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L_p-DUAL MIXED GEOMINIMAL SURFACE AREA

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Abstract. Lutwak (*Adv. Math.*, vol. 118(2), 1996, pp. 244–294) defined the notion of L_p-geominimal surface area based on L_p-mixed volumes. Recently, Wang and Qi (*J. Inequal. Appl.*, vol. 2011, 2011, pp. 1–10) introduced the concept of L_p-dual geominimal surface area based on L_p-dual mixed volumes. In this paper, based on L_p-dual mixed quermassintegrals, we define the concept of L_p-dual mixed geominimal surface area and establish several inequalities for this new notion.

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1. Introduction and main results. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, the set of convex bodies whose centroids are at the origin and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathcal{K}^n_o$, $\mathcal{K}^n_c$ and $\mathcal{K}^n_{os}$, respectively. $\mathcal{S}^n_o$ denotes the set of star bodies (about the origin) in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, and let $V(K)$ denote the n-dimensional volume of a body K. Let B denote the standard Euclidean unit ball in $\mathbb{R}^n$ and write $\omega_n = V(B)$ for its volume. We also note that i denotes any real number in this paper.

The notion of geominimal surface area was introduced by Petty[**17**]. For $K \in \mathcal{K}^n$, the geominimal surface area, $G(K)$, of K is defined by

$$\omega_n^{\frac{1}{n}} G(K) = \inf\{nV_1(K, Q)V(Q^*)^{\frac{1}{n}} : Q \in \mathcal{K}^n\}.$$

Here Q^* denotes the polar of body Q, and $V_1(M, N)$ denotes the mixed volume of $M, N \in \mathcal{K}^n$ (see [**11**]). For other important affine notions of surface area, in particular affine surface area, see [**7**, **8**, **9**, **21**, **25**, **26**].

Both affine surface area and geominimal surface area are unimodular affine invariant functionals of convex hypersurfaces. Isoperimetric inequalities involving geominimal surface area are not only closely related to many isoperimetric inequalities involving affine surface area but, in fact, also clarify the equality conditions of many of these inequalities. Lutwak [**15**] demonstrated that there are extensions of all of the known inequalities involving affine and geominimal surface areas to L_p-affine and L_p-geominimal surface areas which are part of a new L_p-Brunn–Minkowski theory initialized by Lutwak. In particular, Lutwak [**15**] discovered an important relationship between L_p-affine and L_p-geominimal surface areas, which has found a number of applications (see, e.g., [**22**, **28**]).

Based on the notion of L_p-mixed volumes, Lutwak [**15**] introduced L_p-geominimal surface area. For $K \in \mathcal{K}^n_o, p \geq 1$, the L_p-geominimal surface area, $G_p(K)$, of K is defined by

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf\{nV_p(K, Q)V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n\}. \tag{1.1}$$

Here $V_p(M, N)$ denotes the L_p-mixed volume of $M, N \in \mathcal{K}_o^n$ (see [**15**]).

If $p = 1$, $G_p(K)$ is just Petty's geominimal surface area $G(K)$.

A dual theory to the L_p-Brunn–Minkowski theory (i.e. the theory of L_p-mixed volumes and related concepts) was also developed by Lutwak (see [**3**, **6**, **13**, **14**, **19**, **27**]). Based on the notion of L_p-dual mixed volumes, Wang and Qi [**24**] gave a definition of L_p-dual geominimal surface area as follows: For $K \in \mathcal{S}_o^n, p \geq 1$, the L_p-dual geominimal surface area, $\widetilde{G}_{-p}(K)$, of K is defined by

$$\omega_n^{-\frac{p}{n}} \widetilde{G}_{-p}(K) = \inf\{n\widetilde{V}_{-p}(K, Q)V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_{os}^n\}. \tag{1.2}$$

Here $\widetilde{V}_{-p}(M, N)$ denotes the L_p-dual mixed volume of $M, N \in \mathcal{S}_o^n$ (see [**15**]).

For this new notion of L_p-dual geominimal surface area, Wang and Qi [**24**] established the following geometric inequalities.

THEOREM 1A. *If $K \in \mathcal{S}_o^n$, $p \geq 1$, then*

$$\widetilde{G}_{-p}(K) \geq n\omega_n^{-\frac{p}{n}} V(K)^{\frac{n+p}{n}}$$

with equality if and only if K is an ellipsoid centred at the origin.

