase)

$$\frac{1}{T^{9/4}} \sum_{m, n \sim T^{3/2}} \sum_{c \sim T^{3/2}} S(m^2, n^2, c) e\left(\frac{2mn}{c}\right) e(T \log(n/m)) \ll T^{3/2}. \tag{1.10}$$

We split the n and m sums into residue classes modulo c and apply Poisson summation to each. The off-diagonal then equals

$$\frac{1}{T^{9/4}} \sum_{c \sim T^{3/2}} \sum_{k, \ell \in \mathbb{Z}} \frac{1}{c^2} T(k, \ell, c) I(k, \ell, c),$$

where

$$I(k, \ell, c) = \iint e\left(\frac{-kx - \ell y}{c} + T \log x - T \log y\right) w\left(\frac{x}{T^{3/2}}, \frac{y}{T^{3/2}}\right) dx dy$$

for some smooth weight function w which restricts support to $x \sim T^{3/2}, y \sim T^{3/2}$, and

$$T(k, \ell, c) = \sum_{a, b \bmod c} S(a^2, b^2, c) e\left(\frac{2ab + ak + b\ell}{c}\right).$$

We compute this arithmetic sum in §5 and roughly get $T(k, \ell, c) = c^{3/2} \left(\frac{k\ell}{c}\right) e\left(\frac{-k\ell}{4c}\right)$. The integral is computed using stationary phase (see §4 and §8). We see that it is negligibly small unless $k, \ell \sim T$, in which case we get roughly $I(k, \ell, c) = T^2 e\left(\frac{k\ell}{c}\right) (k/\ell)^{iT}$ (see Lemma 9.3 for the rigorous statement). Thus we need to show

$$\frac{1}{T} \sum_{k, \ell \sim T} \sum_{c \sim T^{3/2}} \left(\frac{k}{\ell}\right)^{iT} e\left(\frac{3k\ell}{4c}\right) \left(\frac{k\ell}{c}\right) \ll T^{3/2}.$$

At this point we go beyond previous approaches to the second moment problem [14, 22] by finding cancellation in the c sum. We split the c sum into arithmetic progressions modulo $k\ell$ by quadratic reciprocity and apply Poisson summation, getting that the off-diagonal equals

$$\frac{1}{T} \sum_{k, \ell \sim T} \left(\frac{k}{\ell}\right)^{iT} \sum_{q \in \mathbb{Z}} \frac{1}{k\ell} \sum_{a \bmod k\ell} \left(\frac{a}{k\ell}\right) e\left(\frac{-aq}{k\ell}\right) \int e\left(\frac{3k\ell}{4x} + \frac{qx}{k\ell}\right) w\left(\frac{x}{T^{3/2}}\right) dx. \tag{1.11}$$

This Poisson summation step may be viewed as the key new ingredient in our paper. It leads to a simpler expression in two ways. First, an integration-by-parts argument shows that the q sum can be restricted to $q \sim T$, which is significantly shorter than the earlier c sum of length $T^{3/2}$. A more elaborate stationary-phase analysis of the integral shows that the integral is essentially independent of k and ℓ , which can be seen in rough form by the substitution $x \rightarrow xk\ell$ in expression (1.11). The reader will not actually find an expression like (1.11) in the paper, because we execute Poisson summation in c in the language of Dirichlet series and functional equations. This allows us to more effectively deal with some of the more delicate features of this step (see, e.g., Remark 11.5).

Evaluating the arithmetic sum and using stationary phase to compute the integral in expression (1.11), we get that the off-diagonal equals

$$\frac{1}{T^{3/4}} \sum_{k, \ell \sim T} \sum_{q \sim T} e(\sqrt{q}) \left(\frac{q}{k\ell}\right) \left(\frac{k}{\ell}\right)^{iT} = \frac{1}{T^{3/4}} \sum_{q \sim T} e(\sqrt{q}) \left| \sum_{k \sim T} \left(\frac{q}{k}\right) k^{iT} \right|^2. \tag{1.12}$$

Finally, applying Heath-Brown’s [12] large sieve for quadratic characters, we get that the off-diagonal is $O(T^{5/4+\epsilon})$, which is better than the required bound in expression (1.10).

Now consider Theorem 1.3, which deals with the other extreme end of our problem, where U is small. The treatment of this follows the same plan as just sketched for large values of U , but the details are changed a bit, because the oscillatory factor in expression (1.9) behaves differently. Consider the case $U = 0$ (the central point) and $\Delta = T^{1/5}$, which is the best we can do in Theorem 1.3. In the end, instead of equation (1.12), one arrives roughly at an expression of the form

$$\sum_{q \sim T^{6/5}} e\left(T^{1/2}q^{1/4}\right) \left| \sum_{k \sim T^{3/5}} \frac{\left(\frac{k}{q}\right)}{\sqrt{k}} \right|^2. \tag{1.13}$$

Again, Heath-Brown’s quadratic large sieve is the endgame, giving a bound of $\Delta T^{1+\epsilon} = T^{6/5+\epsilon}$. It is a curious difference that the q sum in expression (1.13) is now actually *longer* than the c sum from which it arose via Poisson summation, in contrast to the situation with $U = T$ presented earlier. However, the gain is that the variables q and k become separated in the exponential-phase factor (indeed, k is completely removed from the phase in expression (1.13)).

1.4. Notational conventions

Throughout, we will follow the epsilon convention, in which ϵ always denotes an arbitrarily small positive constant, but not necessarily the same one from one occurrence to another. As usual, we will write $e(x) = e^{2\pi ix}$ and $e_c(x) = e(x/c)$. For n a positive odd integer, we let $\chi_n(m) = \left(\frac{m}{n}\right)$ denote the Jacobi symbol. If s is complex, an expression of the form $O(p^{-s})$ should be interpreted to mean $O(p^{-\text{Re}(s)})$. This abuse of notation will only be used on occasion with Euler products. We may also write $O(p^{-\min(s,u)})$ in place of $O(p^{-\min(\text{Re}(s), \text{Re}(u))})$.

Upper bounds in terms of the size of U are usually expressed, since U may be 0, in terms of $1 + U$. However, to save clutter, such upper bounds will be written in terms of U only. This is justified at the start of §6.

2. Automorphic forms

2.1. Symmetric-square L -functions

Let u_j be a Hecke–Maass cusp form for the modular group $SL_2(\mathbb{Z})$ with Laplace eigenvalue $1/4 + t_j^2$ and n th Hecke eigenvalue $\lambda_j(n)$. It has an associated symmetric-square L -function defined by $L(\text{sym}^2 u_j, s) = \sum_{n \geq 1} \lambda_{\text{sym}^2 u_j}(n) n^{-s}$, with $\lambda_{\text{sym}^2 u_j}(n) = \sum_{a^2 b = n} \lambda_j(b^2)$.

Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\gamma(\text{sym}^2 u_j, s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s + 2it_j)\Gamma_{\mathbb{R}}(s - 2it_j)$. Then $L(\text{sym}^2 u_j, s)$ has an analytic continuation to \mathbb{C} and satisfies the functional equation $\gamma(\text{sym}^2 u_j, s)L(\text{sym}^2 u_j, s) = \gamma(\text{sym}^2 u_j, 1 - s)L(\text{sym}^2 u_j, 1 - s)$, where the notation for $\gamma(f, s)$ agrees with [13, Chapter 5]. In particular, the analytic conductor of $L(\text{sym}^2 u_j, 1/2 + it)$ equals

$$(1 + |t|)(1 + |t + 2t_j|)(1 + |2t_j - t|). \tag{2.1}$$

2.2. The Kuznetsov formula

Let $h(z)$ be an even, holomorphic function on $|\Im(z)| < \frac{1}{2} + \delta$, with decay $|h(z)| \ll (1 + |z|)^{2-\delta}$, for some $\delta > 0$. Let $\{u_j : j \geq 1\}$ denote an orthonormal basis of Maass cusp forms of level q with Laplace eigenvalue $\frac{1}{4} + t_j^2$ and Fourier expansion

$$u_j(z) = y^{\frac{1}{2}} \sum_{n \neq 0} \rho_j(n) K_{it_j}(2\pi|n|y)e(nx),$$

where $z = x + iy$ and K_{it_j} is the K -Bessel function. At each inequivalent cusp \mathfrak{a} of $\Gamma_0(q)$, let $E_{\mathfrak{a}}(\cdot, \frac{1}{2} + it)$ be the associated Eisenstein series with Fourier expansion

$$E_{\mathfrak{a}}(z, \frac{1}{2} + it) = \delta_{\mathfrak{a}=\infty} y^{\frac{1}{2}+it} + \varphi_{\mathfrak{a}}(\frac{1}{2} + it) y^{\frac{1}{2}-it} + y^{\frac{1}{2}} \sum_{n \neq 0} \tau_{\mathfrak{a}}(n, t) K_{it}(2\pi|n|y)e(nx),$$

where $\varphi_{\mathfrak{a}}(s)$ is meromorphic on \mathbb{C} . These expansions may be found in [13, equations (16.19) and (16.22)].

Lemma 2.1 (Kuznetsov’s formula [13, Theorem 16.3]). *For any $n, m > 0$, we have*

$$\begin{aligned} & \sum_{j \geq 1} \rho_j(n) \bar{\rho}_j(m) \frac{h(t_j)}{\cosh(\pi t_j)} + \sum_{\mathfrak{a}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \tau_{\mathfrak{a}}(n, t) \bar{\tau}_{\mathfrak{a}}(m, t) \frac{h(t) dt}{\cosh(\pi t)} \\ &= \delta_{(n=m)} \int_{-\infty}^{\infty} h(t) t \tanh(\pi t) \frac{dt}{\pi^2} \\ &+ \frac{i}{\pi} \sum_{c \equiv 0 \pmod q} \frac{S(n, m, c)}{c} \int_{-\infty}^{\infty} J\left(\frac{4\pi\sqrt{nm}}{c}, t\right) h(t) t \tanh(\pi t) dt, \end{aligned}$$

where $J(x, t) = \frac{J_{2it}(x) - J_{-2it}(x)}{\sinh(\pi t)}$.

Later we will need to use the Kuznetsov formula for level 2^4 . We will choose our orthonormal basis to include the level 1 Hecke–Maass forms, for which we may write

$$\rho_j(n) \bar{\rho}_j(m) \frac{h(t_j)}{\cosh(\pi t_j)} = \lambda_j(n) \bar{\lambda}_j(m) \frac{h(t_j) |p_j(1)|^2}{\cosh(\pi t_j)},$$

and note that $t_j^{-\epsilon} \ll \frac{|\rho_j(1)|^2}{\cosh(\pi t_j)} \ll t_j^{\epsilon}$ by [11, equation (30)] together with the fact that L^2 -normalization in $\Gamma_0(2^4)$ and $\Gamma_0(1)$ is the same up to a constant factor.

3. The quadratic large sieve

We will have need of Heath-Brown’s large sieve inequality for quadratic characters:

Theorem 3.1 (Heath-Brown [12]). *Set $M, N \gg 1$. Then*

$$\sum_{m \leq M}^* \left| \sum_{n \leq N}^* a_n \left(\frac{n}{m} \right) \right|^2 \ll (M + N)(MN)^\epsilon \sum_{n \leq N} |a_n|^2, \tag{3.1}$$

where the sums are restricted to odd square-free integers.

We will need a corollary of Heath-Brown’s result, namely

$$\sum_{m \leq M} |L(1/2 + it, \chi_m)|^2 \ll \left(M + \sqrt{M(1 + |t|)} \right) (M(1 + |t|))^\epsilon. \tag{3.2}$$

This follows from an approximate functional equation and a simple observation that the square parts of the inner and outer variables are harmless. Similarly, we obtain

$$\sum_{m \leq M} m^{-1/2} |L(1/2 + it, \chi_m)|^2 \ll \left(M^{1/2} + (1 + |t|)^{1/2} \right) (M(1 + |t|))^\epsilon. \tag{3.3}$$

4. Oscillatory integrals

Throughout this paper we will make extensive use of estimates for oscillatory integrals. We will largely rely on the results of [20] (built on [8]), which use the language of families of inert functions. This language gives a concise way to track bounds on derivatives of weight functions. It also has the pleasant property that, loosely speaking, the class of inert functions is closed under application of the stationary-phase method (the precise statement is in Lemma 4.3). We refer the reader to [20] for a more thorough discussion, including examples of applying stationary phase using this language.

Let \mathcal{F} be an index set and $X = X_T : \mathcal{F} \rightarrow \mathbb{R}_{\geq 1}$ be a function of $T \in \mathcal{F}$.

Definition 4.1. A family $\{w_T\}_{T \in \mathcal{F}}$ of smooth functions supported on a product of dyadic intervals in $\mathbb{R}_{>0}^d$ is called X -inert if for each $j = (j_1, \dots, j_d) \in \mathbb{Z}_{\geq 0}^d$ we have

$$C_{\mathcal{F}}(j_1, \dots, j_d) := \sup_{T \in \mathcal{F}} \sup_{(x_1, \dots, x_d) \in \mathbb{R}_{>0}^d} X_T^{-j_1 - \dots - j_d} \left| x_1^{j_1} \dots x_d^{j_d} w_T^{(j_1, \dots, j_d)}(x_1, \dots, x_d) \right| < \infty. \tag{4.1}$$

As an abuse, we might say that a single function is 1-inert (or simply inert), by which we should mean that it is a member of a family of 1-inert functions.

Lemma 4.2 (Integration-by-parts bound [8]). *Suppose that $w = w_T(t)$ is a family of X -inert functions, with compact support on $[Z, 2Z]$, so that for all $j = 0, 1, \dots$ we have the*

bound $w^{(j)}(t) \ll (Z/X)^{-j}$. Also suppose that ϕ is smooth and satisfies, for $j = 2, 3, \dots$, $\phi^{(j)}(t) \ll \frac{Y}{Z^j}$ for some $R \geq 1$ with $Y/X \geq R$ and all t in the support of w . Let

$$I = \int_{-\infty}^{\infty} w(t)e^{i\phi(t)} dt.$$

If $|\phi'(t)| \gg \frac{Y}{Z}$ for all t in the support of w , then $I \ll_A ZR^{-A}$ for A arbitrarily large.

Lemma 4.3 (stationary phase [8, 20]). Suppose w_T is X -inert in t_1, \dots, t_d , supported on $t_1 \asymp Z$ and $t_i \asymp X_i$ for $i = 2, \dots, d$. Suppose that on the support of w_T , $\phi = \phi_T$ satisfies

$$\frac{\partial^{a_1+a_2+\dots+a_d}}{\partial t_1^{a_1} \dots \partial t_d^{a_d}} \phi(t_1, t_2, \dots, t_d) \ll_{C_{\mathcal{F}}} \frac{Y}{Z^{a_1}} \frac{1}{X_2^{a_2} \dots X_d^{a_d}}, \tag{4.2}$$

for all $a_1, \dots, a_d \in \mathbb{N}$ with $a_1 \geq 1$. Suppose $\phi''(t_1, t_2, \dots, t_d) \gg \frac{Y}{Z^2}$ (here and later, ϕ' and ϕ'' denote the derivative with respect to t_1), for all t_1, t_2, \dots, t_d in the support of w_T , and for each t_2, \dots, t_d in the support of ϕ there exists $t_0 \asymp Z$ such that $\phi'(t_0, t_2, \dots, t_d) = 0$. Suppose that $Y/X^2 \geq R$ for some $R \geq 1$. Then

$$I = \int_{\mathbb{R}} e^{i\phi(t_1, \dots, t_d)} w_T(t_1, \dots, t_d) dt_1 = \frac{Z}{\sqrt{Y}} e^{i\phi(t_0, t_2, \dots, t_d)} W_T(t_2, \dots, t_d) + O_A(ZR^{-A}), \tag{4.3}$$

for some X -inert family of functions W_T , and where $A > 0$ may be taken to be arbitrarily large. The implied constant in equation (4.3) depends only on A and on $C_{\mathcal{F}}$ defined in formula (4.1).

The fact that W_T is inert with respect to the same variables as w_T (with the exception of t_1 , of course) is highly convenient. In practice, we may often temporarily suppress certain variables from the notation. This is justified, provided that the functions satisfy the inertness condition in terms of these variables. We also remark that if $d = 1$, then $W_T(t_2, \dots, t_d)$ is a constant.

The following remark will be helpful for using Lemma 4.3 in an iterative fashion. First note that t_0 is the unique function of t_2, \dots, t_d which solves $\phi'(t_1, \dots, t_d) = 0$ when viewed as an equation in t_1 . In other words, t_0 is defined implicitly by $\phi'(t_0, \dots, t_d) = 0$. In practice it might be an unwelcome task to explicitly solve for t_0 , and the following discussion will aid in avoiding this issue. Let

$$\Phi(t_2, \dots, t_d) = \phi(t_0, t_2, \dots, t_d), \tag{4.4}$$

so by the chain rule,

$$\frac{\partial}{\partial t_j} \Phi(t_2, \dots, t_d) = \phi'(t_0, t_2, \dots, t_d) \frac{\partial t_0}{\partial t_j} + \frac{\partial}{\partial t_j} \phi(t_0, \dots, t_j) = \frac{\partial}{\partial t_j} \phi(t_0, \dots, t_j), \tag{4.5}$$

and so on for higher derivatives. Hence the derivatives of Φ have the same bounds as those on ϕ (supposing uniformity with respect to the first variable t_1).

As a simple yet useful consequence of this, if ϕ satisfies formula (4.2) (with Z replaced by X_1 , say) as well as $\frac{\partial^2}{\partial t_j^2} \phi(t_1, \dots, t_d) \gg \frac{Y}{X_j^2} \geq R \geq 1$ for $j = 1, 2, \dots, k$, uniformly for all t_1, \dots, t_d in the support of w_T , then

$$\int_{\mathbb{R}^k} e^{i\phi(t_1, \dots, t_d)} w_T(t_1, \dots, t_d) dt_1 \cdots dt_k = \frac{X_1 \cdots X_k}{Y^{k/2}} e^{i\phi(\mathbf{v}_0; t_{k+1}, \dots, t_d)} W_T(t_{k+1}, \dots, t_d) + O\left(\frac{X_1 \cdots X_k}{R^A}\right), \tag{4.6}$$

where $\mathbf{v}_0 \in \mathbb{R}^k$ is the solution to $\nabla\phi(\mathbf{v}_0; t_{k+1}, \dots, t_d) = 0$, with the derivative being with respect to the first k variables only (i.e., the first k entries of $\nabla\phi$ are zero). Here we have trivially integrated each error term over any remaining variables of integration; the arbitrarily large power of R savings nicely allows for this crude treatment of the error terms.

The following is an Archimedean analogue of the well-known change-of-basis formula from additive to multiplicative characters (compare with [13, equation (3.11)]):

Lemma 4.4. *Suppose that w_T is 1-inert, supported on $x \asymp X$ where $X \gg 1$. Then*

$$e^{-ix} w_T(x) = X^{-1/2} \int_{-t \asymp X} v(t) x^{it} dt + O(X^{-100}), \tag{4.7}$$

where $v(t) = v_X(t)$ is some smooth function satisfying $v(t) \ll 1$. Moreover, $v(t) = e^{-it \log(|t|/e)} W(t)$ for some 1-inert function W supported on $-t \asymp X$.

Proof. Let $f(x) = e^{-ix} w_T(x)$. By Mellin inversion,

$$f(x) = \int_{(\sigma)} \frac{\tilde{f}(-s)}{2\pi i} x^s ds, \quad \text{where } \tilde{f}(-s) = \int_0^\infty e^{-ix} x^{-s} w_T(x) \frac{dx}{x}. \tag{4.8}$$

Take $\sigma = 0$, so $s = it$. Lemma 4.2 implies that $\tilde{f}(-it)$ is very small outside of the interval $-t \asymp X$. For $-t \asymp X$, Lemma 4.3 gives

$$\tilde{f}(-it) = X^{-1/2} e^{-it \log(|t|/e)} W(t) + O(X^{-200}), \tag{4.9}$$

where W is a 1-inert function supported on $-t \asymp X$. □

For later use, we record some simple consequences of the previous lemmas.

Lemma 4.5. *Let $v(t) = e^{-it \log(|t|/e)} W(t)$ for some 1-inert function W supported on $-t \asymp X$ with $X \gg 1$. Let $\gamma(s) = \pi^{-s/2} \Gamma\left(\frac{s+\kappa}{2}\right)$ for $\kappa \in \{0, 1\}$. Let $D(s) = \sum_{n=1}^\infty a_n n^{-s}$ be a Dirichlet series absolutely convergent for $\text{Re}(s) = 0$ with $\max_{t \in \mathbb{R}} |D(it)| \leq A$ for some $A \geq 0$. Let c_1, c_2, c_3 be some real numbers (which may vary with X) with $0 \leq c_1 \ll 1$ and $|c_2|X^3 + |c_3| \ll X^{1-\delta}$ for some $\delta > 0$. For any $Y > 0$, we have*

$$X^{-1/2} \int_{-\infty}^\infty v(t) e^{-c_1 it \log|t| + c_2 it^3} Y^{it} D(it) dt \ll_{v,A} 1 \tag{4.10}$$

and

$$X^{-1/2} \int_{-\infty}^{\infty} v(t) e^{-c_1 it \log|t| + c_2 it^3} \frac{\gamma(1/2 - i(t + c_3))}{\gamma(1/2 + i(t + c_3))} Y^{it} D(it) dt \ll_{v,A} 1. \tag{4.11}$$

The bounds depend only on v and A .

Proof. Expanding out the Dirichlet series, and exchanging summation and integration, it suffices to prove the result with $D(s) = 1$. We first consider formula (4.10), which is an oscillatory integral with phase

$$\phi(t) = -(1 + c_1)t \log|t| + t \log(eY) + c_2 t^3.$$

Note that the leading phase points in the direction $-t \log|t|$. For $|t| \asymp X$ we have $\phi'(t) = -(1 + c_1) \log|t| + \log(Y) - c_1 + O(X^{-\delta})$. Lemma 4.2 shows that the left-hand side of formula (4.10) is very small unless $\log Y = (1 + c_1) \log X + O(1)$, for a sufficiently large implied constant. On the other hand, if $\log Y = (1 + c_1) \log X + O(1)$, then $\phi'(t) = -(1 + c_1) \log(|t|/X) - (1 + c_1) \log X + \log(Y) + O(1) = O(1)$. We may then use Lemma 4.3 to show the claimed bound (4.10).

For the second bound (4.11), we first observe that by Stirling’s formula we have

$$\frac{\gamma(1/2 - i(t + c_3))}{\gamma(1/2 + i(t + c_3))} = W(t) e^{-i(t+c_3) \log|t+c_3| + cit} + O(X^{-200})$$

for some 1-inert function W and some $c \in \mathbb{R}$. With the phase of this gamma ratio pointing in the same direction as $-t \log|t|$, we can repeat the same argument as before to show square-root cancellation. □

We end this section with some heuristic motivation for the bound in formula (4.11), and how it is related to expression (1.11) from the sketch. Let w be a fixed inert function, $C \gg 1$, and $P := A/C \gg 1$. By Poisson summation, we have

$$S := \sum_{c=1}^{\infty} e\left(-\frac{A}{c}\right) w(c/C) = \sum_q \int_{-\infty}^{\infty} e\left(-\frac{A}{t} - qt\right) w(t/C) dt. \tag{4.12}$$

Integration by parts and stationary phase tells us that the sum is essentially supported on $q \asymp \frac{A}{C^2}$, in which case the integral is bounded by $\frac{C}{\sqrt{P}}$. An alternative (and admittedly more roundabout!) way to accomplish this same goal is to use Lemma 4.4 with $x = 2\pi \frac{A}{c}$ and the functional equation of the Riemann zeta function (shifting contours appropriately). The dual sum will have a test function of the form on the left-hand side of formula (4.11) (with $c_3 = 0$, in fact), and the bound in formula (4.11) is consistent with the simpler Fourier analysis just presented. The reader may wonder, then, why we have proceeded in this more complicated fashion if the Fourier approach is simpler. The answer is that the actual sums we encounter in this paper are arithmetically much more intricate than the simplified one presented in formula (4.12). The Mellin-transform approach is better suited to handling the more complicated arithmetical features that are present in our problem, so on the whole, taking into account both the analytic and arithmetic aspects of the problem, it is simpler.

5. Character sum evaluations

We need the following elementary character sum calculations. Define the Gauss sum

$$G\left(\frac{a}{c}\right) = \sum_{x(\bmod c)} e_c(ax^2). \tag{5.1}$$

We need to evaluate $G(a/c)$. It is well known (see, e.g., [13, equations (3.22) and (3.38)]) that

$$G\left(\frac{a}{c}\right) = \left(\frac{a}{c}\right) \epsilon_c \sqrt{c}, \quad \epsilon_c = \begin{cases} 1, & c \equiv 1 \pmod{4}, \\ i, & c \equiv 3 \pmod{4}, \end{cases} \tag{5.2}$$

provided $(2a, c) = 1$. The case with c even is treated as follows. Let $\delta \in \{0, 1\}$ indicate the parity of the highest power of 2 dividing c , as follows: If $2^{v_2} \parallel c$ then let

$$\delta \equiv v_2 \pmod{2}. \tag{5.3}$$

From the context, this should not be confused with usages where δ is a small positive constant or the $\delta(P)$ function, which equals 1 when a statement P is true and 0 otherwise.

Lemma 5.1. *Suppose $c = 2^k c_o$ with $k \geq 2$, c_o odd, and δ is as in equation (5.3). Suppose also $(a, c) = 1$. Then*

$$G\left(\frac{a}{c}\right) = \epsilon_{c_o} c^{1/2} \left(\frac{a2^\delta}{c_o}\right) \begin{cases} 1 + e_4(ac_o), & \delta = 0, \\ 2^{1/2} e_8(ac_o), & \delta = 1. \end{cases} \tag{5.4}$$

Proof. First we note that if $c = c_1 c_2$ with $(c_1, c_2) = 1$, then

$$G\left(\frac{a}{c_1 c_2}\right) = G\left(\frac{ac_2}{c_1}\right) G\left(\frac{ac_1}{c_2}\right). \tag{5.5}$$

Suppose that $c = 2^k$ with $k \geq 2$. Let j be an integer so that $2j \geq k$, and write $x = u + 2^j v$ with u running modulo 2^j and v running modulo 2^{k-j} . Then

$$G\left(\frac{a}{2^k}\right) = \sum_{u(\bmod 2^j)} e_{2^k}(au^2) \sum_{v(\bmod 2^{k-j})} e_{2^{k-j-1}}(auv). \tag{5.6}$$

The inner sum over v vanishes unless $u \equiv 0 \pmod{2^{k-j-1}}$, so we change variables $u = 2^{k-j-1} r$, with r now running modulo 2^{2j-k+1} . This gives

$$G\left(\frac{a}{2^k}\right) = 2^{k-j} \sum_{r(\bmod 2^{2j-k+1})} e_{2^{2j-k+2}}(ar^2). \tag{5.7}$$

In the case that k is even, we make the choice $j = k/2$, giving

$$G\left(\frac{a}{2^k}\right) = 2^{k/2} \sum_{r(\bmod 2)} e_4(ar^2) = 2^{k/2}(1 + e_4(a)). \tag{5.8}$$

If k is odd, we take $j = \frac{k+1}{2}$, giving now

$$G\left(\frac{a}{2^k}\right) = 2^{\frac{k-1}{2}} \sum_{r \pmod{2^2}} e_{2^3}(ar^2) = 2^{\frac{k+1}{2}} e_8(a). \tag{5.9}$$

Assembling these facts and using equation (5.2) completes the proof. □

Lemma 5.2. *Let χ be a Dirichlet character modulo q , and suppose $d \mid q$ and $(a, d) = 1$. Let*

$$S_\chi(a, d, q) = \sum_{\substack{n \pmod{q} \\ n \equiv a \pmod{d}}} \chi(n). \tag{5.10}$$

Suppose that χ is induced by the primitive character χ^ modulo q^* , and write $\chi = \chi^* \chi_0$ where χ_0 is trivial modulo q_0 , with $(q_0, q^*) = 1$. Then $S_\chi(a, d, q) = 0$ unless $q^* \mid d$, in which case*

$$S_\chi(a, d, q) = \frac{q}{d} \chi^*(a) \prod_{\substack{p \mid q_0 \\ p \nmid d}} \left(1 - \frac{1}{p}\right). \tag{5.11}$$

Proof. Suppose $q = q_1 q_2$ with $(q_1, q_2) = 1$ and correspondingly factor $d = d_1 d_2$ and $\chi = \chi_1 \chi_2$ with χ_i modulo q_i . The Chinese remainder theorem gives $S_\chi(a, d, q) = S_{\chi_1}(a, d_1, q_1) S_{\chi_2}(a, d_2, q_2)$. Writing $d = d^* d_0$ where $d^* \mid q^*$ and $d_0 \mid q_0$, we apply this with $q_1 = q^*$, $q_2 = q_0$, $\chi_1 = \chi^*$, $\chi_2 = \chi_0$, $d_1 = d^*$, and $d_2 = d_0$. By the multiplicativity of the right-hand side of equation (5.11), it suffices to prove it for χ^* and χ_0 .

By [13, equation (3.9)], $S_{\chi^*}(a, d^*, q^*) = 0$ unless $q^* \mid d^*$, in which case it is given by equation (5.11), so this case is done.

For the χ_0 part, we simply use Möbius inversion, giving

$$S_{\chi_0}(a, d_0, q_0) = \sum_{\ell \mid q_0} \mu(\ell) \sum_{\substack{n \pmod{q_0/\ell} \\ \ell n \equiv a \pmod{d_0}}} 1. \tag{5.12}$$

Since $(a, d_0) = 1$ by assumption, this means that we may assume $(\ell, d_0) = 1$, and then n is uniquely determined modulo d_0 , which divides q_0/ℓ , giving

$$S_{\chi_0}(a, d_0, q_0) = \frac{q_0}{d_0} \sum_{\substack{\ell \mid q_0 \\ (\ell, d_0) = 1}} \frac{\mu(\ell)}{\ell} = \frac{q_0}{d_0} \prod_{\substack{p \mid q_0 \\ p \nmid d_0}} \left(1 - \frac{1}{p}\right). \tag{5.13}$$

□

For $a, b, c \in \mathbb{Z}$ with $c \geq 1$, define

$$T(a, b; c) = \sum_{x, y \pmod{c}} S(x^2, y^2; c) e_c(2xy + ax + by). \tag{5.14}$$

For c_o odd, write its prime factorization as $c_o = \prod_p p^{a_p} \prod_q q^{b_q}$, where each a_p is odd and each b_q is even. Let $c^* = \prod_p p$ and $c_\square = \prod_q q$. Then c^* is the conductor of the Jacobi symbol $\left(\frac{\cdot}{c_o}\right)$.

Lemma 5.3. *Set $a, b, c \in \mathbb{Z}$, with $c \geq 1$. Suppose $c = 2^j c_o$ with $j \geq 4$ and c_o odd, with δ defined as in equation (5.3). Define $a' = \frac{a}{(a,c)}$, $b' = \frac{b}{(b,c)}$. Then $T(a, b; c) = 0$ unless $4 \mid (a, b)$ and $(a, c) = (b, c)$, in which case*

$$T(a, b; c) = \left(a, \frac{c}{2^{2+\delta}}\right) c^{3/2} e_c(-ab/4) \left(\frac{a'b'}{c^*}\right) g_\delta(a', b', c_o) \delta\left(c^* \mid \frac{c_o}{(a, c_o)}\right) \prod_{p \mid c_\square, p \nmid \frac{c_o}{(a, c_o)}} (1 - p^{-1}), \tag{5.15}$$

where g_δ is some function depending on a', b', c_o modulo $2^{2+\delta}$ that additionally depends on $\left(\frac{2^j}{(a, 2^j)}, 2^{2+\delta}\right)$. In particular, we have $T(0, b; c) \ll c^{5/2} \delta(c^* = 1) \delta(c \mid b)$.

Proof. We have

$$T(a, b; c) = \sum_{t \pmod{c}}^* \sum_{x, y \pmod{c}} e_c\left(t(x + \bar{t}y)^2 + ax + by\right). \tag{5.16}$$

Changing variables $x \rightarrow x - \bar{t}y$ and evaluating the resulting y sum by orthogonality, we deduce

$$T(a, b; c) = c \sum_{\substack{t \pmod{c} \\ bt \equiv a \pmod{c}}}^* \sum_{x \pmod{c}} e_c(tx^2 + ax). \tag{5.17}$$

The congruence in the sum implies that $T(a, b; c) = 0$ unless $(a, c) = (b, c)$, a condition that we henceforth assume. Changing variables $x \rightarrow x + c/2$ also shows that $T(a, b; c) = 0$ unless $2 \mid a$, so we assume this condition also.

Write c uniquely as $c = c_1 c_2$, where $c \mid c_1^2$, $c_2 \mid c_1$, and c_1/c_2 is square-free (another way to see this factorization is by writing c uniquely as AB^2 with A square-free; then $c_1 = AB$ and $c_2 = B$). Observe that $2^2 \mid c_2$ from $2^4 \mid c$. Let $x = x_1 + c_1 x_2$, and let $Q(x) = tx^2 + ax$. Note that

$$Q(x_1 + c_1 x_2) = Q(x_1) + Q'(x_1)c_1 x_2 + \frac{Q''(x_1)}{2} c_1^2 x_2^2 \equiv Q(x_1) + Q'(x_1)c_1 x_2 \pmod{c}. \tag{5.18}$$

Thus

$$\begin{aligned} \sum_{x \pmod{c}} e_c(Q(x)) &= \sum_{x_1 \pmod{c_1}} e_c(Q(x_1)) \sum_{x_2 \pmod{c_2}} e_{c_2}(Q'(x_1)x_2) \\ &= c_2 \sum_{\substack{x_1 \pmod{c_1} \\ Q'(x_1) \equiv 0 \pmod{c_2}}} e_c(Q(x_1)). \end{aligned} \tag{5.19}$$

In our case, $Q'(x_1) = 2tx_1 + a$, so the congruence means $2x_1 \equiv -\bar{t}a \pmod{c_2}$. Since $2 \mid a$ and $2 \mid c_2$, this is equivalent to $x_1 \equiv -\bar{t}\frac{a}{2} \pmod{c_2/2}$. Writing $x_1 = -\bar{t}\frac{a}{2} + \frac{c_2}{2}v$, with v

running modulo $2\frac{c_1}{c_2}$, we obtain

$$\sum_{x(\bmod c)} e_c(Q(x)) = c_2 e_c(-\bar{t}a^2/4) \sum_{v(\bmod 2\frac{c_1}{c_2})} e\left(\frac{tv^2}{4c_1/c_2}\right). \tag{5.20}$$

While the exponential in the inner sum has modulus $4c_1/c_2$, the sum is only over $0 \leq v \leq 2(c_1/c_2) - 1$. However, observe that the exponential has the same values at $1 \leq -v \leq 2(c_1/c_2)$, so that the inner sum is half of a Gauss sum. Thus

$$T(a, b; c) = c \frac{c_2}{2} \sum_{\substack{t(\bmod c) \\ bt \equiv a(\bmod c)}}^* e_c(-\bar{t}a^2/4) G\left(\frac{t}{4c_1/c_2}\right). \tag{5.21}$$

By Lemma 5.1, we deduce

$$T(a, b; c) = c^{3/2} \epsilon_{c_o} \sum_{\substack{t(\bmod c) \\ bt \equiv a(\bmod c)}}^* e_c(-\bar{t}a^2/4) \left(\frac{t2^\delta}{c_o}\right) \begin{cases} 1 + e_4(tc_o), & \delta = 0, \\ 2^{1/2} e_8(tc_o), & \delta = 1. \end{cases} \tag{5.22}$$

This formulation contains a few additional observations. We have used the fact that the Jacobi symbol $\left(\frac{t}{(c_1/c_2)_o}\right)$ agrees with $\left(\frac{t}{c_o}\right)$ for t coprime to c , where n_o is the odd part of an integer n . We have also used the fact that $(c_1/c_2)_o$ and c_o have the same values modulo 8. Thus we can replace $\epsilon_{(c_1/c_2)_o}$, $e_4(t(c_1/c_2)_o)$, and $e_8(t(c_1/c_2)_o)$ with ϵ_{c_o} , $e_4(tc_o)$, and $e_8(tc_o)$, respectively. These observations can easily be checked by using multiplicativity to reduce to the case when c is a power of an odd prime. If $c = p^l$, then $c_1/c_2 = 1$ when l is even and $c_1/c_2 = p$ when l is odd.