THEOREM 1B. *If $K \in \mathcal{K}_c^n$, $1 \leq p < n$, then*

$$\widetilde{G}_{-p}(K)\widetilde{G}_{-p}(K^*) \leq (n\omega_n)^2$$

with equality if and only if K is an ellipsoid.

THEOREM 1C. *If $K \in \mathcal{S}_o^n$, $1 \leq p < q$, then*

$$\left(\frac{\widetilde{G}_{-p}(K)^n}{n^n V(K)^{n+p}}\right)^{\frac{1}{p}} \leq \left(\frac{\widetilde{G}_{-q}(K)^n}{n^n V(K)^{n+q}}\right)^{\frac{1}{q}}.$$

The quantity

$$\left(\frac{\widetilde{G}_{-p}(K)^n}{n^n V(K)^{n+p}}\right)^{\frac{1}{p}}$$

is called the L_p-dual geominimal surface area ratio of $K \in \mathcal{S}_o^n$.

Wang and Leng in [**23**] extended the notion of L_p-dual mixed volume to a family of L_p-dual mixed quermassintegrals. The main aim of this paper is to define a corresponding notion of L_p-dual mixed geominimal surface areas based on L_p-dual mixed quermassintegrals, and to extend the above inequalities to the entire family of these new L_p-dual mixed geominimal surface areas.

For $K \in \mathcal{S}_o^n$, $p \geq 1$ and $0 \leq i < n$, we define the L_p-dual mixed geominimal surface area, $\widetilde{G}_{-p,i}(K)$, of K by

$$\omega_n^{-\frac{p}{n-i}} \widetilde{G}_{-p,i}(K) = \inf\{n\widetilde{W}_{-p,i}(K, Q)\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n\}. \tag{1.3}$$

Here $\widetilde{W}_i(M)$ denotes the dual quermassintegrals of $M \in \mathcal{S}_o^n$, and $\widetilde{W}_{-p,i}(M, N)$ denotes the L_p-dual mixed quermassintegrals of $M, N \in \mathcal{S}_o^n$ (see Section 2).

From definitions (1.2) and (1.3) and formula (2.12), it follows that

$$\widetilde{G}_{-p,0}(K) = \widetilde{G}_{-p}(K).$$

The main results can be stated as follows: First, we establish the extended form of Theorem 1A, given by Theorem 1.1, and also obtain Theorem 1.2, which is the dual form of Theorem 1.1.

Theorem 1.1. *If $K \in \mathcal{S}_o^n$, $p \geq 1$ and $0 < i < n$, then*

$$\widetilde{G}_{-p,i}(K) \geq n\omega_n^{-\frac{p}{n-i}} \widetilde{W}_i(K)^{\frac{n+p-i}{n-i}},$$

with equality if and only if K is a ball centred at the origin.

Theorem 1.2. *If $K \in \mathcal{K}_c^n$, $p \geq 1$ and $0 < i < n$, then*

$$\widetilde{G}_{-p,i}(K) \leq n\omega_n^{\frac{2n-2i+p}{n-i}} \widetilde{W}_i(K^*)^{-\frac{n+p-i}{n-i}}$$

with equality if and only if K is a ball centred at the origin.

Moreover, we obtain the extended versions of Theorems 1B and 1C.

Theorem 1.3. *If $K \in \mathcal{K}_c^n$, $p \geq 1$ and $0 < i < n - p$, then*

$$\widetilde{G}_{-p,i}(K)\widetilde{G}_{-p,i}(K^*) \leq (n\omega_n)^2$$

with equality if and only if K is a ball centred at the origin.

Theorem 1.4. *If $K \in \mathcal{S}_o^n$, $1 < p < q$ and $0 < i < n$, then*

$$\left(\frac{\widetilde{G}_{-p,i}(K)^{n-i}}{n^{n-i}\widetilde{W}_i(K)^{n+p-i}}\right)^{\frac{1}{p}} \leq \left(\frac{\widetilde{G}_{-q,i}(K)^{n-i}}{n^{n-i}\widetilde{W}_i(K)^{n+q-i}}\right)^{\frac{1}{q}}.$$

We call

$$\left(\frac{\widetilde{G}_{-p,i}(K)^{n-i}}{n^{n-i}\widetilde{W}_i(K)^{n+p-i}}\right)^{\frac{1}{p}}$$

the L_p-dual mixed geominimal surface area ratio of $K \in \mathcal{S}_o^n$.

Finally, we obtain the following Brunn–Minkowski-type inequality for L_p-dual mixed geominimal surface area.