Next we turn to the t sum in equation (5.22). Suppose first that $2 \parallel a$. Let $a' = \frac{a}{(a,c)}$ and $b' = \frac{b}{(a,c)}$. The congruence $bt \equiv a \pmod{c}$ uniquely determines t modulo $c/(a,c)$, since it is equivalent to $\bar{t} \equiv \bar{b}'\bar{a}' \pmod{c/(a,c)}$. Now in the t sum, one can pair up \bar{t} with $\bar{t} + c/2$ and observe that the corresponding values of the exponential $e_c(-\bar{t}a^2/4)$ will cancel out, since $e_c(-(c/2)a^2/4) = -1$. Also, the values of $\left(\frac{t}{c_o}\right) = \left(\frac{\bar{t}}{c_o}\right)$, $e_4(tc_o) = e_4(\bar{t}c_o)$, and $e_8(tc_o) = e_8(\bar{t}c_o)$ remain the same under $\bar{t} \rightarrow \bar{t} + c/2$, since by assumption $2^4 \mid c$. Therefore, $T(a, b; c)$ vanishes unless $4 \mid a$ (and hence $4 \mid b$), which we now assume to be the case. This allows the convenient simplification $e_c(-\bar{t}a^2/4) = e_c(-ab/4)$.

Breaking up the t sum into congruence classes modulo $2^{2+\delta}$, to uniquely determine $e_{2^{2+\delta}}(tc_o)$, we obtain

$$T(a, b; c) = c^{3/2} \epsilon_{c_o} e_c(-ab/4) \sum_{v(\bmod 2^{2+\delta})}^* \left\{ \begin{matrix} 1 + e_4(vc_o) \\ 2^{1/2} e_8(vc_o) \end{matrix} \right\} \sum_{\substack{t(\bmod c) \\ t \equiv \bar{b}'a'(\bmod \frac{c}{(a,c)}) \\ t \equiv v(\bmod 2^{2+\delta})}}^* \left(\frac{t2^\delta}{c_o}\right). \tag{5.23}$$

For the congruence $t \equiv \bar{b}'a' \pmod{\frac{c}{(a,c)}}$ to be consistent with $t \equiv v \pmod{2^{2+\delta}}$, it is necessary and sufficient that $v \equiv \bar{b}'a' \pmod{\left(\frac{c}{(a,c)}, 2^{2+\delta}\right)}$.

Recall that $c = 2^j c_o$, where $j \geq 4$. Factoring the moduli in the sum, we have

$$\sum_{\substack{t \pmod{c} \\ t \equiv \bar{b}'a' \pmod{\frac{c}{(a,c)}} \\ t \equiv v \pmod{2^{2+\delta}}}^* \left(\frac{t}{c_o}\right) = \left(\sum_{\substack{t \pmod{c_o} \\ t \equiv \bar{b}'a' \pmod{\frac{c_o}{(a,c_o)}}}^* \left(\frac{t}{c_o}\right) \right) \left(\sum_{\substack{t \pmod{2^j} \\ t \equiv \bar{b}'a' \pmod{\frac{2^j}{(a,2^j)}} \\ t \equiv v \pmod{2^{2+\delta}}}^* 1 \right). \tag{5.24}$$

The sum modulo 2^j , by the Chinese remainder theorem and the fact that the condition $(t, 2) = 1$ is automatic because $(v, 2) = 1$, equals

$$\frac{2^j}{\left[\frac{2^j}{(a,2^j)}, 2^{2+\delta}\right]} = \frac{2^j \left(\frac{2^j}{(a,2^j)}, 2^{2+\delta}\right)}{\frac{2^j}{(a,2^j)} 2^{2+\delta}} = (a, 2^{j-2-\delta}),$$

provided of course that $v \equiv \bar{b}'a' \pmod{\left(\frac{2^j}{(a,2^j)}, 2^{2+\delta}\right)}$. Therefore, we have that $T(a, b; c)$ equals

$$c^{3/2} \epsilon_{c_o} e_c(-ab/4) (a, 2^{j-2-\delta}) \sum_{\substack{v \pmod{2^{2+\delta}} \\ v \equiv \bar{b}'a' \pmod{\left(\frac{2^j}{(a,2^j)}, 2^{2+\delta}\right)}}^* \left\{ \begin{matrix} 1 + e_4(vc_o) \\ 2^{1/2} e_8(vc_o) \end{matrix} \right\} \sum_{\substack{t \pmod{c_o} \\ t \equiv \bar{b}'a' \pmod{\frac{c_o}{(a,c_o)}}}^* \left(\frac{t2^\delta}{c_o}\right). \tag{5.25}$$

By Lemma 5.2, with $q = c_o$, $d = \frac{c_o}{(a,c_o)}$, $a = \bar{b}'a'$, $q^* = c^*$, and $q_0 = c_\square$, we have

$$\sum_{\substack{t \pmod{c_o} \\ t \equiv \bar{b}'a' \pmod{\frac{c_o}{(a,c_o)}}}^* \left(\frac{t}{c_o}\right) = (a, c_o) \left(\frac{a'b'}{c^*}\right) \delta\left(c^* \mid \frac{c_o}{(a,c_o)}\right) \prod_{\substack{p \mid c_\square \\ p \nmid \frac{c_o}{(a,c_o)}}} (1 - p^{-1}). \tag{5.26}$$

Inserting equation (5.26) into expression (5.25) and simplifying a bit using $(a, c_o) (a, 2^{j-2-\delta}) = (a, \frac{c}{2^{2+\delta}})$, we deduce that $T(a, b; c)$ equals

$$c^{3/2} \epsilon_{c_o} e_c(-ab/4) \left(a, \frac{c}{2^{2+\delta}}\right) \left(\frac{a'b'2^\delta}{c^*}\right) \prod_{\substack{p \mid c_\square \\ p \nmid \frac{c_o}{(a,c_o)}}} (1 - p^{-1}) \sum_{\substack{v \pmod{2^{2+\delta}} \\ v \equiv \bar{b}'a' \pmod{\left(\frac{2^j}{(a,2^j)}, 2^{2+\delta}\right)}}^* \left\{ \begin{matrix} 1 + e_4(vc_o) \\ 2^{1/2} e_8(vc_o) \end{matrix} \right\} \tag{5.27}$$

times the delta function that c^* divides $\frac{c_o}{(a,c_o)}$. The inner sum over v is a function of a', b', c_o modulo $2^{2+\delta}$ that additionally depends on $\left(\frac{2^j}{(a,2^j)}, 2^{2+\delta}\right)$. In addition, $\left(\frac{2^\delta}{c^*}\right)$ is a function of c_o modulo $2^{2+\delta}$. □

6. Start of proof

Set $0 \leq U \leq (2 - \delta)T$. By an approximate functional equation, dyadic decomposition of unity, and Cauchy's inequality, we have

$$\mathcal{M} := \sum_{T < t_j < T + \Delta} |L(\text{sym}^2 u_j, 1/2 + iU)|^2 \ll \max_{1 \ll N \ll N_{\max}} \frac{T^\varepsilon}{N} \sum_{T < t_j < T + \Delta} \left| \sum_n \frac{\lambda_j(n^2)}{n^{iU}} w_N(n) \right|^2, \tag{6.1}$$

where $w_N(x)$ is supported on $x \asymp N$ and satisfies $w_N^{(j)}(x) \ll_j N^{-j}$ and $N_{\max} = (U + 1)^{1/2} T^{1+\varepsilon}$. To save some clutter in the notation, we want to simply write U instead of $U + 1$ in all estimates involving U . The reader may accept this as a convention or, when $0 \leq U \leq 1$, we can write $n^{-iU} w_N(n) = n^{-i(U+1)} n^i w_N(n)$ and absorb n^i into $w_N(n)$ by redefining the weight function. Thus we can henceforth assume that $U \geq 1$.

Next we insert a weight

$$h(t) = \frac{t^2 + \frac{1}{4}}{T^2} \left[\exp\left(-\frac{(t-T)^2}{\Delta^2}\right) + \exp\left(-\frac{(t+T)^2}{\Delta^2}\right) \right], \tag{6.2}$$

write $\lambda_j(n^2) = \rho_j(n^2) / \rho_j(1)$, and overextend (by positivity) the spectral sum to an orthonormal basis of all cusp forms of level 2^4 , embedding the level 1 forms. This embedding trick, introduced for the purpose of simplifying the 2-part of the exponential sum in Lemma 5.3, is motivated from [4, p. 4]. We also form the obvious Eisenstein series variant on the sum. This leads to the following inequality (see the remarks following Lemma 2.1):

$$\begin{aligned} \mathcal{M} \ll & \max_{1 \ll N \ll N_{\max}} \frac{T^\varepsilon}{N} \left(\sum_{u_j \text{ level } 2^4} \frac{h(t_j)}{\cosh(\pi t_j)} \left| \sum_n \frac{\rho_j(n^2)}{n^{iU}} w_N(n) \right|^2 \right. \\ & \left. + \sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{h(t)}{\cosh(\pi t)} \left| \sum_n \frac{\tau_{a,it}(n^2)}{n^{iU}} w_N(n) \right|^2 dt \right). \end{aligned} \tag{6.3}$$

Opening the square and applying the Kuznetsov formula, we obtain

$$\mathcal{M} \ll \Delta T^{1+\varepsilon} + \max_{1 \ll N \ll N_{\max}} T^\varepsilon |\mathcal{S}(H)|, \tag{6.4}$$

where

$$\mathcal{S}(H) = \frac{1}{N} \sum_{c \equiv 0 \pmod{2^4}} \sum_{m,n} \frac{S(m^2, n^2; c)}{cm^{iU} n^{-iU}} w_N(m) w_N(n) H\left(\frac{4\pi mn}{c}\right), \tag{6.5}$$

$$H(x) = i \int_{-\infty}^{\infty} J(x, t) t \tanh(\pi t) h(t) dt, \tag{6.6}$$

and $J(x, t)$ is as defined in Lemma 2.1.

By [17, equation (3.10)], we get $H(x) \ll \frac{\Delta}{T} x^2$ for $x \leq 1$. Using this with $x = 4\pi mn/c$, we can truncate c at some large power of T , say $c \leq T^{100}$, with an acceptable error term.

Using [10, section 8.411 equation 11] and the fact that the integrand in equation (6.6) is an even function of t , one can derive as in [17, equation (3.13)] that $H(x) = \frac{2}{\pi} \text{Re}(H_0(x))$, where

$$H_0(x) = \int_{-\infty}^{\infty} e^{ix \cosh v} \int_{-\infty}^{\infty} e^{-2ivt} t \tanh(\pi t) h(t) dt dv. \tag{6.7}$$

The inner t -integral is

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-2ivt} t \tanh(\pi t) \frac{t^2 + \frac{1}{4}}{T^2} \left(\exp\left(-\frac{(t-T)^2}{\Delta^2}\right) + \exp\left(-\frac{(t+T)^2}{\Delta^2}\right) \right) dt \\ = \Delta T (e^{-2ivT} + e^{2ivT}) g(\Delta v), \end{aligned} \tag{6.8}$$

where $g(y) = g_{\Delta, T}(y)$ behaves like a fixed (even) Schwartz-class function; namely, it satisfies the derivative bounds $g^{(j)}(y) \ll_{j, A} (1 + |y|)^{-A}$, for any $j, A \in \mathbb{Z}_{\geq 0}$. Hence

$$H_0(x) = 2\Delta T \int_{-\infty}^{\infty} e^{ix \cosh v} e^{-2ivT} g(\Delta v) dv. \tag{6.9}$$

From this, we can write the real part of $H_0(x)$ as a linear combination of $H_{\pm}(x)$, where

$$H_{\pm}(x) = \Delta T \int_{-\infty}^{\infty} e^{\pm ix \cosh v - 2ivT} g(\Delta v) dv = \Delta T e^{\pm ix} \int_{-\infty}^{\infty} e^{\pm ix(\cosh v - 1) - 2ivT} g(\Delta v) dv. \tag{6.10}$$

Then formula (6.4) becomes

$$\mathcal{M} \ll \Delta T^{1+\varepsilon} + \max_{\substack{1 \ll N \ll N_{\max} \\ \pm}} T^{\varepsilon} |\mathcal{S}(H_{\pm})|. \tag{6.11}$$

It suffices to bound $\mathcal{S}(H_+)$, as the argument for $\mathcal{S}(H_-)$ is similar. For convenience, let us write this as $H_+(x) = \Delta T e^{ix} K_+(x)$, where

$$K_+(x) = \int_{-\infty}^{\infty} e^{ix(\cosh v - 1) - 2ivT} g(\Delta v) dv. \tag{6.12}$$

Finally, we apply a dyadic partition of unity to the c sum. To summarize, we have shown

$$\mathcal{S}(H_+) = \frac{\Delta T}{N} \sum_C \sum_{c \equiv 0 \pmod{2^j}} \sum_{m, n} \frac{S(m^2, n^2; c) e_c(2mn)}{cm^i n^{-iU}} w(m, n, c) K_+ \left(\frac{4\pi mn}{c} \right) + O(T^{-100}), \tag{6.13}$$

where the first sum is a sum over integers C equal to $2^{j/2}$ for $0 \leq j \leq 300 \log T$ and $w(x_1, x_2, x_3) = w_{N, C}(x_1, x_2, x_3)$ is 1-inert and supported on $x_1 \asymp x_2 \asymp N$ and $c \asymp C$.

We may approximate $H_+(x)$ quite well by truncating the integral at $|v| \leq \Delta^{-1}T^\varepsilon$, and then use an integration-by-parts argument to see that $H_+(x)$ is very small unless

$$x \gg \Delta T^{1-\varepsilon}. \tag{6.14}$$

For more details of an alternative approach, see [17, pp. 76–77]. In our situation, where $x \asymp \frac{N^2}{C}$, we conclude that we may assume

$$C \ll T^\varepsilon \frac{N^2}{\Delta T} \ll T^\varepsilon \frac{UT}{\Delta}. \tag{6.15}$$

For our purposes it is inconvenient to develop the v -integral further at this early stage. However, we do record the following slight refinement that is useful for large values of x :

Lemma 6.1. *Suppose that*

$$x \gg T^{2-\varepsilon}. \tag{6.16}$$

Then

$$K_+(x) = \int_{|v| \ll x^{-1/2}T^\varepsilon} e^{ix(\cosh(v)-1)-2iTv} g(\Delta v)\eta(v)dv + O((xT)^{-100}), \tag{6.17}$$

where η is supported on $|v| \ll x^{-1/2}T^\varepsilon$ and satisfies property (4.1) for a 1-inert function.

Proof. This follows from the integration-by-parts lemma 4.2. □

7. Double Poisson summation

Next we apply Poisson summation to the m and n sums in equation (6.13), giving

$$\mathcal{S}(H_+) = \frac{\Delta T}{N} \sum_C \sum_{c \equiv 0 \pmod{2^4}} \sum_{k, \ell} \frac{T(-k, \ell; c)}{c^3} I(k, \ell, c) + O(T^{-100}), \tag{7.1}$$

where

$$I(k, \ell, c) = \int_0^\infty \int_0^\infty x^{-iU} y^{iU} e_c(kx - \ell y) K_+ \left(\frac{4\pi xy}{c} \right) w(x, y, c) dx dy. \tag{7.2}$$

By Lemma 5.3, $T(-k, \ell; c) = 0$ unless $(k, c) = (\ell, c)$ and $4 \mid (k, \ell)$, in which case

$$\begin{aligned} & T(-k, \ell, c) \\ &= c^{3/2} (k, 2^{-2-\delta}c) e_c(k\ell/4) \left(\frac{k'\ell'}{c^*} \right) g_\delta(k', \ell', c_o) \delta \left(c^* \mid \frac{c_o}{(k, c_o)} \right) \prod_{p \mid c_\square, p \nmid \frac{c_o}{(k, c_o)}} (1 - p^{-1}), \end{aligned} \tag{7.3}$$

where $k' = \frac{k}{(k, c)}$, $\ell' = \frac{\ell}{(\ell, c)}$, δ was defined in equation (5.3), and other notation is carried over from Lemma 5.3 (here the function g_δ has the same properties as the one appearing in Lemma 5.3, but may not agree with it).