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THEOREM 1.5. *If $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \geq 0$ (not both zero) and $p \geq 1$, then for $0 \leq i < n$*

$$\widetilde{G}_{-p,i}(\lambda \cdot K +_{-p} \mu \cdot L)^{-\frac{p}{n+p-i}} \geq \lambda \widetilde{G}_{-p,i}(K)^{-\frac{p}{n+p-i}} + \mu \widetilde{G}_{-p,i}(L)^{-\frac{p}{n+p-i}},$$

with equality if and only if K and L are dilates.

Here $\lambda \cdot K +_{-p} \mu \cdot L$ denotes the L_p-harmonic radial combination of K and L (see (2.4)). For log-concavity properties of other important geometric functionals we refer to [**1**, **16**, **20**].

The proofs of Theorems 1.1–1.4 will be given in Section 3 of this paper. In Section 4, we give the proof of Theorem 1.5.

2. Preliminaries.

2.1. Radial functions and polars of star bodies and convex bodies. If K is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \longrightarrow [0, \infty)$, is defined by (see [**2**, **18**])

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda \cdot u \in K\}, \quad u \in S^{n-1}.$$

If ρ_K is continuous and positive, then K will be called a star body. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $K \in \mathcal{K}_o^n$, the polar body, K^*, of K is defined by (see [**2**, **18**])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in K\}. \tag{2.1}$$

Clearly,

$$(K^*)^* = K. \tag{2.2}$$

For $K \in \mathcal{K}_o^n$ and its polar body, the well-known Blaschke–Santaló inequality can be stated as follows (see [**12**]).

THEOREM 2A. *If $K \in \mathcal{K}_c^n$, then*

$$V(K)V(K^*) \leq \omega_n^2 \tag{2.3}$$

with equality if and only if K is an ellipsoid.

2.2. Dual quermassintegrals and L_p-dual mixed quermassintegrals. For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-harmonic radial combination, $\lambda \cdot K +_{-p} \mu \cdot L \in \mathcal{S}_o^n$, of K and L is defined by (see [**15**])

$$\rho(\lambda \cdot K +_{-p} \mu \cdot L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}. \tag{2.4}$$

For $K \in \mathcal{S}_o^n$ and any real i, the dual quermassintegrals, $\widetilde{W}_i(K)$, of K are defined by (see [**10**])

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} dS(u). \tag{2.5}$$

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Obviously,

$$\widetilde{W}_0(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n dS(u) = V(K). \tag{2.6}$$

Based on L_p-harmonic radial combinations of star bodies, Wang and Leng [**23**] introduced the notion of L_p-dual mixed quermassintegrals as follows: For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, $\varepsilon > 0$, real $i \neq n$, the L_p-dual mixed quermassintegrals, $\widetilde{W}_{-p,i}(K, L)$, of K and L are defined by (see [**23**])

$$\frac{n-i}{-p}\widetilde{W}_{-p,i}(K,L) = \lim_{\varepsilon \longrightarrow 0^+} \frac{\widetilde{W}_i(K \widetilde{+}_{-p} \varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon}. \tag{2.7}$$

The above definition and Hospital's rule give the following integral representation of L_p-dual mixed quermassintegrals (see [**23**]):

$$\widetilde{W}_{-p,i}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p-i}(u)\rho_L^{-p}(u)dS(u), \tag{2.8}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}. From (2.8) and definition (2.5), we get

$$\widetilde{W}_{-p,i}(K,K) = \widetilde{W}_i(K). \tag{2.9}$$

The Minkowski inequality for L_p-dual mixed quermassintegrals states the following (see [**23**]).

THEOREM 2B. *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, and $i \neq n$, then for $i < n$ or $n < i < n+p$,*

$$\widetilde{W}_{-p,i}(K,L) \geq \widetilde{W}_i(K)^{\frac{n+p-i}{n-i}}\widetilde{W}_i(L)^{-\frac{p}{n-i}}; \tag{2.10}$$

for $i > n+p$, inequality (2.10) is reversed. Equality holds in either case if and only if K and L are dilates. For $i = n+p$, (2.10) holds with equality for all K and L.