Write

$$c = 2^\lambda c_o, \quad k = 2^\nu k_o, \quad \ell = 2^\gamma \ell_o, \tag{7.4}$$

with $(k_o \ell_o c_o, 2) = 1$. The condition $(k, c) = (\ell, c)$ now becomes $\min(\lambda, \nu) = \min(\lambda, \gamma)$, and $(k_o, c_o) = (\ell_o, c_o)$. The condition $4 \mid (k, \ell)$ now means $\nu, \gamma \geq 2$. We also write

$$c_o = qr_1^2 r_2^2, \tag{7.5}$$

where q is square-free, $r_1 \mid q^\infty$, and $(q, r_2) = 1$. With this notation, $c^* = q$ and c_\square shares the same prime factors as r_2 . Note that $\frac{c_o}{(k, c_o)} = \frac{qr_1^2}{(k_o, qr_1^2)} \frac{r_2^2}{(k_o, r_2^2)}$. Thus the condition $c^* \mid \frac{c_o}{(k, c_o)}$ means $q \mid \frac{qr_1^2}{(k_o, qr_1^2)}$, which is equivalent to $(k_o, qr_1^2) \mid r_1^2$. Then

$$\mathcal{S}(H_+) = \sum_C \frac{\Delta T}{NC^{3/2}} \sum_{\substack{\nu, \gamma \geq 2, \lambda \geq 4 \\ \min(\lambda, \nu) = \min(\lambda, \gamma)}} (2^\nu, 2^{\lambda-2-\delta}) \sum_{(r_1 r_2, 2)=1} \sum_{q: r_1 \mid q^\infty} \sum_{\substack{(k_o \ell_o, 2)=1 \\ (q, 2r_2)=1 \\ (k_o, qr_1^2) \mid r_1^2}} \left(\prod_{p \mid r_2, p \nmid \frac{r_2^2}{(k_o, r_2^2)}} (1 - p^{-1}) \right) \left(\frac{k' \ell'}{c^*} \right) (k_o, c_o) e_c(k\ell/4) g_\delta(k', \ell', c_o) I(k, \ell, c) + O(T^{-100}), \tag{7.6}$$

where in places to simplify the notation we have not displayed the substituted values, such as $c_o = qr_1^2 r_2^2$. We remark that the statement that $g_\delta(k', \ell', c_o)$ depends additionally on $\left(\frac{c}{(a, c)}, 2^{2+\delta}\right)$ means it depends on $(2^{\lambda-\min(\lambda, \nu)}, 2^{2+\delta})$. In particular, g_δ depends additionally on λ, ν , but only lightly, in the sense that it falls into the four following cases:

$$\text{i) } \lambda \leq \nu, \quad \text{ii) } \lambda = \nu + 1, \quad \text{iii) } \lambda = \nu + 2, \quad \text{iv) } \lambda \geq \nu + 3. \tag{7.7}$$

Next we want to give a variable name to (k_o, c_o) , etc. We have $(k_o, c_o) = (k_o, qr_1^2) (k_o, r_2^2)$, and similarly $(\ell_o, c_o) = (\ell_o, qr_1^2) (k_o, r_2^2)$. Let

$$(k_o, qr_1^2) = (\ell_o, qr_1^2) = g_1 \quad \text{and} \quad (k_o, r_2^2) = (\ell_o, r_2^2) = g_2. \tag{7.8}$$

Here g_1 runs over divisors of r_1^2 and g_2 runs over divisors of r_2^2 . Let

$$k_o = g_1 g_2 k'_o \quad \text{and} \quad \ell_o = g_1 g_2 \ell'_o, \tag{7.9}$$

where $(k'_o \ell'_o, q \frac{r_1^2}{g_1}) = 1$ and $(k'_o \ell'_o, \frac{r_2^2}{g_2}) = 1$. In our context, the presence of the Jacobi symbol $\left(\frac{k' \ell'}{q}\right)$ means that we may automatically assume $(k'_o \ell'_o, q) = 1$, which implies $(k'_o \ell'_o, q \frac{r_1^2}{g_1}) = 1$. Note that $k' = k'_o 2^{\nu-\min(\nu, \lambda)}$ and $\ell' = \ell'_o 2^{\gamma-\min(\gamma, \lambda)}$. We also apply quadratic reciprocity,

giving $\left(\frac{k'_o \ell'_o}{q}\right) = \left(\frac{q}{k'_o \ell'_o}\right)$ times a function depending on k'_o, ℓ'_o, q' modulo 8 (which alters only the definition of g). Making these substitutions, we obtain

$$\begin{aligned} \mathcal{S}(H_+) &= \sum_C \frac{\Delta T}{NC^{3/2}} \sum_{\substack{\nu, \gamma \geq 2, \lambda \geq 4 \\ \min(\lambda, \nu) = \min(\lambda, \gamma)}} (2^\nu, 2^{\lambda-2-\delta}) \sum_{(r_1 r_2, 2)=1} \sum_{\substack{g_1 | r_1^2 \\ g_2 | r_2^2}} g_1 g_2 \prod_{p|r_2, p \nmid \frac{r_2^2}{g_2}} (1-p^{-1}) \\ &\quad \sum_{\substack{q: r_1 | q^\infty \\ (q, 2r_2)=1}}^* \sum_{\substack{(k'_o \ell'_o, 2)=1 \\ (k'_o \ell'_o, \frac{r_2^2}{g_2})=1}} \left(\frac{q}{k'_o \ell'_o}\right) e_c(k\ell/4) g_{\lambda, \nu, \gamma, \delta}(k'_o, \ell'_o, q) I(k, \ell, c) + O(T^{-100}), \end{aligned} \tag{7.10}$$

where $g_{\lambda, \nu, \gamma, \delta}$ is some new function modulo 8.

Finally, we decompose g into Dirichlet characters modulo 8 and break up the sum according to the four cases in formula (7.7), leading to a formula of the form

$$|\mathcal{S}(H_+)| \ll \max_{\substack{\eta_1, \eta_2, \eta_3 \\ \text{cases in (7.7)}}} |\mathcal{S}_\eta(H_+)|, \tag{7.11}$$

where

$$\begin{aligned} \mathcal{S}_\eta(H_+) &= \sum_C \frac{\Delta T}{NC^{3/2}} \sum_{\substack{\nu, \gamma \geq 2, \lambda \geq 4 \\ \min(\lambda, \nu) = \min(\lambda, \gamma) \\ \text{one of (7.7) holds}}} (2^\nu, 2^{\lambda-2-\delta}) \sum_{(r_1 r_2, 2)=1} \sum_{\substack{g_1 | r_1^2 \\ g_2 | r_2^2}} g_1 g_2 \prod_{p|r_2, p \nmid \frac{r_2^2}{g_2}} (1-p^{-1}) \\ &\quad \sum_{\substack{q: r_1 | q^\infty \\ (q, 2r_2)=1}}^* \sum_{\substack{(k'_o \ell'_o, 2)=1 \\ (k'_o \ell'_o, \frac{r_2^2}{g_2})=1}} \eta_1(k'_o) \eta_2(\ell'_o) \eta_3(q) \left(\frac{q}{k'_o \ell'_o}\right) e_c(k\ell/4) I(k, \ell, c) + O(T^{-100}). \end{aligned} \tag{7.12}$$

8. The behavior of $I(k, \ell, c)$

The purpose of this section is to develop the analytic properties of $I(k, \ell, c)$. We begin with a few reduction steps. Inserting equation (6.12) into equation (7.2), we have

$$I(k, \ell, c) = \int_{-\infty}^{\infty} g(\Delta v) e^{-2ivT} \int_0^\infty \int_0^\infty x^{-iU} y^{iU} e_c(kx - \ell y + 2xy(\cosh v - 1)) w(x, y, c) dx dy dv. \tag{8.1}$$

Let $A, B > 0, \epsilon \geq 0$ be real numbers and N and U as before, and consider the integral

$$I(A, B, U, \epsilon, N) = \int_{\mathbb{R}^2} e^{i\phi(x, y)} w_N(x, y, \cdot) dx dy, \tag{8.2}$$

where w_N is 1-inert, supported on $x \asymp y \asymp N$ with $N \gg 1$, and

$$\phi(x, y) = -U \log x + U \log y + Ax - By + \epsilon xy. \tag{8.3}$$

In our case,

$$A = \frac{2\pi k}{c}, \quad B = \frac{2\pi \ell}{c}, \quad \epsilon = \epsilon(v) = 4\pi \frac{\cosh v - 1}{c}, \tag{8.4}$$

and then

$$I(k, \ell, c) = \int_{-\infty}^{\infty} g(\Delta v) e^{-2ivT} I(A, B, U, \epsilon(v), N) dv. \tag{8.5}$$

Note that in our study of $I(A, B, U, \epsilon, N)$, we may assume throughout that $\epsilon > 0$, because $\epsilon(v) = 0$ if and only if $v = 0$, a set of measure 0 for the v -integral of $I(k, \ell, c)$.

Moreover, we may wish to assume that $w_N(x, y) = w_N(x, y, \cdot)$ depends on some unspecified finite list of additional variables that are held suppressed in the notation. In this situation we will assume that w_N is 1-inert in terms of all the variables, not just x and y .

Lemma 8.1. *Suppose that $\epsilon N^2 = o(U)$, with $U \rightarrow \infty$.*

1. *Then $I(A, B, U, \epsilon, N) \ll_C NU^{-C}$ with $C > 0$ arbitrarily large, unless*

$$A \asymp B \asymp \frac{U}{N}. \tag{8.6}$$

2. *In the range (8.6), we have*

$$I = \frac{N^2}{U} e^{i\phi(x_0, y_0)} W(\cdot) + O(N^2 U^{-C}), \tag{8.7}$$

where (x_0, y_0) is the unique solution to $\nabla\phi(x_0, y_0) = \mathbf{0}$ and W is 1-inert in terms of any suppressed variables on which w_N may depend.

3. *Supposing formula (8.6) holds, $\phi(x_0, y_0)$ has the asymptotic expansion*

$$\phi(x_0, y_0) = U \log(A/B) + \sum_{j=0}^J c_j U \left(\frac{\epsilon U}{AB}\right)^{1+2j} + O\left(U \left(\frac{\epsilon U}{AB}\right)^{3+2J}\right), \tag{8.8}$$

for some absolute constants c_j .

Note that formula (8.6) implies $\frac{\epsilon U}{AB} \asymp \frac{\epsilon N^2}{U} = o(1)$, so that equation (8.8) is an asymptotic expansion. We also remark that the assumption $\epsilon N^2 = o(U)$ means that the dominant part of ϕ comes from $-U \log x + U \log y$, and ϵxy is a smaller perturbation.

Proof. The integration-by-parts lemma (Lemma 4.2) shows that the integral is small unless formula (8.6) holds. Assuming it does hold, Lemma 4.3 may be iteratively applied (using the remarks following Lemma 4.3), which gives the form (8.7), with a 1-inert function W .

It only remains to derive the Taylor expansion for $\phi(x_0, y_0)$. We have

$$\phi(Ux/A, Uy/B) = U \log(A/B) + U \Phi(x, y), \tag{8.9}$$

where

$$\Phi(x, y) = -\log x + \log y + x - y + \delta xy \quad \text{and} \quad \delta = \frac{\epsilon U}{AB} = o(1). \tag{8.10}$$

By a simple calculation, we have $\nabla\Phi(x_0, y_0) = \mathbf{0}$ if and only if $x_0 = 1 - \delta x_0 y_0$ and $y_0 = 1 + \delta x_0 y_0$. Thus

$$x_0 + y_0 = 2 \quad \text{and} \quad y_0 - x_0 = 2\delta x_0 y_0. \tag{8.11}$$

Letting $r_0 = x_0 y_0$, we see that it satisfies the relation $r_0 = (1 - \delta r_0)(1 + \delta r_0) = 1 - \delta^2 r_0^2$. Solving this explicitly, we see that r_0 is an even function of δ , analytic for $|\delta| < 1/2$. Note that $r_0 = 1 - \delta^2 + O(\delta^4)$. Then we have

$$\Phi(x_0, y_0) = \log(y_0/x_0) + x_0 - y_0 + \delta x_0 y_0 = \log\left(\frac{1 + \delta r_0}{1 - \delta r_0}\right) - \delta r_0, \tag{8.12}$$

which is an odd function of δ , with power series expansion of the form $\Phi(x_0, y_0) = \delta - \frac{1}{3}\delta^3 + \dots$. Translating back to the original notation gives equation (8.8). \square

Lemma 8.2. *Suppose that $\frac{U}{\epsilon N^2} = o(1)$.*

1. *Then $I(A, B, U, \epsilon, N) \ll_C N^{-C}$ with $C > 0$ arbitrarily large, unless*

$$|A| \asymp |B| \asymp \epsilon N, \quad A < 0, \quad B > 0. \tag{8.13}$$

2. *Assuming formula (8.13), then*

$$I = \frac{1}{\epsilon} e^{i\phi(x_0, y_0)} W(\cdot) + O(N^2 U^{-C}), \tag{8.14}$$

where (x_0, y_0) is the unique solution to $\nabla\phi(x_0, y_0) = \mathbf{0}$ and W is 1-inert in terms of any suppressed variables on which w_N may depend.

3. *Finally, $\phi(x_0, y_0)$ has the following Taylor expansion:*

$$\phi(x_0, y_0) = \frac{AB}{\epsilon} \left[\sum_{j=0}^J c_j \left(\frac{U\epsilon}{AB}\right)^{2j} + O\left(\frac{U\epsilon}{AB}\right)^{2J+2} \right] + U \log\left(\frac{-A}{B}\right), \tag{8.15}$$

with certain absolute constants c_j .

The condition $U = o(\epsilon N^2)$ means that the dominant phase in ϕ is ϵxy , and the phase $-U \log x + U \log y$ is a perturbation.

Proof. Considering the x -integral, Lemma 4.2 shows that $I \ll N^{-C}$ unless

$$\left| \frac{A}{\epsilon N} + \frac{y}{N} \right| \ll \frac{U}{\epsilon N^2} = o(1). \tag{8.16}$$

Since $1 \ll \frac{y}{N} \ll 1$ (with certain absolute implied constants), this means that $|A| \asymp |\epsilon|N$, with A having the opposite sign of ϵ (i.e., $A < 0$). Similarly, considering the y -integral shows that I is small unless $|B| \asymp \epsilon N$, with B having the same sign as ϵ (i.e., $B > 0$).

Next we wish to apply Lemma 4.3 to I . There is a minor technical issue from the fact that the second derivative with respect to x (or y) of ϵxy vanishes, even though this should be viewed as the dominant phase. This issue may be circumvented by a simple change of variable to diagonalize this quadratic form. Precisely, if we let $x = u + v$ and

$y = u - v$, then

$$\varphi(u, v) := \phi(u + v, u - v) = \epsilon u^2 + \alpha u - \epsilon v^2 + \beta v + U \log \left(\frac{u - v}{u + v} \right), \tag{8.17}$$

for certain α, β whose values are immaterial. Then a simple calculation gives

$$\frac{\partial^2}{\partial u^2} \varphi(u, v) = 2\epsilon + U \left(\frac{-1}{(u - v)^2} + \frac{1}{(u + v)^2} \right) = 2\epsilon (1 + O(\epsilon^{-1} N^{-2} U)) \gg |\epsilon|. \tag{8.18}$$

A similar calculation shows $\left| \frac{\partial^2}{\partial v^2} \varphi(u, v) \right| \gg |\epsilon|$. Once we know that stationary phase can be applied after this linear change of variables, we can then revert back to the original variables x, y , giving

$$I = \frac{1}{\epsilon} e^{i\phi(x_0, y_0)} W_T(\cdot) + O(N^{-C}), \tag{8.19}$$

where $\nabla \phi(x_0, y_0) = \mathbf{0}$. We have

$$\phi(Bx/\epsilon, -Ay/\epsilon) = \frac{-AB}{\epsilon} \Phi(x, y) + U \log \left(\frac{-A}{B} \right), \tag{8.20}$$

where

$$\Phi(x, y) = xy - x - y + \delta \log(y/x) \quad \text{and} \quad \delta = \frac{U\epsilon}{AB} \asymp \frac{U}{\epsilon N^2} = o(1). \tag{8.21}$$

A simple calculation shows $\nabla \Phi(x_0, y_0) = \mathbf{0}$ if and only if

$$x_0 = 1 - \frac{\delta}{y_0}, \quad y_0 = 1 + \frac{\delta}{x_0}. \tag{8.22}$$

Solving these explicitly, we obtain

$$x_0 = \frac{1 - 2\delta + \sqrt{1 + 4\delta^2}}{2}, \quad y_0 = \frac{1 + 2\delta + \sqrt{1 + 4\delta^2}}{2}, \tag{8.23}$$

and thus

$$\Phi(x_0, y_0) = -\frac{1 + \sqrt{1 + 4\delta^2}}{2} - \delta \log \left(\frac{1 + 2\delta + \sqrt{1 + 4\delta^2}}{1 - 2\delta + \sqrt{1 + 4\delta^2}} \right) = -\sum_{j=0}^{\infty} c_j \delta^j, \tag{8.24}$$

which is analytic in δ for $|\delta| < 1/2$, and also even with respect to δ . □

Remark. Lemmas 8.1 and 8.2 have some close similarities. In both cases, the stationary-phase method may be applied, and the stationary point can be explicitly found by solving a quadratic equation. In each case, only one of the two roots is relevant, and the other is outside the support of the test function. We expect, but did not confirm rigorously, that when $U \asymp \epsilon N^2$, which is a range that is not needed in this paper, then both roots of the quadratic equation are relevant. This situation is more complicated because the two roots may approach each other, in which case a cubic Taylor approximation to the phase function is more applicable (as with the Airy function, for instance).

9. Cleaning up some terms

In this section we take the opportunity to deal with some ranges of parameters for which relatively easy methods suffice. This will simplify our exposition for the more difficult cases.

With the aid of the analysis from §8 we can now treat some ranges of c .

Lemma 9.1. *The contribution to $\mathcal{S}(H_+)$ from $C \ll \frac{N^2}{T^2} T^\epsilon$ is bounded by $\Delta T^{1+\epsilon}$.*

Proof. Let \mathcal{S} be the contribution to $\mathcal{S}(H_+)$ from $C \ll \frac{N^2}{T^2} T^\epsilon$. Since $x \asymp \frac{N^2}{C}$, the assumed upper bound on C means $x \gg T^{2-\epsilon}$, so that the conditions to apply Lemma 6.1 are in effect. Applying equation (6.17) to equation (7.2), we deduce

$$I(k, \ell, c) = \int_{|v| \ll x^{-1/2} T^\epsilon} e^{-2iTv} g(\Delta v) \eta(v) I(A, B, U, \epsilon(v), N) dv + O(T^{-50}), \tag{9.1}$$

with parameters as given in equation (8.4). Under the present assumptions, we have $\epsilon \ll \frac{v^2}{c} \ll \frac{T^{2\epsilon}}{xc} \asymp \frac{T^{2\epsilon}}{N^2}$. Therefore, in the notation of equation (8.4), we have $\epsilon N^2 \ll T^{2\epsilon}$.

First consider the case where $U \gg T^{3\epsilon}$. In this case, $\epsilon N^2 = o(U)$, and so Lemma 8.1 implies that $I(A, B, U, \epsilon, N) \ll U^{-1} N^2$ and is very small unless $A \asymp B \asymp \frac{U}{N}$. Translating notation, we may assume $|k| \asymp |\ell| \asymp \frac{CU}{N}$, and in particular k and ℓ are nonzero. Integrating trivially over v , we deduce

$$I(k, \ell, c) \ll \frac{NC^{1/2} T^\epsilon}{U} \left(1 + \frac{|k|N}{CU}\right)^{-100} \left(1 + \frac{|\ell|N}{CU}\right)^{-100}. \tag{9.2}$$

Inserting this bound into equation (7.10), we obtain

$$\begin{aligned} |\mathcal{S}| &\ll \frac{\Delta T T^\epsilon}{UC} \sum_{\substack{\nu, \gamma \geq 2, \lambda \geq 4 \\ \min(\lambda, \nu) = \min(\lambda, \gamma)}} (2^\nu, 2^{\lambda-2-\delta}) \\ &\sum_{\substack{k'_o, \ell'_o \neq 0 \\ r_1, r_2}} \sum_{\substack{g_1 | r_1^2 \\ g_2 | r_2^2}} \sum_{g_1 g_2} \sum_{\substack{q^\infty \equiv 0 \pmod{r_1} \\ q \asymp \frac{C}{2^\lambda r_1^2 r_2^2}}} \left(1 + \frac{|k'_o 2^\nu g_1 g_2| N}{CU}\right)^{-100} \left(1 + \frac{|\ell'_o 2^\gamma g_1 g_2| N}{CU}\right)^{-100}. \end{aligned} \tag{9.3}$$

Estimating the sum trivially, and simplifying using $C \ll \frac{N^2}{T^2} T^\epsilon$ and $N \ll N_{\max} \ll U^{1/2} T^{1+\epsilon}$, we deduce

$$|\mathcal{S}| \ll \frac{\Delta T}{N} \frac{C^2 U}{N} T^\epsilon \ll \frac{\Delta U N^2}{T^3} T^\epsilon \ll \Delta T \frac{U^2}{T^2} T^\epsilon, \tag{9.4}$$

which is acceptable, since $U \ll T$.

Next we indicate the changes needed to handle the case $U \ll T^{3\epsilon}$. Integration by parts (Lemma 4.2) shows that $I(A, B, U, \epsilon, N)$ is very small unless $A, B \ll \frac{T^{3\epsilon}}{N}$, or equivalently, $|k|, |\ell| \ll \frac{C}{N} T^{3\epsilon}$. Using $C \ll \frac{N^2}{T^2} T^\epsilon$ and $N \ll N_{\max} \ll T^{1+3\epsilon}$, this means that we only need to consider $k = \ell = 0$. A trivial bound implies $I(0, 0, c) \ll NC^{1/2} T^\epsilon$.

Using the final sentence of Lemma 5.3, we see that the contribution to \mathcal{S} from $k = \ell = 0$ is bounded by

$$\frac{\Delta T}{NC^{3/2}} \frac{NC^{1/2}T^\epsilon}{U} \sum_{r_2 \succ C^{1/2}} C \ll \frac{\Delta T}{U} T^\epsilon C^{1/2} \ll \frac{\Delta N}{U} T^\epsilon \ll \Delta T^{1+\epsilon}. \tag{9.5}$$

□

In light of Lemma 9.1, for the rest of the paper we can assume that

$$C \gg \frac{N^2}{T^2} T^\epsilon. \tag{9.6}$$

Lemma 9.2. *Suppose formula (9.6) holds, and let*

$$V_0 = \frac{TC}{N^2}. \tag{9.7}$$

Then with $x = \frac{4\pi mn}{c} \asymp \frac{N^2}{C}$, we have

$$K_+(x) = \int_{v \asymp V_0} e^{ix(\cosh(v)-1)-2iTv} g(\Delta v) \eta(v) dv + O((xT)^{-100}), \tag{9.8}$$

where η is a 1-inert function supported on $v \asymp V_0$.

Before proving the lemma, we record a simple consequence of it which follows from inserting equation (9.8) into equation (7.2) (valid under the assumption (9.6), which is in effect):

$$I(k, \ell, c) = \int_{v \asymp V_0} e^{ix(\cosh(v)-1)-2iTv} g(\Delta v) \eta(v) I(A, B, U, \epsilon(v), N) dv + O(T^{-50}). \tag{9.9}$$

Proof. In the definition of $K_+(x)$ given by equation (6.12), we first apply a smooth dyadic partition of unity to the region $100V_0 \leq |v| \ll \Delta^{-1}T^\epsilon = o(1)$. Consider a piece of this partition, with, say, $Z \leq |v| \leq 2Z$. We may apply Lemma 4.2 with both Y and R taking the value xZ^2 (and $x \asymp \frac{N^2}{C}$). Note that $xZ^2 \gg \frac{N^2V_0^2}{C} \gg T^\epsilon$, so any such dyadic piece is very small.

Next we consider the portion of the integral with $|v| \leq \frac{V_0}{100}$. The version of the integration-by-parts bound stated in Lemma 4.2 is a simplified variant of [8, Lemma 8.1] (localized to a dyadic interval, etc.) which does not directly apply. However, the more general [8, Lemma 8.1] can be used to show that this portion of the integral is also small. The statement of [8, Lemma 8.1] contains a list of parameters (X, U, R, Y, Q) – not to be confused with the notation from this paper – which in our present context take the values $(1, V_0, T, N^2/C, 1)$. It suffices to use [8, Lemma 8.1] to show that the integral is very small, provided $\frac{QR}{\sqrt{Y}} \rightarrow \infty$ and $RU \rightarrow \infty$. Here QR/\sqrt{Y} takes the form $\frac{T\sqrt{C}}{N} \gg T^{\epsilon/2}$, and $RU = V_0T \gg T^\epsilon$, using the assumption (9.6). The remaining part of the integral is displayed in equation (9.8). □

Lemma 9.3. *Suppose that the conditions of Theorem 1.1 hold, along with formula (6.15). Then*

$$I(k, \ell, c) = \frac{NC^{1/2}}{U} \left(\frac{k}{\ell}\right)^{iU} \exp\left(-\frac{2\pi iT^2 k \ell}{U^2 c}\right) W(\cdot) + O(T^{-100}), \tag{9.10}$$

where W is 1-inert (in k, ℓ , and c , as well as all suppressed variables), and supported on

$$k \asymp \ell \asymp \frac{CU}{N}. \tag{9.11}$$

Proof. We begin by making some simple deductions from the conditions of Theorem 1.1. First we note that formula (1.2) directly implies $U\Delta \geq T^{1+\delta}$. Since formula (6.15) holds, we additionally deduce

$$C \ll \frac{UN^2}{T^2} T^{-\delta}, \tag{9.12}$$

for some $\delta > 0$. Another consequence of formula (1.2) is that

$$\frac{T^3}{U^2 \Delta^3} \ll T^{-2\delta}. \tag{9.13}$$

From the fact that $U \ll T$, we also deduce that (for some $\delta > 0$)

$$\Delta \gg T^{1/3+\delta}. \tag{9.14}$$

Now we pick up with equation (9.9). Using equation (9.7), the condition (9.12) means that $\frac{\epsilon N^2}{U} \asymp \frac{V_0^2 N^2}{CU} \asymp \frac{T^2 C}{UN^2} \ll T^{-\delta}$, so that the conditions of Lemma 8.1 are met. This gives an asymptotic formula for the inner integral $I(A, B, U, \epsilon(v), N)$ for all $v \asymp V_0$. In particular, we deduce that $I(k, \ell, c)$ is very small unless formula (9.11) holds, a condition that we henceforth assume is in place. Note that by formula (8.6),

$$\frac{\epsilon U}{AB} = \frac{(\cosh v - 1)Uc}{\pi k \ell} \asymp \frac{UCV_0^2}{k \ell} \asymp \frac{UC(TC/N^2)^2}{(CU/N)^2} = \frac{T^2 C}{UN^2} \ll \frac{T}{U\Delta} T^\epsilon, \tag{9.15}$$

since $k \asymp \ell \asymp \frac{CU}{N}$, $v \asymp V_0$, and $C \ll \frac{N^2}{\Delta T} T^\epsilon$ (recalling formula (6.15)). Therefore,

$$U \left(\frac{\epsilon U}{AB}\right)^3 \ll U \left(\frac{T}{U\Delta}\right)^3 T^\epsilon \ll \frac{T^3}{U^2 \Delta^3} T^\epsilon \ll T^{-\delta'}, \tag{9.16}$$

for some $\delta' > 0$. This calculation shows that in equation (8.8), the terms with $j \geq 1$ can be absorbed into the inert weight function. This is where we use the condition (1.2), which can likely be relaxed to $U\Delta \gg T^{1+\delta}$, since this condition is sufficient to show that equation (8.8) is a good asymptotic expansion. Therefore,

$$I(k, \ell, c) = \frac{N^2}{U} \left(\frac{k}{\ell}\right)^{iU} \int_{v \asymp V_0} \exp\left(-2iTv + i\frac{U^2 c(\cosh v - 1)}{\pi k \ell}\right) W(v, \cdot) dv, \tag{9.17}$$

plus a small error term, where $W(v, \cdot)$ is 1-inert with respect to k, ℓ, c , and all other suppressed variables. Next we can apply $\cosh(v) - 1 = v^2/2 + O(v^4)$ and absorb the v^4

terms into the inert weight function, using formulas (6.15) and (9.14) as follows:

$$\frac{U^2 CV_0^4}{k\ell} \asymp \frac{C^3 T^4}{N^6} \ll \frac{T}{\Delta^3} T^{3\epsilon} \ll T^{-\delta'}. \tag{9.18}$$

Finally, by stationary phase we obtain the desired estimate. □

Next we simplify our expression for $I(k, \ell, c)$ under the conditions of Theorem 1.3, when U is small.

Lemma 9.4. *Suppose that the conditions of Theorem 1.3 hold, as well as formula (9.6). Then $I(k, \ell, c)$ is very small unless*

$$-k \asymp \ell \asymp \frac{C^2 T^2}{N^3}, \tag{9.19}$$

in which case

$$I(k, \ell, c) = \frac{N^4}{CT^2} (-k/\ell)^{iU} e_c(-k\ell/12) \int_{v \asymp V_0} e^{-2ivT + \frac{2\pi i k \ell}{cv^2}} W(v, \cdot) dv + O(T^{-100}), \tag{9.20}$$

for some function $W(v, \cdot)$ that is 1-inert with respect to k, ℓ, c , and all other suppressed variables.

Remark. Although it is possible to also evaluate the asymptotic of the v -integral in equation (9.20), we prefer to save this step for later (§10).

Proof. We again pick up with equation (9.9) (recall also the definition (8.2)), which takes the form

$$I(k, \ell, c) = \int_{v \asymp V_0} \eta(v) g(\Delta v) e^{-2ivT} I\left(\frac{2\pi k}{c}, \frac{2\pi \ell}{c}, U, \epsilon, N\right) dv, \tag{9.21}$$

with $\epsilon = \epsilon(v) = 4\pi \frac{\cosh(v)-1}{c} \asymp \frac{V_0^2}{C} \asymp \frac{CT^2}{N^4}$, for all $v \asymp V_0$. Since formula (9.6) holds, this means that $\frac{U}{\epsilon N^2} \asymp \frac{UN^2}{T^2 C} \ll T^{-\epsilon}$, so that the conditions of Lemma 8.2 are met. This directly implies that $I(k, \ell, c)$ is very small unless formula (9.19) holds. Note that

$$\frac{AB}{\epsilon} = \frac{\pi k \ell}{c(\cosh v - 1)}, \quad \left| \frac{AB}{\epsilon} \right| \left(\frac{U\epsilon}{AB} \right)^2 = \left| \frac{U^2 \epsilon}{AB} \right| \asymp \frac{U^2 N^2}{CT^2} \ll T^{-\epsilon}. \tag{9.22}$$

The latter calculation shows that the terms with $j \geq 1$ in equation (8.15) may be absorbed into the inert weight function. We thus conclude that

$$I(k, \ell, c) = \frac{N^4}{CT^2} (-k/\ell)^{iU} \int_{v \asymp V_0} e^{-2ivT + \frac{\pi i k \ell}{c(\cosh v - 1)}} W(v, \cdot) dv + O(T^{-100}). \tag{9.23}$$

Finally we observe the Taylor/Laurent approximation

$$\frac{1}{\cosh v - 1} = \frac{2}{v^2} - \frac{1}{6} + O(v^2), \tag{9.24}$$

and that

$$\frac{k\ell}{c} v^2 \asymp \frac{C^5 T^6}{N^{10}} \ll \frac{T}{\Delta^5} T^\epsilon \ll T^{-\delta'} \tag{9.25}$$

for some $\delta' > 0$, where we have used $C \ll \frac{N^2}{\Delta T} T^\epsilon$ from formula (6.15). This lets us absorb the lower-order terms in the Taylor expansion into the inert weight function. Therefore, equation (9.20) holds. \square

10. Mellin inversion

We recall that we have the expression (7.10), which contains a smooth (yet oscillatory) weight function of the form

$$f(k, \ell, c) = e_c(k\ell/4)I(k, \ell, c). \tag{10.1}$$

In the conditions of Theorem 1.1, I is given by Lemma 9.3, whereas in the conditions of Theorem 1.3, I is given by Lemma 9.4. In both cases, the function f is very small except when k and ℓ are fixed into dyadic intervals. We may therefore freely insert an inert weight function that enforces this condition.

First consider the setting relevant for Theorem 1.1. The function f has phase as given in Lemma 9.3, modified to include $e_c(k\ell/4)$, which is strictly smaller in size due to the assumption $U \leq (2 - \delta)T$. We apply Lemma 4.4 to the phase function and Mellin inversion to the inert part. We therefore obtain

$$f(k, \ell, c) = \frac{\Phi}{\sqrt{P}} \left(\frac{2^\nu k'_o}{2^\gamma \ell'_o} \right)^{iU} \int_{-t \asymp P} \int \int \int \left(\frac{T^2 g_1^2 g_2^2 k'_o \ell'_o}{U^2 q r_1^2 r_2^2 2^{\lambda - \nu - \gamma}} \right)^s \left(1 - \frac{U^2}{4T^2} \right)^s v(t) \tilde{w}(u_1, u_2, u_3) \\ \times \left(\frac{C}{q r_1^2 r_2^2 2^\lambda} \right)^{u_1} \left(\frac{K}{k'_o g_1 g_2 2^\nu} \right)^{u_2} \left(\frac{K}{\ell'_o g_1 g_2 2^\gamma} \right)^{u_3} du_1 du_2 du_3 ds, \tag{10.2}$$

plus a small error term, where $s = it$ and

$$\Phi = \frac{N\sqrt{C}}{U}, \quad P = \frac{CT^2}{N^2}, \quad K = \frac{CU}{N}. \tag{10.3}$$

By standard Mellin inversion of an inert function, the function \tilde{w} is entire and has rapid decay on any vertical line. However, we do not specify the vertical contour in this integral (or in several instances to follow). Also, we have absorbed constants such as $\frac{1}{2\pi i}$ and the like into the weight functions. We recall that $k = 2^\nu g_1 g_2 k'_o$, $\ell = 2^\gamma g_1 g_2 \ell'_o$, and $c = 2^\lambda q r_1^2 r_2^2$. We recall from Lemma 4.4 that $v(t)$ is supported on $-t \asymp P$, is $O(1)$, and has phase $e^{-it \log(|t|/e)}$.

We can also apply these steps to I given by Lemma 9.4, which will have a similar structure but with an extra v -integral. We obtain

$$f(k, \ell, c) = \frac{\Phi_0}{\sqrt{P}} \int_{v \asymp V_0} e^{-2ivT} \left(\frac{-2^\nu k'_o}{2^\gamma \ell'_o} \right)^{iU} \int_{-t \asymp P} \int \int \int \left(\frac{g_1^2 g_2^2 |k'_o| \ell'_o}{q r_1^2 r_2^2 2^{\lambda - \nu - \gamma}} \right)^s \\ \times \left(\frac{1}{v^2} + \frac{1}{6} \right)^s v(t) \tilde{w}(u_1, u_2, u_3) \\ \times \left(\frac{C}{q r_1^2 r_2^2 2^\lambda} \right)^{u_1} \left(\frac{K}{|k'_o| g_1 g_2 2^\nu} \right)^{u_2} \left(\frac{K}{\ell'_o g_1 g_2 2^\gamma} \right)^{u_3} du_1 du_2 du_3 ds dv, \tag{10.4}$$

plus a small error term, where this time

$$\Phi_0 = \frac{N^4}{CT^2}, \quad P = \frac{CT^2}{N^2}, \quad K = \frac{C^2T^2}{N^3}, \quad V_0 = \frac{CT}{N^2}. \tag{10.5}$$

Here, $\tilde{w}(u_1, u_2, u_3)$ is implicitly an inert function of v . It is the Mellin transform (in the suppressed variables, but not in v) of the function $W(v, \cdot)$ which was introduced in Lemma 9.4.

At this point, we finally asymptotically evaluate the v -integral. We are considering

$$\int_{v \asymp V_0} e^{-2ivT - 2s \log v + s \log(1 + \frac{v^2}{6})} W(v, \cdot) dv, \tag{10.6}$$

where we recall $s = it$ and $-t \asymp P$. We first observe that $s \log(1 + \frac{v^2}{6}) = sv^2/6 + O(sv^4)$, and note that

$$|sv^4| \asymp PV_0^4 \ll \frac{T^{1+\epsilon}}{\Delta^5} \ll T^{-\delta}, \tag{10.7}$$

by the assumption $\Delta \gg T^{1/5+\epsilon}$. Therefore, the term with sv^4 can be absorbed into the inert weight function at no cost. We are therefore considering an oscillatory integral with phase $\phi(v) = -2vT - 2t \log v + tv^2/6$. It is easy to see that $|\phi''(v)| \asymp \frac{P}{V_0^2}$ throughout the support of the test function, and that there exists a stationary point at v_0 satisfying

$$-2T - \frac{2t}{v_0} + \frac{tv_0}{3} = 0. \tag{10.8}$$

We explicitly calculate

$$v_0 = \frac{2T - 2T\sqrt{1 + \frac{2t^2}{3T^2}}}{2t/3} = \frac{-t}{T} + a' \frac{t^3}{T^3} + O\left(\frac{P^5}{T^5}\right) \tag{10.9}$$

for some constant a' . We observe that $\frac{P^5}{T^4} \ll \frac{T^{1+\epsilon}}{\Delta^5} \ll T^{-\delta}$, so quantities of this size (or smaller) may be safely discarded. For later use, we note in passing that $\frac{P^2}{T^2} \ll \frac{T^\epsilon}{\Delta^2} \ll T^{-\delta}$. We conclude

$$\phi(v_0) = -2t \log(|s|/T) + 2t + a \frac{t^3}{T^2} + O\left(\frac{P^5}{T^4}\right) \tag{10.10}$$

for some new constant a . Therefore,

$$\int_{v \asymp V_0} e^{-2ivT - 2it \log v + it \log(1 + \frac{v^2}{6})} w(v, \cdot) dv = \frac{V_0}{\sqrt{P}} e^{-2it \log(\frac{|t|}{eT})} e^{ia \frac{t^3}{T^2}} W(\cdot) \tag{10.11}$$

for some inert function W and constant a . We deduce a formula for f in the form

$$f(k, \ell, c) = \frac{\Phi}{\sqrt{P}} \left(\frac{-k'_o}{\ell'_o}\right)^{iU} \int_{-t \asymp P} \int \int \int \left(\frac{g_1^2 g_2^2 |k'_o| \ell'_o}{qr_1^2 r_2^2 2^{\lambda-\nu-\gamma}}\right)^s v(t) e^{-2it \log(\frac{|t|}{eT}) + ia \frac{t^3}{T^2}} \tilde{w}(u_1, u_2, u_3) \\ \times \left(\frac{C}{qr_1^2 r_2^2 2^\lambda}\right)^{u_1} \left(\frac{K}{|k'_o| g_1 g_2 2^\nu}\right)^{u_2} \left(\frac{K}{\ell'_o g_1 g_2 2^\gamma}\right)^{u_3} du_1 du_2 du_3 ds dv, \tag{10.12}$$

where now

$$\Phi = \frac{N^4 V_0}{CT^2 P^{1/2}} = \frac{N^3}{C^{1/2} T^2}, \quad P = \frac{CT^2}{N^2}, \quad K = \frac{C^2 T^2}{N^3}, \quad V_0 = \frac{CT}{N^2}. \tag{10.13}$$

This expression for $f(k, \ell, c)$ is similar enough to equation (10.2) that we can proceed in parallel. We mainly focus on the proof of Theorem 1.1.