Recall that Lutwak [**15**] introduced the L_p-dual mixed volume as follows: For $K, L \in \mathcal{S}_o^n$, $p \geq 1$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K, L)$, of K and L is defined by

$$\frac{n}{-p}\widetilde{V}_{-p}(K,L) = \lim_{\varepsilon \longrightarrow 0^+} \frac{V(K \widetilde{+}_{-p} \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.11}$$

From (2.11), (2.6) and (2.7), we see that

$$\widetilde{W}_{-p,0}(K,L) = \widetilde{V}_{-p}(K,L). \tag{2.12}$$

3. Proofs of Theorems 1.1–1.4. In this section, we will prove Theorems 1.1–1.4. First, we need the following lemma for the proofs of Theorems 1.1–1.2.

LEMMA 3.1 ([**10**]). *If $K \in \mathcal{K}_o^n$ and $0 \leq i < n$, then*

$$\widetilde{W}_i(K) \leq \omega_n^{\frac{i}{n}} V(K)^{\frac{n-i}{n}}, \tag{3.1}$$

with equality for $0 < i < n$ if and only if K is a ball centred at the origin.

Proof of Theorem 1.1. From definitions (1.3), (2.10) and (3.1) and the Blaschke–Santaló inequality (2.3), we have that for $0 < i < n$

$$\begin{aligned}\omega_n^{-\frac{p}{n-i}}\widetilde{G}_{-p,i}(K) &= \inf\{n\widetilde{W}_{-p,i}(K,Q)\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n\}\\ &\geq \inf\{n\widetilde{W}_i(K)^{\frac{n+p-i}{n-i}}[\widetilde{W}_i(Q)\widetilde{W}_i(Q^*)]^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n\}\\ &\geq \inf\{n\widetilde{W}_i(K)^{\frac{n+p-i}{n-i}}[\omega_n^{\frac{2i}{n}}(V(Q)V(Q^*))^{\frac{n-i}{n}}]^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n\}\\ &\geq n\omega_n^{-\frac{2p}{n-i}}\widetilde{W}_i(K)^{\frac{n+p-i}{n-i}},\end{aligned}$$

i.e.

$$\widetilde{G}_{-p,i}(K) \geq n\omega_n^{-\frac{p}{n-i}}\widetilde{W}_i(K)^{\frac{n+p-i}{n-i}}. \tag{3.2}$$

By the equality conditions of (2.10), (3.1) and (2.3), we see that equality holds in (3.2) if and only if K is a ball centred at the origin. □

Proof of Theorem 1.2. From definition (1.3) and inequality (2.10), we have for $0 < i < n$

$$\begin{aligned}&\omega_n^{-\frac{p}{n-i}}\widetilde{W}_i(K^*)^{\frac{n+p-i}{n-i}}\widetilde{G}_{-p,i}(K)\\ &\quad= \inf\{n\widetilde{W}_{-p,i}(K,Q)\widetilde{W}_i(K^*)^{\frac{n+p-i}{n-i}}\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n\}\\ &\quad\leq \inf\{n\widetilde{W}_{-p,i}(K,Q)\widetilde{W}_{-p,i}(K^*,Q^*) : Q \in \mathcal{K}_c^n\}.\end{aligned} \tag{3.3}$$

Since $K \in \mathcal{K}_c^n$, taking $Q = K$, it follows from formulas (3.1), (3.3) and (2.3) that

$$\begin{aligned}&\omega_n^{-\frac{p}{n-i}}\widetilde{W}_i(K^*)^{\frac{n+p-i}{n-i}}\widetilde{G}_{-p,i}(K)\\ &\quad\leq \inf\{n\widetilde{W}_i(K)\widetilde{W}_i(K^*) : K \in \mathcal{K}_c^n\}\\ &\quad\leq \inf\{n\omega_n^{\frac{2i}{n}}[V(K)V(K^*)]^{\frac{n-i}{n}} : K \in \mathcal{K}_c^n\}\\ &\quad= n\omega_n^2,\end{aligned}$$

i.e.

$$\widetilde{G}_{-p,i}(K) \leq n\omega_n^{\frac{2n-2i+p}{n-i}}\widetilde{W}_i(K^*)^{-\frac{n+p-i}{n-i}}. \tag{3.4}$$

By the equality conditions of (2.10), (3.1) and (2.3), we see that equality holds in (3.4) if and only if K is a ball centred at the origin. □

Proof of Theorem 1.3. From definition (1.3), it follows that for any $Q \in \mathcal{K}_c^n$

$$\omega_n^{-\frac{p}{n-i}}\widetilde{G}_{-p,i}(K) \leq n\widetilde{W}_{-p,i}(K,Q)\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}}. \tag{3.5}$$