Inserting equation (10.2) into equation (7.12), we obtain

$$\mathcal{S}_\eta(H_+) = \sum_C \frac{\Delta T}{NC^{3/2}} \frac{\Phi}{\sqrt{P}} \int_{-t \asymp P} \int \int \int \left(\frac{T^2}{U^2} - \frac{1}{4} \right)^s v(t) \tilde{w}(u_1, u_2, u_3) C^{u_1} K^{u_2+u_3} Z(s, u_1, u_2, u_3) du_1 du_2 du_3 ds, \tag{10.14}$$

where $Z = Z_\eta$ is defined by

$$Z(s, u_1, u_2, u_3) = \sum_{\substack{\nu, \gamma \geq 2, \lambda \geq 4 \\ \min(\lambda, \nu) = \min(\lambda, \gamma) \\ \text{one of (7.7) holds}}} \frac{(2^\nu, 2^{\lambda-2-\delta})}{2^{\lambda(u_1+s) + \nu(u_2-iU-s) + \gamma(u_3+iU-s)}} \sum_{(r_1 r_2, 2)=1} \sum_{\substack{g_1 | r_1^2 \\ g_2 | r_2^2}} \sum_{\substack{q: r_1 | q^\infty \\ (q, 2r_2)=1}}^* \sum_{\substack{(k'_o \ell'_o, 2)=1 \\ (k'_o \ell'_o, \frac{r_2^2}{g_2^2})=1}} \frac{\left(\frac{q}{k'_o \ell'_o} \right) \eta_1(k'_o) \eta_2(\ell'_o) \eta_3(q) \prod_{p|r_2, p \nmid \frac{r_2^2}{g_2^2}} (1-p^{-1})}{(k'_o)^{u_2-iU-s} (\ell'_o)^{u_3+iU-s} q^{u_1+s} (r_1^2 r_2^2)^{u_1+s} (g_1 g_2)^{u_2+u_3-2s-1}}. \tag{10.15}$$

We initially suppose that $\text{Re}(s) = 0$ and $\text{Re}(u_i) = 2$ for each i , securing absolute convergence of the sum. An obvious modification, using equation (10.12) in place of equation (10.4), gives the corresponding formula for U small, namely

$$\mathcal{S}_\eta(H_+) = \sum_C \frac{\Delta T}{NC^{3/2}} \frac{\Phi}{\sqrt{P}} \int_{-t \asymp P} \int \int \int e^{-2it \log(\frac{|t|}{eT}) + ia \frac{t^3}{T^2}} v(t) \tilde{w}(u_1, u_2, u_3) C^{u_1} K^{u_2+u_3} Z(s, u_1, u_2, u_3) du_1 du_2 du_3 ds, \tag{10.16}$$

where the parameters correspond with equation (10.13) and the formula for Z is slightly different (multiplied by $\eta_1(-1)$ to account for changing variables $k'_o \rightarrow -k$, with $k \geq 1$).

11. Properties of the Dirichlet series Z

In this section, we pause the development of $\mathcal{S}_\eta(H_+)$ and entirely focus on the Dirichlet series Z .

11.1. Initial factorization

Throughout this section we assume that $\text{Re}(s) = 0$. For simplicity of notation only, we also take $\eta = (\eta_1, \eta_2, \eta_3)$ to be trivial, as the same proof works in the general case.

Definition 11.1. Let \mathcal{D}_0 be the set of $(s, u_1, u_2, u_3) \in \mathbb{C}^4$ with $\text{Re}(s) = 0$, and

$$\text{Re}(u_1) > 1, \quad \text{Re}(u_2) > 1, \quad \text{Re}(u_3) > 1. \tag{11.1}$$

It is easy to see that the multiple sum (10.15) defining Z converges absolutely on \mathcal{D}_0 . We will work initially in \mathcal{D}_0 , and progressively develop analytic properties (meromorphic continuation, bounds, etc.) to larger regions. The largest domain in which we work is the following:

Definition 11.2. Let \mathcal{D}_∞ be the set of $(s, u_1, u_2, u_3) \in \mathbb{C}^4$ with $\operatorname{Re}(s) = 0$, and

$$\operatorname{Re}(u_2) > 1/2, \quad \operatorname{Re}(u_3) > 1/2, \quad \operatorname{Re}(u_1) + \min(\operatorname{Re}(u_2), \operatorname{Re}(u_3)) > 1. \tag{11.2}$$

Obviously, $\mathcal{D}_0 \subset \mathcal{D}_\infty$.

The following notation will be useful throughout this section. Suppose that \mathcal{D} is a subset of $(s, u_1, u_2, u_3) \in \mathbb{C}^4$ defined by $\operatorname{Re}(s) = 0$ and by finitely many equations of the form $L(\operatorname{Re}(u_1), \operatorname{Re}(u_2), \operatorname{Re}(u_3)) > c$, where $c \in \mathbb{R}$ and L is linear with nonnegative coefficients. For $\sigma > 0$, define \mathcal{D}^σ by replacing each such equation by $L(\operatorname{Re}(u_1), \operatorname{Re}(u_2), \operatorname{Re}(u_3)) \geq c + \sigma$. The nonnegativity condition means $\mathcal{D}^\sigma \subseteq \mathcal{D}$ for any $\sigma > 0$.

As a notational convenience, we write k and ℓ instead of k'_0 and ℓ'_0 in equation (10.15) (since there should be no danger of confusion with the original k and ℓ variables). In the domain \mathcal{D}_0 , we may take the sums over k and ℓ to the outside, giving

$$Z(s, u_1, u_2, u_3) = Z^{(2)}(s, u_1, u_2, u_3) \sum_{(k\ell, 2)=1} \frac{Z_{k,\ell}(s, u_1, u_2, u_3)}{k^{u_2-iU-s} \ell^{u_3+iU-s}}, \tag{11.3}$$

where

$$Z_{k,\ell}(s, u_1, u_2, u_3) = \sum_{(r_1 r_2, 2)=1} \sum_{\substack{g_1 | r_1^2 \\ g_2 | r_2^2 \\ \left(\frac{r_1^2}{g_1^2}, k\ell\right)=1}} \sum_{\substack{q: r_1 | q^\infty \\ (q, 2r_2)=1}}^* \frac{\left(\frac{q}{k\ell}\right) \prod_{p|r_2, p \nmid \frac{r_2^2}{g_2^2}} (1-p^{-1})}{q^{u_1+s} (r_1^2 r_2^2)^{u_1+s} (g_1 g_2)^{u_2+u_3-2s-1}} \tag{11.4}$$

and

$$Z^{(2)}(s, u_1, u_2, u_3) = \sum_{\substack{\nu, \gamma \geq 2, \lambda \geq 4 \\ \min(\lambda, \nu) = \min(\lambda, \gamma) \\ \text{one of (7.7) holds}}} \frac{(2^\nu, 2^{\lambda-2-\delta})}{2^{\lambda(u_1+s) + \nu(u_2-iU-s) + \gamma(u_3+iU-s)}}. \tag{11.5}$$

We first focus on properties of $Z_{k,\ell}$, and then turn to $Z^{(2)}$.

11.2. Continuation of $Z_{k,\ell}$

Note that $Z_{k,\ell}$ has an Euler product, say $Z_{k,\ell} = \prod_{p \neq 2} Z_{k,\ell}^{(p)}$. It is convenient to define

$$\alpha = u_2 + u_3 - 2s - 1, \quad \beta = u_1 + s. \tag{11.6}$$

Note that formula (11.1) implies $\operatorname{Re}(\alpha) > 1$ and $\operatorname{Re}(\beta) > 1$. It is also convenient to observe that

$$(s, u_1, u_2, u_3) \in \mathcal{D}_\infty \implies \operatorname{Re}(2\alpha + 2\beta) > 1 \quad \text{and} \quad \operatorname{Re}(\alpha + 2\beta) > 1. \tag{11.7}$$

We evaluate $Z_{k,\ell}^{(p)}$ explicitly as follows.

Lemma 11.3. *Suppose that $\text{Re}(\beta) > 0$ and $\text{Re}(\alpha + \beta) > 0$. For $p \nmid 2k\ell$, we have*

$$Z_{k,\ell}^{(p)}(s, u_1, u_2, u_3) = \frac{1 + p^{-\alpha-2\beta} - p^{-1-2\alpha-2\beta} + \chi(p)p^{-1-2\alpha-3\beta}}{(1 - \chi(p)p^{-\beta})(1 - p^{-2\alpha-2\beta})}, \tag{11.8}$$

where $\chi(n) = \chi_{k\ell}(n) = \left(\frac{n}{k\ell}\right)$. For $p \mid k\ell$, we have

$$Z_{k,\ell}^{(p)}(s, u_1, u_2, u_3) = \frac{1 - p^{-1-2\alpha-2\beta}}{1 - p^{-2\alpha-2\beta}}. \tag{11.9}$$

Proof. For $(p, 2k\ell) = 1$, we have, using the convention $\infty \cdot 0 = 0$,

$$Z^{(p)}(\alpha, \beta) = \sum_{\min(r_1, r_2)=0} \sum_{\substack{0 \leq g_1 \leq 2r_1 \\ 0 \leq g_2 \leq 2r_2}} (1 - p^{-1})^{\delta_{g_2=2r_2}>0} \sum_{\substack{0 \leq q \leq 1 \\ \infty \cdot q \geq r_1 \\ \min(q, r_2)=0}} \frac{\chi(p^q)}{p^{\beta(q+2r_1+2r_2)+\alpha(g_1+g_2)}}. \tag{11.10}$$

We write this as $\sum_{r_2=0} + \sum_{r_2 \geq 1}$, where the latter terms force $q = r_1 = 0$. We have

$$\begin{aligned} \sum_{r_2 \geq 1} &= \sum_{r_2=1}^{\infty} p^{-2\beta r_2} \left(\sum_{0 \leq g_2 \leq 2r_2-1} p^{-\alpha g_2} + (1 - p^{-1}) p^{-2\alpha r_2} \right) \\ &= \sum_{r_2=1}^{\infty} p^{-2\beta r_2} \left(\frac{1 - p^{-2\alpha r_2}}{1 - p^{-\alpha}} + (1 - p^{-1}) p^{-2\alpha r_2} \right). \end{aligned} \tag{11.11}$$

This evaluates as

$$(1 - p^{-\alpha})^{-1} \left(\frac{p^{-2\beta}}{1 - p^{-2\beta}} - \frac{p^{-2\beta-2\alpha}}{1 - p^{-2\alpha-2\beta}} \right) + (1 - p^{-1}) \frac{p^{-2\alpha-2\beta}}{1 - p^{-2\alpha-2\beta}}, \tag{11.12}$$

which simplifies as

$$p^{-2\beta} \frac{1 + p^{-\alpha}}{(1 - p^{-2\beta})(1 - p^{-2\alpha-2\beta})} + (1 - p^{-1}) \frac{p^{-2\alpha-2\beta} (1 - p^{-2\beta})}{(1 - p^{-2\alpha-2\beta})(1 - p^{-2\beta})}. \tag{11.13}$$

In turn, this becomes

$$\frac{p^{-2\beta}}{(1 - p^{-2\alpha-2\beta})(1 - p^{-2\beta})} [1 + p^{-\alpha} + (1 - p^{-1}) p^{-2\alpha} (1 - p^{-2\beta})]. \tag{11.14}$$

Likewise, we compute

$$\sum_{r_2=0} = \sum_{r_1=0}^{\infty} \sum_{0 \leq g_1 \leq 2r_1} \sum_{\substack{0 \leq q \leq 1 \\ \infty \cdot q \geq r_1}} \frac{\chi(p^q)}{p^{\beta(q+2r_1)+\alpha g_1}} = 1 + \sum_{r_1=0}^{\infty} \sum_{0 \leq g_1 \leq 2r_1} \frac{\chi(p)}{p^{\beta(1+2r_1)+\alpha g_1}}, \tag{11.15}$$

by separating out the cases $q = 0$ and $q = 1$. We calculate this as

$$1 + \chi(p)p^{-\beta} \sum_{r_1=0}^{\infty} p^{-2\beta r_1} \frac{1 - p^{-\alpha(2r_1+1)}}{1 - p^{-\alpha}}, \tag{11.16}$$

which can be expressed as

$$1 + \frac{\chi(p)p^{-\beta}}{1-p^{-\alpha}} \left(\frac{1}{1-p^{-2\beta}} - \frac{p^{-\alpha}}{1-p^{-2\alpha-2\beta}} \right) = 1 + \frac{\chi(p)p^{-\beta} (1+p^{-\alpha-2\beta})}{(1-p^{-2\beta})(1-p^{-2\alpha-2\beta})}. \tag{11.17}$$

Putting the two calculations together, we obtain

$$\begin{aligned} & Z^{(p)}(\alpha, \beta) \\ &= \frac{(1-p^{-2\beta})(1-p^{-2\alpha-2\beta}) + \chi(p)p^{-\beta}(1+p^{-\alpha-2\beta}) + p^{-2\beta}(1+p^{-\alpha} + (1-p^{-1})(p^{-2\alpha} - p^{-2\alpha-2\beta}))}{(1-p^{-2\beta})(1-p^{-2\alpha-2\beta})}. \end{aligned}$$

Distributing out the numerator and canceling like terms, we obtain

$$Z^{(p)}(\alpha, \beta) = \frac{(1 + \chi(p)p^{-\beta})(1 + p^{-\alpha-2\beta}) - p^{-1-2\alpha-2\beta}(1 - p^{-2\beta})}{(1 - p^{-2\beta})(1 - p^{-2\alpha-2\beta})}. \tag{11.18}$$

Simplifying gives equation (11.8).

Next we need to consider the primes $p \mid k\ell$. At such a prime we must have $(q, p) = 1$ (or else $(\frac{q}{k\ell}) = 0$), which implies $r_1 = 1$ and $g_2 = r_2^2$. Thus

$$Z^{(p)}(s, u_1, u_2, u_3) = \sum_{r_2 \geq 0} \frac{(1-p^{-1})^{\delta_{r_2 > 0}}}{p^{r_2(2\beta+2\alpha)}} = \frac{1-p^{-1-2\alpha-2\beta}}{1-p^{-2\alpha-2\beta}}. \tag{11.19}$$

□

Define the Dirichlet series

$$D(\alpha, \beta, \chi_{k\ell}) = \sum_{(n, 2)=1} \frac{\mu^2(n)}{n^{\alpha+2\beta}} \sum_{abc=n} \frac{\mu(b)\chi_{k\ell}(c)}{b^{1+\alpha}c^{1+\alpha+\beta}}, \tag{11.20}$$

which is absolutely convergent for $\text{Re}(\alpha + 2\beta) > 1$ and $\text{Re}(\alpha + \beta) > 0$ (observe that these conditions hold on \mathcal{D}_∞ , by formula (11.7)). Note the Euler product formula

$$D(\alpha, \beta, \chi_{k\ell}) = \prod_{p \neq 2} (1 + p^{-\alpha-2\beta}(1 - p^{-1-\alpha} + \chi_{k\ell}(p)p^{-1-\alpha-\beta})). \tag{11.21}$$

Putting together equations (11.8) and (11.9), we deduce (initially) in the region \mathcal{D}_0

$$Z_{k,\ell}(s, u_1, u_2, u_3) = L(\beta, \chi_{k\ell}) \frac{\zeta(2\alpha + 2\beta)}{(1 - 2^{-2\alpha-2\beta})^{-1}} D(\alpha, \beta, \chi_{k\ell}) (1 - \chi_{k\ell}(2)2^{-\beta}) \prod_{p \mid k\ell} a_p, \tag{11.22}$$

where

$$a_p = \frac{1 - p^{-1-2\alpha-2\beta}}{1 + p^{-\alpha-2\beta} - p^{-1-2\alpha-2\beta}}. \tag{11.23}$$

Note that in \mathcal{D}_∞ , we have

$$a_p = 1 + O(p^{-1}). \tag{11.24}$$

Lemma 11.4. *The series $Z_{k,\ell}(s, u_1, u_2, u_3)$ has meromorphic continuation to the domain \mathcal{D}_∞ . In this region, $Z_{k,\ell}$ has a polar line only at $\beta = 1$ which occurs if and only if $\chi_{k\ell}$ is trivial.*

Proof. This follows from equation (11.22), using formula (11.7). □

Remark 11.5. Observe the nice simplification in the passage from equation (11.18) to equation (11.8), in which a factor of $(1 - p^{-2\beta})$ is canceled from the numerator and denominator. This reveals that there is no $\zeta(2\beta)^{-1}$ -type factor in equation (11.22), which would have infinitely many poles in the domain \mathcal{D}_∞ .

11.3. Evaluation of $Z^{(2)}$

Recall that $Z^{(2)}$ has four cases, corresponding to formula (7.7).

Lemma 11.6. *In cases (i)–(iii) of formula (7.7), the function $Z^{(2)}$ initially defined by equation (11.5) in the region (11.1) extends to a bounded analytic function on \mathcal{D}_∞ .*

Proof. This follows from brute-force computation with geometric series. For case (i), we have

$$Z^{(2)} = \frac{(1 - 2^{-(u_2 - iU - s)})^{-1} (1 - 2^{-(u_3 + iU - s)})^{-1}}{2^{2+\delta} 2^{4(\alpha+\beta)} (1 - 2^{-\alpha-\beta})}, \tag{11.25}$$

which satisfies the claimed properties by inspection. Cases (ii) and (iii) are easier, and give $Z^{(2)} = 2^{-1-\delta-3\alpha-4\beta} (1 - 2^{-\alpha-\beta})^{-1}$ and $Z^{(2)} = 2^{-\delta-2\alpha-4\beta} (1 - 2^{-\alpha-\beta})^{-1}$, respectively. In case (ii), to see the boundedness on \mathcal{D}_∞ , note $2^{-3\alpha-4\beta} = 2^{-2\alpha-2\beta} 2^{-\alpha-2\beta}$, and recall formula (11.7). □

When $Z^{(2)}$ is given by case (iv) – which, recall, restricts the summation to $\lambda \geq \nu + 3$ – it is convenient to split the sum into two pieces according to the size of $\lambda - \nu$. For any integer $L \geq 3$, write $Z^{(2)} = Z_{\leq L}^{(2)} + Z_{> L}^{(2)}$, where $Z_{\leq L}^{(2)}$ restricts to $\lambda - \nu \leq L$ and $Z_{> L}^{(2)}$ restricts to $\lambda - \nu > L$.

Lemma 11.7. *In case (iv), $Z_{\leq L}^{(2)}$ extends to an analytic function on \mathcal{D}_∞ , wherein it satisfies the bound*

$$\left| Z_{\leq L}^{(2)} \right| \ll L (2^{-L\beta} + 1). \tag{11.26}$$

The tail $Z_{> L}^{(2)}$ is analytic on \mathcal{D}_0 , wherein it satisfies the bound

$$\left| Z_{> L}^{(2)} \right| \ll 2^{-L\beta}. \tag{11.27}$$

Proof. Since $\lambda \geq \nu + 3$, then $\min(\lambda, \nu) = \nu$, and the condition $\min(\lambda, \nu) = \min(\lambda, \gamma)$ means $\gamma = \nu$. Therefore,

$$Z_{\leq L}^{(2)} = \sum_{\nu \geq 2} \sum_{\nu+3 \leq \lambda \leq \nu+L} \frac{2^\nu}{2^{\lambda\beta + \nu(\alpha+1)}} = \sum_{\nu \geq 2} \sum_{3 \leq \mu \leq L} \frac{1}{2^{(\nu+\mu)\beta + \nu\alpha}} = \frac{2^{-2\alpha-2\beta}}{(1 - 2^{-\alpha-\beta})} \sum_{3 \leq \mu \leq L} 2^{-\mu\beta}. \tag{11.28}$$

From this representation we easily read off its analytic continuation and the bound (11.26). For the tail, we may modify the previous calculation to give

$$Z_{>L}^{(2)} = \frac{2^{-2\alpha-2\beta}}{(1-2^{-\alpha-\beta})} \sum_{\mu \geq L+1} 2^{-\mu\beta} = \frac{2^{-2\alpha-2\beta}}{(1-2^{-\alpha-\beta})} \frac{2^{-\beta(L+1)}}{(1-2^{-\beta})}, \tag{11.29}$$

from which we immediately read off the desired properties. □

Remark. Note that $Z_{>L}^{(2)}$ does not analytically continue to \mathcal{D}_∞ , since equation (11.29) has poles on the line $\text{Re}(\beta) = 0$. This explains the reason for splitting $Z^{(2)}$ into these two pieces.

To unify the notation, in cases (i)–(iii) we define $Z_{>L}^{(2)} = 0$ and $Z_{\leq L}^{(2)} = Z^{(2)}$. Corresponding to this decomposition of $Z^{(2)}$, we likewise write

$$Z = Z_{\leq L} + Z_{>L}. \tag{11.30}$$

With this definition, the statement of Lemma 11.7 holds in cases (i)–(iii) as well. In this way we may henceforth unify the exposition for all four cases.

11.4. Continuation of $Z_{\leq L}$

It is now useful to define another domain.

Definition 11.8. Let \mathcal{D}_1 be the set of $(s, u_1, u_2, u_3) \in \mathbb{C}^4$ with $\text{Re}(s) = 0$, $\text{Re}(u_2) > 1$, and $\text{Re}(u_3) > 1$, and satisfying

$$\begin{cases} \text{Re}(u_1) + \min(\text{Re}(u_2), \text{Re}(u_3)) > 3/2 \\ \text{Re}(u_1) + 2\min(\text{Re}(u_2), \text{Re}(u_3)) > 3. \end{cases} \tag{11.31}$$

Note that $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \mathcal{D}_\infty$.

Lemma 11.9. *The series (11.3) converges absolutely on $\mathcal{D}_1 \cap \{\beta \neq 1\}$ (and uniformly on compact subsets), which furnishes meromorphic continuation of the function $Z_{\leq L}$ to this domain. Moreover, the residue at $\beta = 1$ of $Z_{\leq L}$ is bounded for $\text{Re}(u_2), \text{Re}(u_3) > 1$.*

Proof. We return to equation (11.3) and use the representation (11.22), valid in \mathcal{D}_0 . The results from §11.3 give the analytic continuation of $Z_{\leq L}^{(2)}$ to \mathcal{D}_∞ (and hence, \mathcal{D}_1). Since $L(\beta, \chi_{k\ell})$ has a pole at $\beta = 1$ when $\chi_{k\ell}$ is trivial, we suppose $|\beta - 1| \geq \sigma > 0$, and will claim bounds with an implied constant that may depend on σ . For $0 \leq \text{Re}(\beta) = \text{Re}(u_1) \leq 1$, we have the convexity bound $|L(\beta, \chi_{k\ell})| \ll_{\text{Im}(\beta), \sigma, \varepsilon} (k\ell)^{\frac{1-\text{Re}(\beta)}{2} + \varepsilon}$ (with an implied constant depending at most polynomially on β). One easily checks that equation (11.3) converges absolutely for $\min(\text{Re}(u_2), \text{Re}(u_3)) + \frac{\text{Re}(\beta)}{2} > \frac{3}{2}$, which is one of the inequalities stated in formula (11.31). Similarly, for $\text{Re}(\beta) \leq 0$ we use the convexity bound $|L(\beta, \chi_{k\ell})| \ll (k\ell)^{\frac{1}{2} - \text{Re}(\beta) + \varepsilon}$ to see the absolute convergence for $\text{Re}(u_1) + \min(\text{Re}(u_2), \text{Re}(u_3)) > 3/2$. The uniform convergence on compact subsets is immediate, and so the meromorphic continuation follows.

Finally, to see the size of the residue, we simply note from equation (11.22) that $\text{Res}_{\beta=1} Z_{k,\ell} \ll (k\ell)^\varepsilon$ for $\text{Re}(u_2), \text{Re}(u_3) \geq 1$. In addition, the pole exists only if $k\ell$ is a

square. Moreover, $Z_{\leq L}^{(2)}$ is bounded at this point. From equation (11.3) we may then easily see the absolute convergence of the sum of these residues over k, ℓ . \square

11.5. Functional equation

Next we investigate how $Z_{k, \ell}$ and $Z_{\leq L}$ behave after an application of the functional equation of $L(\beta, \chi_{k\ell})$. Suppose that $\chi_{k\ell}$ is induced by the primitive character χ^* of conductor $(k\ell)^*$. We have

$$\Lambda(s, \chi^*) = ((k\ell)^*)^{s/2} \gamma(s) L(s, \chi^*) = \Lambda(1-s, \chi^*), \tag{11.32}$$

where $\gamma(s) = \pi^{-s/2} \Gamma\left(\frac{s+\kappa}{2}\right)$, with $\kappa \in \{0, 1\}$ reflecting the parity of χ . We therefore deduce the asymmetric form of the functional equation:

$$L(s, \chi_{k\ell}) = ((k\ell)^*)^{\frac{1}{2}-s} \frac{\gamma(1-s)}{\gamma(s)} L(1-s, \chi_{k\ell}) \prod_{p|k\ell} \frac{(1-\chi^*(p)p^{-s})}{(1-\chi^*(p)p^{s-1})}. \tag{11.33}$$

Lemma 11.10. *In $\mathcal{D}_\infty \cap \{\text{Re}(\beta) < 0\}$, we have*

$$Z_{k, \ell} = ((k\ell)^*)^{\frac{1}{2}-\beta} \frac{\gamma(1-\beta)}{\gamma(\beta)} D(\alpha, \beta, \chi_{k\ell}) \frac{\zeta(2\alpha+2\beta)}{(1-2^{-2\alpha-2\beta})^{-1}} (1-2^{-\beta} \chi_{k\ell}(2)) \sum_{q=1}^\infty \frac{\left(\frac{q}{k\ell}\right)}{q^{1-\beta}} \prod_{p|k\ell} \frac{(1-\chi^*(p)p^{-\beta})}{(1-\chi^*(p)p^{\beta-1})} \prod_{p|k\ell} a_p. \tag{11.34}$$

Proof. Lemma 11.4 implies that the expression (11.22) for $Z_{k, \ell}$ is analytic on $\mathcal{D}_\infty \cap \{\beta \neq 1\}$. With the assumption $\text{Re}(\beta) < 0$, we may apply the functional equation and express $L(1-\beta, \chi_{k\ell})$ in terms of its absolutely convergent Dirichlet series, which is equation (11.34). \square

Having applied the functional equation to $Z_{k, \ell}$, the plan of action is to now insert this expression into the definition of $Z_{\leq L}$ and reverse the orders of summation, bringing k and ℓ to the inside. The outcome of this step is recorded with the following:

Lemma 11.11. *On $\mathcal{D}_1 \cap \{\text{Re}(\beta) < 0\}$, $Z_{\leq L}$ is a finite linear combination of absolutely convergent expressions of the form*

$$Z_{\leq L}^{(2)} \frac{\gamma(1-\beta)}{\gamma(\beta)} \frac{\zeta(2\alpha+2\beta)}{(1-2^{-2\alpha-2\beta})^{-1}} \frac{(1 \pm 2^{-\beta})}{(1 \pm 2^{\beta-1})} \sum_{(q, 2)=1} q^{\beta-1} \nu_1(q) A_q, \tag{11.35}$$

with $A_q = A_q(s, u_1, u_2, u_3, U, \nu_2, \dots, \nu_6)$ defined by

$$A_q = \sum_{(abc, 2)=1} \frac{\mu^2(abc) \nu_2(c)}{(abc)^{\alpha+2\beta}} \frac{\mu(b)}{b^{1+\alpha} c^{1+\alpha+\beta}} \sum_{(k\ell, 2)=1} \frac{\left(\frac{k\ell}{cq}\right) \nu_3(k) \nu_4(\ell) ((k\ell)^*)^{\frac{1}{2}-\beta}}{k^{u_2-iU-s} \ell^{u_3+iU-s}} \prod_{p|k\ell} \frac{(1-\chi_p((k\ell)^*) \nu_5(p) p^{-\beta})}{(1-\chi_p((k\ell)^*) \nu_6(p) p^{\beta-1})} \prod_{p|k\ell} a_p, \tag{11.36}$$

and where the ν_i run over Dirichlet characters modulo 8.

Observe that equation (11.36) converges absolutely on \mathcal{D}_1 .

Proof. Applying Lemma 11.10 into equation (11.3), which is valid on $\mathcal{D}_1 \cap \{\text{Re}(\beta) < 0\}$ by Lemma 11.9, and applying the Dirichlet series expansion of $D(\alpha, \beta, \chi_{k\ell})$ given in equation (11.20), we deduce

$$\begin{aligned}
 Z_{\leq L}(s, u_1, u_2, u_3) &= Z_{\leq L}^{(2)} \frac{\zeta(2\alpha + 2\beta)}{(1 - 2^{-2\alpha - 2\beta})^{-1}} \sum_{(k\ell, 2)=1} \frac{((k\ell)^*)^{\frac{1}{2} - \beta}}{k^{u_2 - iU - s} \ell^{u_3 + iU - s}} \frac{\gamma(1 - \beta)}{\gamma(\beta)} \\
 (1 - \chi_{k\ell}(2)2^{-\beta}) \sum_{(abc, 2)=1} \frac{\mu^2(abc)}{(abc)^{\alpha + 2\beta}} \frac{\mu(b)}{b^{1 + \alpha} c^{1 + \alpha + \beta}} \sum_{q=1}^{\infty} \frac{\left(\frac{qc}{k\ell}\right)}{q^{1 - \beta}} \prod_{p|k\ell} \frac{(1 - \chi^*(p)p^{-\beta})}{(1 - \chi^*(p)p^{\beta - 1})} \prod_{p|k\ell} a_p,
 \end{aligned} \tag{11.37}$$

where recall $a_p = 1 + O(p^{-1})$ on \mathcal{D}_{∞} , and $\chi^* = \chi_{k\ell}^*$ is the primitive character induced by $\chi_{k\ell}(n) = \left(\frac{n}{k\ell}\right)$ (so $\chi^*(n) = \left(\frac{n}{(k\ell)^*}\right)$).

We next wish to focus on the sums over k and ℓ . One small issue is that the parity of the character $\chi_{k\ell}$ (and hence the formula for $\gamma(s)$) may vary. However, the parity depends only on k and ℓ modulo 8. Also, q may be even, but we can factor out the 2-part of q and directly evaluate its summation. Likewise, we can apply quadratic reciprocity (again!) to give that $\left(\frac{qc}{k\ell}\right)$ equals $\left(\frac{k\ell}{qc}\right)$ times a function that depends only on q, c, k, ℓ modulo 4. Similarly, we have that $\chi_{k\ell}^*(p)$ equals $\chi_p((k\ell)^*)$ up to a function modulo 4. We can then use multiplicative Fourier/Mellin decomposition modulo 8 to express $Z_{\leq L}$ as a finite linear combination, with bounded coefficients, of sums of the form claimed in the statement of the lemma. \square

Next we develop some of the analytic properties of A_q . For notational convenience, we consider the case with all $\nu_i = 1$, as the general case is no more difficult. We expand the Euler product over $p | k\ell$ involving χ^* into its Dirichlet series and reverse the orders of summation (taking k, ℓ to the inside), giving

$$A_q = \sum_{(abcde, 2)=1} \frac{\mu^2(abc)\mu(b)\mu(d)}{(abc)^{\alpha + 2\beta} b^{1 + \alpha} c^{1 + \alpha + \beta} d^{\beta} e^{1 - \beta}} A_{q, c, d, e}, \tag{11.38}$$

where

$$A_{q, c, d, e} = \sum_{\substack{k\ell \equiv 0 \pmod{d} \\ (k\ell)^{\infty} \equiv 0 \pmod{e} \\ (k\ell, 2)=1}} \frac{\left(\frac{k\ell}{cq}\right) \left(\frac{(k\ell)^*}{de}\right) ((k\ell)^*)^{\frac{1}{2} - \beta}}{k^{u_2 - iU - s} \ell^{u_3 + iU - s}} \prod_{p|k\ell} a_p. \tag{11.39}$$

Lemma 11.12. *The function $A_{q, c, d, e}$ has meromorphic continuation to \mathcal{D}_{∞} , in the form*

$$A_{q, c, d, e} = L\left(u_1 + u_2 - iU - \frac{1}{2}, \chi_{qcde}\right) L\left(u_1 + u_3 - iU - \frac{1}{2}, \chi_{qcde}\right) C(\cdot), \tag{11.40}$$

where $C = C_q(c, d, e, s, u_1, u_2, u_3, U)$ is a Dirichlet series analytic on \mathcal{D}_{∞} and satisfying the bound $C \ll ((de)')^{-2\min \text{Re}(u_2, u_3) + \varepsilon}$ on \mathcal{D}_{∞} .

Proof. We initially work on \mathcal{D}_1 , where the sum defining $A_{q,c,d,e}$ converges absolutely. Now $A_{q,c,d,e}$ has an Euler product, taking the form $A_{q,c,d,e} = \prod_{(p,2)=1} A_{q,c,d,e}^{(p)}$, say, where

$$A_{q,c,d,e}^{(p)} = \sum_{\substack{k+\ell \geq v_p(d) \\ \infty \cdot (k+\ell) \geq v_p(e)}} \frac{\left(\frac{p^{k+\ell}}{cq}\right) \left(\frac{(p^{k+\ell})^*}{de}\right) \left((p^{k+\ell})^*\right)^{\frac{1}{2}-\beta}}{p^{k(u_2-iU-s)+\ell(u_3+iU-s)}} a_{p^{k+\ell}}, \tag{11.41}$$

where v_p is the p -adic valuation and where we set $a_{p^0} = 1$ and $a_{p^j} = a_p$ for $j \geq 1$.

For the forthcoming estimates, we recall our convention from §1.4 that an expression of the form $O(p^{-s})$ should be interpreted to mean $O(p^{-\text{Re}(s)})$. If $p \nmid de$, then by separating the cases with $k+\ell$ odd and $k+\ell$ even we obtain

$$\begin{aligned} A_{q,c,d,e}^{(p)} &= 1 + \left(\frac{p}{qcde}\right) \left[\frac{1}{p^{u_1+u_2-iU-\frac{1}{2}}} + \frac{1}{p^{u_1+u_3+iU-\frac{1}{2}}}\right] a_p + O\left(p^{-\min(2u_2, 2u_3)}\right) \\ &= 1 + \left(\frac{p}{qcde}\right) \left[\frac{1}{p^{u_1+u_2-iU-\frac{1}{2}}} + \frac{1}{p^{u_1+u_3+iU-\frac{1}{2}}}\right] \\ &\quad + O\left(p^{-\min(2u_2, 2u_3)}\right) + O\left(\frac{p^{-1}}{p^{u_1+\min(u_2, u_3)-\frac{1}{2}}}\right) \\ &= \frac{1 + O\left(p^{-\min(2u_2, 2u_3)}\right) + O\left(\frac{p^{-1}}{p^{u_1+\min(u_2, u_3)-\frac{1}{2}}}\right) + O\left(p^{-2(u_1+\min(u_2, u_3)-\frac{1}{2})}\right)}{\left(1 - \chi_{qcde}(p)p^{-u_1-u_2+iU+\frac{1}{2}}\right) \left(1 - \chi_{qcde}(p)p^{-u_1-u_3-iU+\frac{1}{2}}\right)}. \end{aligned}$$

Note that on $\mathcal{D}_\infty^\sigma$, the O -term is of size $O(p^{-1-\sigma})$, and hence

$$\prod_{p \nmid de} A_{q,c,d,e}^{(p)} = L\left(u_1 + u_2 - iU - \frac{1}{2}, \chi_{qcde}\right) L\left(u_1 + u_3 - iU - \frac{1}{2}, \chi_{qcde}\right) B, \tag{11.42}$$

where $B = B(q,c,d,e,s,u_1,u_2,u_3,U)$ is an Euler product that is absolutely convergent and bounded on $\mathcal{D}_\infty^\sigma$.