Since $K \in \mathcal{K}_c^n$, taking $Q = K$ in (3.5), we get that

$$\omega_n^{-\frac{p}{n-i}}\widetilde{G}_{-p,i}(K) \leq n\widetilde{W}_{-p,i}(K,K)\widetilde{W}_i(K^*)^{-\frac{p}{n-i}},$$

i.e.

$$\omega_n^{-\frac{p}{n-i}}\widetilde{G}_{-p,i}(K) \leq n\widetilde{W}_i(K)\widetilde{W}_i(K^*)^{-\frac{p}{n-i}}. \tag{3.6}$$

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Similarly,

$$\omega_n^{-\frac{p}{n-i}} \widetilde{G}_{-p,i}(K^*) \leq n\widetilde{W}_i(K^*)\widetilde{W}_i(K)^{-\frac{p}{n-i}}. \tag{3.7}$$

Combining (3.6) and (3.7), we get for $0 < i < n - p$

$$\begin{aligned}\omega_n^{-\frac{2p}{n-i}} \widetilde{G}_{-p,i}(K)\widetilde{G}_{-p,i}(K^*) &\leq n^2[\omega_n^{\frac{2i}{n}}(V(K)V(K^*))^{\frac{n-i}{n}}]^{\frac{n-i-p}{n-i}} \\ &\leq n^2\omega_n^{\frac{2(n-i-p)}{n-i}},\end{aligned}$$

i.e.

$$\widetilde{G}_{-p,i}(K)\widetilde{G}_{-p,i}(K^*) \leq (n\omega_n)^2. \tag{3.8}$$

By the equality conditions of (2.3) and (3.1), we see that equality holds in (3.8) if and only if K is a ball centred at the origin. □

Proof of Theorem 1.4. From (2.8) and $1 \leq p < q$, we have by Hölder's inequality

$$\begin{aligned}\widetilde{W}_{-p,i}(K, Q) &= \frac{1}{n}\int_{S^{n-1}} \rho_K(u)^{n+p-i}\rho_Q(u)^{-p}dS(u) \\ &= \frac{1}{n}\int_{S^{n-1}} \left[\rho_K(u)^{n+q-i}\rho_Q(u)^{-q}\right]^{\frac{p}{q}} \left[\rho_K(u)^{n-i}\right]^{\frac{q-p}{q}} dS(u) \\ &\leq \widetilde{W}_{-q,i}(K, Q)^{\frac{p}{q}}\widetilde{W}_i(K)^{\frac{q-p}{q}},\end{aligned}$$

i.e.

$$\left(\frac{\widetilde{W}_{-p,i}(K, Q)}{\widetilde{W}_i(K)}\right)^{\frac{1}{p}} \leq \left(\frac{\widetilde{W}_{-q,i}(K, Q)}{\widetilde{W}_i(K)}\right)^{\frac{1}{q}}. \tag{3.9}$$

Thus, from definition (1.3) and (3.9), we get for $0 < i < n$

$$\begin{aligned}&\frac{1}{\omega_n}\left(\frac{\widetilde{G}_{-p,i}(K)^{n-i}}{n^{n-i}\widetilde{W}_i(K)^{n+p-i}}\right)^{\frac{1}{p}} \\ &\quad = \inf\left\{\left(\frac{\widetilde{W}_{-p,i}(K, Q)}{\widetilde{W}_i(K)}\right)^{\frac{n-i}{p}} \frac{\widetilde{W}_i(Q^*)^{-1}}{\widetilde{W}_i(K)} : Q \in \mathcal{K}_c^n\right\} \\ &\quad \leq \inf\left\{\left(\frac{\widetilde{W}_{-q,i}(K, Q)}{\widetilde{W}_i(K)}\right)^{\frac{n-i}{q}} \frac{\widetilde{W}_i(Q^*)^{-1}}{\widetilde{W}_i(K)} : Q \in \mathcal{K}_c^n\right\} \\ &\quad \leq \frac{1}{\omega_n}\left(\frac{\widetilde{G}_{-q,i}(K)^{n-i}}{n^{n-i}\widetilde{W}_i(K)^{n+q-i}}\right)^{\frac{1}{q}},\end{aligned}$$

that is

$$\left(\frac{\widetilde{G}_{-p,i}(K)^{n-i}}{n^{n-i}\widetilde{W}_i(K)^{n+p-i}}\right)^{\frac{1}{p}} \leq \left(\frac{\widetilde{G}_{-q,i}(K)^{n-i}}{n^{n-i}\widetilde{W}_i(K)^{n+q-i}}\right)^{\frac{1}{q}}.$$ □

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4. Brunn–Minkowski-type inequalities. In this section we study Brunn–Minkowski-type inequalities for L_p-dual mixed geominimal surface areas. First, we prove Theorem 1.5. Next, we show that for L_p-radial combination of star bodies, we also have a Brunn–Minkowski-type inequality. The proof of Theorem 1.5 requires the following lemma.