If $p \mid de$, then $\left(\frac{(p^{k+\ell})^*}{de}\right) = 0$ unless $(p^{k+\ell})^* = 1$, so we can assume that $k+\ell$ is even (and positive, hence ≥ 2). From such primes we obtain $A_{q,c,d,e}^{(p)} = O(p^{-\min(2u_2, 2u_3)})$, and hence

$$\prod_{p \mid de} A_{q,c,d,e}^{(p)} \ll ((de)')^{-2 \min \text{Re}(u_2, u_3) + \varepsilon}, \tag{11.43}$$

where $(de)' = \prod_{p \mid de} p$. Putting the estimates together, we deduce (initially in \mathcal{D}_1) the representation (11.40), where C is analytic on \mathcal{D}_∞ . Thus $A_{q,c,d,e}$ inherits the meromorphic continuation to \mathcal{D}_∞ as well. \square

Definition 11.13. Let \mathcal{D}_2 be the set of $(s, u_1, u_2, u_3) \in \mathbb{C}^4$ with $\operatorname{Re}(s) = 0, \operatorname{Re}(u_2) > 1/2, \operatorname{Re}(u_3) > 1/2$, and satisfying

$$\operatorname{Re}(u_1) + \min(\operatorname{Re}(u_2), \operatorname{Re}(u_3)) > 3/2. \tag{11.44}$$

One easily checks that $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}_\infty$.

Lemma 11.14. *The function A_q has meromorphic continuation to $\mathcal{D}_2 \cap \{\operatorname{Re}(u_1) < 1/2\}$.*

Proof. We (initially) work in the domain \mathcal{D}_1 , where the absolute convergence is ensured. Substituting equation (11.40) into equation (11.36) and letting $cde = r$, we obtain

$$A_q = \sum_{(r,2)=1} L(u_1 + u_2 - iU - \frac{1}{2}, \chi_{qr}) L(u_1 + u_3 + iU - \frac{1}{2}, \chi_{qr}) D(q, r), \tag{11.45}$$

where

$$D(q, r) = \sum_{\substack{(ab,2)=1 \\ cde=r}} \frac{\mu^2(abc)\mu(b)\mu(d)}{(abc)^{\alpha+2\beta} b^{1+\alpha} c^{1+\alpha+\beta} d^\beta e^{1-\beta}} C_q(\cdot). \tag{11.46}$$

We claim that $D(q, r)$ is analytic on $\mathcal{D}_\infty \cap \{\operatorname{Re}(u_1) < 1/2\}$ and therein satisfies the bound

$$D(q, r) \ll q^\varepsilon r^{\beta-1+\varepsilon} \prod_{p|r} p^{-2u_1-2\min(u_2, u_3)+1}. \tag{11.47}$$

We now prove this claim. Recall formula (11.7), which in particular immediately shows the absolute convergence in \mathcal{D}_∞ of the free sum over a, b in equation (11.46). Hence

$$|D(q, r)| \ll r^\varepsilon \sum_{cde=r} \frac{\mu^2(c)\mu^2(d)((de)')^{-2\min(u_2, u_3)}}{c^{1+2\alpha+3\beta} d^\beta e^{1-\beta}} = \sum_{cd|r} \frac{\mu^2(c)\mu^2(d)((r/c)')^{-2\min(u_2, u_3)}}{c^{1+2\alpha+3\beta} d^\beta (\frac{r}{cd})^{1-\beta}}. \tag{11.48}$$

One may now check formula (11.47) by brute force, prime by prime (by multiplicativity).

A consequence of formula (11.47) is that on $\mathcal{D}_\infty \cap \{\operatorname{Re}(u_1) < 1/2\}$ we have the bound $D(q, p^k) \ll p^{k\varepsilon} p^{-1-\frac{k}{2}}$, for p prime and $k \geq 1$, which extends multiplicatively. Therefore, $\sum_r |D(q, r)| < \infty$ on $\mathcal{D}_\infty \cap \{\operatorname{Re}(u_1) < 1/2\}$. The Dirichlet L -functions appearing in equation (11.45) are at most $O((qr)^\varepsilon)$ on \mathcal{D}_2 . Therefore, equation (11.45) gives the meromorphic continuation of A_q as stated in the lemma. \square

Lemma 11.15. *On $\mathcal{D}_2 \cap \{\operatorname{Re}(u_1) < 0\}$, the function $Z_{\leq L}$ extends to a meromorphic function, on which it is a finite linear combination of absolutely convergent sums of the form*

$$Z_{\leq L}^{(2)} \frac{\gamma(1-\beta)}{\gamma(\beta)} \sum_{(r,2)=1}^* \sum_{(q,2)=1}^* \frac{c_{q,r}}{q^{1-\beta}} L(u_1 + u_2 - iU - \frac{1}{2}, \chi_{qr}\nu) L(u_1 + u_3 - iU - \frac{1}{2}, \chi_{qr}\nu'), \tag{11.49}$$

where ν, ν' are Dirichlet characters modulo 8 and \sum^* means that the sum runs only over square-free integers. Here $c_{q,r}$ is a Dirichlet series depending on s, u_1, u_2, u_3, U that

is analytic on $\mathcal{D}_\infty \cap \{\text{Re}(u_1) < 1/2\}$, wherein it satisfies the bound

$$c_{q,r} \ll r^{-u_1 - 2\min(u_2, u_3)} (qr)^\varepsilon. \tag{11.50}$$

Proof. We work initially on the domain $\mathcal{D}_1 \cap \{\text{Re}(u_1) < 0\}$, so that Lemma 11.11 may be applied, giving expression (11.35). Now Lemma 11.14 may be invoked to give that $Z_{\leq L}$ is a linear combination of terms of the form

$$Z_{\leq L}^{(2)} \frac{\gamma(1-\beta)}{\gamma(\beta)} \frac{\zeta(2\alpha+2\beta)}{(1-2^{-2\alpha-2\beta})^{-1}} \frac{(1 \pm 2^{-\beta})}{(1 \pm 2^{\beta-1})} \sum_{(q,2)=1} q^{\beta-1} \sum_{(r,2)=1} L(u_1+u_2-iU-\frac{1}{2}, \chi_{qr}\nu) L(u_1+u_3+iU-\frac{1}{2}, \chi_{qr}\nu') D(q,r), \tag{11.51}$$

which converges absolutely on $\mathcal{D}_2 \cap \{\text{Re}(\beta) < 0\}$. This gives the claimed meromorphic continuation of $Z_{\leq L}$.

Next we show the claimed form (11.49), which closely resembles expression (11.51) except that we need to restrict q and r to be square-free. Toward this end, replace q by qq_2^2 and r by rr_2^2 , where the new q and r are square-free. Note that $L(s, \chi_{qrq_2^2r_2^2})$ agrees with $L(s, \chi_{qr})$ up to finitely many Euler factors that are bounded by $O((qr)^\varepsilon)$ for $\text{Re}(s) > 1/2$. These finite Euler products can be incorporated into the definition of $D(q,r)$, which still satisfies formula (11.47) on $\mathcal{D}_\infty \cap \{\text{Re}(u_1) < 1/2\}$. Then we need to check the convergence in the sums over q_2 and r_2 . To this end, we first note simply that $\sum_{(q_2,2)=1} q_2^{2(\beta-1)} = \zeta(2-2\beta)(1-2^{-2+2\beta})$, which is analytic and bounded for $\text{Re}(u_1) \leq 1/2 - \sigma$. For r_2 , we have from formula (11.47) that

$$\sum_{r_2} |D(q, rr_2^2)| \ll \sum_{r_2} q^\varepsilon (rr_2^2)^{\beta-1+\varepsilon} \prod_{p|r} p^{-2u_1-2\min(u_2, u_3)+1} \ll (qr)^\varepsilon r^{-u_1-2\min(u_2, u_3)}. \tag{11.52}$$

Finally, this gives the meromorphic continuation of $Z_{\leq L}$ to $\mathcal{D}_2 \cap \{\text{Re}(u_1) < 0\}$ with the coefficients $c_{q,r}$ analytic on $\mathcal{D}_\infty \cap \{\text{Re}(u_1) < 1/2\}$ and satisfying formula (11.50). \square

12. Completion of the proof of Theorem 1.1

Recall that the off-diagonal of $\sum_{T < t_j < T+\Delta} |L(\text{sym}^2 u_j, 1/2 + iU)|^2$ is a sum which we have been studying in dyadic intervals $n \asymp m \asymp N$ and $c \asymp C$. Recall that $N \ll U^{1/2} T^{1+\varepsilon}$, $C \ll \frac{N^2 T^\varepsilon}{\Delta T}$, and $C \gg \frac{N^2 T^\varepsilon}{T^2}$, originating from formulas (6.1), (6.15), and (9.6). We also defined certain parameters Φ, P, K which can be found in equation (10.3), but for convenience we recall here $\Phi = \frac{N\sqrt{C}}{U}$, $P = \frac{CT^2}{N^2}$, $K = \frac{CU}{N}$. Aided by the properties of Z developed in the previous section, we are now ready to finish the proof of Theorem 1.1. We pick up from expression (10.14), where we begin with $\text{Re}(u_1) = \text{Re}(u_2) = \text{Re}(u_3) = 2$. Next we write $Z = Z_{\leq L} + Z_{>L}$, and choose L so that $2^L \asymp CT^\varepsilon$. To bound the contribution from $Z_{>L}$, we shift u_1 far to the right, and use the bound (11.27). In terms of u_1 , we get a bound of size $O\left((C/2^L)^{\text{Re}(u_1)}\right) \ll T^{-\varepsilon \text{Re}(u_1)}$ which is negligible. Next we focus on $Z_{\leq L}$.

We begin by shifting u_1 to the line $-\varepsilon$, which is allowed by Lemma 11.9. There is a pole of $Z_{\leq L}$ at $\beta = u_1 + s = 1$, with bounded residue. However, since $\text{Im}(s) \asymp P$ and $P \gg T^\varepsilon$,

the weight function is very small at this height and the contribution from such poles are negligible. Thus we obtain

$$\begin{aligned}
 \mathcal{S}(H_+) &= \sum_C \frac{\Delta T}{NC^{3/2}} \frac{\Phi}{\sqrt{P}} \int_{-t \asymp P} \int \int \int \left(\frac{T^2}{U^2} - \frac{1}{4}\right)^s v(t) \tilde{w}(u_1, u_2, u_3) C^{u_1} K^{u_2+u_3} \frac{\gamma(1-u_1-s)}{\gamma(u_1+s)} \\
 &Z_{\leq L}^{(2)} \sum_r^* \sum_q^* q^{u_1+s-1} c_{q,r} L\left(u_1+u_2-iU-\frac{1}{2}, \chi_{qr}\right) L\left(u_1+u_3-iU-\frac{1}{2}, \chi_{qr}\right) du_1 du_2 du_3 ds,
 \end{aligned}
 \tag{12.1}$$

plus a small error term, as well as additional terms with the characters twisted modulo 8. Since all our estimates hold verbatim for these additional twists, we suppress this from the notation. Next we want to truncate the sums over q and r . To do so, we move u_1 far to the left, keeping $\text{Re}(u_2) = \text{Re}(u_3) = -\text{Re}(u_1) + 100$. Note that this remains in the domain \mathcal{D}'_2 and that $\text{Re}(u_1) < 0$, so that the conditions of Lemma 11.15 remain in place to apply expression (11.49). Also, note that the coefficients $c_{q,r}$ are $O(r^{-100})$ here. Moreover, we observe by Stirling that

$$\left| \frac{\gamma(1-u_1-s)}{\gamma(u_1+s)} \right| \ll P^{\frac{1}{2}-\text{Re}(u_1)}.
 \tag{12.2}$$

In terms of the u_1 -variable, the integrand in equation (12.1) is bounded by some fixed polynomial in T times

$$\left(\frac{Cq}{PK^2}\right)^{\text{Re}(u_1)}.
 \tag{12.3}$$

Therefore, we may truncate q at $q \leq Q$, where

$$Q = \frac{PK^2}{C} T^\varepsilon.
 \tag{12.4}$$

After enforcing this condition and reversing the orders of summation (taking r, q to the outside of the integrals), we shift the contours of integration so that $\text{Re}(u_1) = 1/2 - \varepsilon$ and $\text{Re}(u_2) = \text{Re}(u_3) = 1/2 + \varepsilon$; this is allowed by Lemma 11.15, as these contour shifts may be done in such a way that we remain in the domain $\mathcal{D}_\infty \cap \{\text{Re}(u_1) < 1/2\}$, on which $c_{q,r}$ is analytic. Moreover, we observe from formula (11.26) that $Z_{\leq L}^{(2)} \ll L \ll T^\varepsilon$ on this contour. We then bound everything with absolute values, obtaining

$$\begin{aligned}
 \mathcal{S}(H_+) &\ll T^\varepsilon \max_C \frac{\Delta T}{NC^{3/2}} \frac{\Phi}{\sqrt{P}} \int \int \int \max_{\substack{x>0 \\ q,r \ll Q}} \left| \int_{-t \asymp P} x^{it} \frac{\gamma(1-u_1-it)}{\gamma(u_1+it)} v(t) c_{q,r} dt \right| \\
 &|\tilde{w}(u_1, u_2, u_3)| C^{1/2} K \sum_{q \leq Q}^* q^{-1/2} \left| L\left(u_1+u_2-iU-\frac{1}{2}, \chi_q\right) \right|^2 du_1 du_2 du_3.
 \end{aligned}
 \tag{12.5}$$

By Lemma 4.5, keeping in mind that $c_{q,r}$ is given by a Dirichlet series uniformly bounded in t by formula (11.50), we have

$$\max_{x>0} \left| P^{-1/2} \int x^{it} \frac{\gamma(1-u_1-it)}{\gamma(u_1+it)} v(t) c_{q,r} dt \right| \ll 1. \tag{12.6}$$

Applying formula (3.3), we then obtain

$$S(H_+) \ll T^\varepsilon \max_C \frac{\Delta T}{NC} \Phi K \left(Q^{1/2} + U^{1/2} \right), \quad Q = \frac{PK^2}{C} T^\varepsilon. \tag{12.7}$$

Therefore, we obtain

$$S(H_+) \ll T^\varepsilon \max_C \frac{\Delta T}{NC} \Phi K \left(\frac{P^{1/2}K}{C^{1/2}} + U^{1/2} \right) \ll T^\varepsilon \max_C \frac{\Delta T}{NC} \frac{N\sqrt{C}}{U} \frac{CU}{N} \left(\frac{TCU}{N^2} + U^{1/2} \right). \tag{12.8}$$

Using $C \ll \frac{N^2}{\Delta T} T^\varepsilon$, this simplifies as

$$S(H_+) \ll T^\varepsilon \max_C \frac{\Delta T}{N} \sqrt{C} \left(\frac{TCU}{N^2} + U^{1/2} \right) \ll T^\varepsilon \left(\frac{T^{1/2}U}{\Delta^{1/2}} + (\Delta TU)^{1/2} \right). \tag{12.9}$$

By formula (6.11) and the remark following it, this implies

$$\sum_{T < t_j < T + \Delta} |L(\text{sym}^2 u_j, 1/2 + iU)|^2 \ll T^\varepsilon \left(\Delta T + \frac{T^{1/2}U}{\Delta^{1/2}} \right). \tag{12.10}$$

We have $\Delta T \gg \frac{T^{1/2}U}{\Delta^{1/2}}$ if and only if $\Delta \gg \frac{U^{2/3}}{T^{1/3}}$. This inequality holds because one of the conditions of Theorem 1.1 requires $\Delta \gg \frac{T}{U^{2/3}}$, and $\frac{T}{U^{2/3}} \gg \frac{U^{2/3}}{T^{1/3}}$ because $T \gg U$.

13. Proving Theorem 1.3

For the proof of Theorem 1.3, the parameters Φ, P, K are given in equation (10.13), which for convenience we recall take the form $\Phi = \frac{N^3}{\sqrt{C}T^2}$, $P = \frac{CT^2}{N^2}$, $K = \frac{C^2T^2}{N^3}$. The bounds on N and C are the same as recollected in §12. The overall idea is to follow the same steps as in §12, but picking up with equation (10.16) instead of equation (10.14). The only structural difference between the two formulas is the additional phase of the form

$$e^{-2it \log(\frac{|t|}{eT}) + ia \frac{t^3}{T^2}}. \tag{13.1}$$

Here the cubic term is of size $O(PT^{-\delta})$, as mentioned after equation (10.9). This affects only the argument in bounding formula (12.6), but Lemma 4.5 is applicable (using the previous remark that the cubic term is of lower order) and gives the same bound with the additional phase from before. Referring to formula (12.7), we thus obtain

$$S(H_+) \ll T^\varepsilon \max_C \frac{\Delta T}{NC} \Phi K \left(\frac{P^{1/2}K}{C^{1/2}} + U^{1/2} \right) \ll T^\varepsilon \max_C \frac{\Delta T}{NC} \frac{N^3}{C^{1/2}T^2} \frac{C^2T^2}{N^3} \left(\frac{T^3C^2}{N^4} + 1 \right). \tag{13.2}$$

Using $C \ll \frac{N^2}{\Delta T} T^\varepsilon$, this simplifies as

$$\mathcal{S}(H_+) \ll T^\varepsilon \max_C \frac{\Delta T}{N} C^{1/2} \left(\frac{T^3 C^2}{N^4} + 1 \right) \ll T^\varepsilon \left(\frac{T^{3/2}}{\Delta^{3/2}} + (\Delta T)^{1/2} \right). \quad (13.3)$$

Thus in all, by formula (6.11) and the remark following it, we obtain

$$\sum_{T < t_j < T + \Delta} |L(\text{sym}^2 u_j, 1/2)|^2 \ll T^\varepsilon \left(\Delta T + \frac{T^{3/2}}{\Delta^{3/2}} \right). \quad (13.4)$$

The second term is smaller than the first term if and only if $\Delta \gg T^{1/5}$.

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