LEMMA 4.1. *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, $\lambda, \mu \geq 0$ (not both zero) and $0 \leq i < n$, then for any $Q \in \mathcal{S}_o^n$,*

$$\widetilde{W}_{-p,i}(\lambda \cdot K +_{-p} \mu \cdot L, Q)^{-\frac{p}{n+p-i}} \geq \lambda \widetilde{W}_{-p,i}(K, Q)^{-\frac{p}{n+p-i}} + \mu \widetilde{W}_{-p,i}(L, Q)^{-\frac{p}{n+p-i}}, \tag{4.1}$$

with equality if and only if K and L are dilates.

Proof. Since $0 \leq i < n$, we have $-(n+p-i)/p < 0$. Hence, by (2.8) and Minkowski's integral inequality (see [**5**]), we have for any $Q \in \mathcal{S}_o^n$,

$$\begin{aligned}
&\widetilde{W}_{-p,i}(\lambda \cdot K +_{-p} \mu \cdot L, Q)^{-\frac{p}{n+p-i}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \rho(\lambda \cdot K +_{-p} \mu \cdot L, u)^{n+p-i} \rho(Q, u)^{-p} dS(u)\right]^{-\frac{p}{n+p-i}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left(\rho(\lambda \cdot K +_{-p} \mu \cdot L, u)^{-p} \rho(Q, u)^{\frac{p^2}{n+p-i}}\right)^{-\frac{n+p-i}{p}} dS(u)\right]^{-\frac{p}{n+p-i}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \left((\lambda\rho(K, u)^{-p} + \mu\rho(L, u)^{-p})\rho(Q, u)^{\frac{p^2}{n+p-i}}\right)^{-\frac{n+p-i}{p}} dS(u)\right]^{-\frac{p}{n+p-i}} \\
&\geq \lambda \left[\frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n+p-i} \rho(Q, u)^{-p} dS(u)\right]^{-\frac{p}{n+p-i}} \\
&\quad + \mu \left[\frac{1}{n}\int_{S^{n-1}} \rho(L, u)^{n+p-i} \rho(Q, u)^{-p} dS(u)\right]^{-\frac{p}{n+p-i}} \\
&= \lambda \widetilde{W}_{-p,i}(K, Q)^{-\frac{p}{n+p-i}} + \mu \widetilde{W}_{-p,i}(L, Q)^{-\frac{p}{n+p-i}}.
\end{aligned}$$

That is,

$$\widetilde{W}_{-p,i}(\lambda \cdot K +_{-p} \mu \cdot L, Q)^{-\frac{p}{n+p-i}} \geq \lambda \widetilde{W}_{-p,i}(K, Q)^{-\frac{p}{n+p-i}} + \mu \widetilde{W}_{-p,i}(L, Q)^{-\frac{p}{n+p-i}}.$$

By the equality conditions of Minkowski's integral inequality, we see that equality holds in (4.1) if and only if K and L are dilates. □

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Proof of Theorem 1.5. From definition (1.3) and Lemma 4.1, we obtain for $0 \le i < n$

$$
\begin{aligned}
&\left[\omega_n^{-\frac{p}{n-i}} \widetilde{G}_{-p,i}(\lambda \cdot K +_{-p} \mu \cdot L)\right]^{-\frac{p}{n+p-i}} \\
&\quad = \inf\left\{\left[n\widetilde{W}_{-p,i}(\lambda \cdot K +_{-p} \mu \cdot L, Q)\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}}\right]^{-\frac{p}{n+p-i}} : Q \in \mathcal{K}_c^n\right\} \\
&\quad = \inf\left\{\left[n\widetilde{W}_{-p,i}(\lambda \cdot K +_{-p} \mu \cdot L, Q)\right]^{-\frac{p}{n+p-i}} \left[\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}}\right]^{-\frac{p}{n+p-i}} : Q \in \mathcal{K}_c^n\right\} \\
&\quad \ge \inf\left\{\left[\lambda\left(n\widetilde{W}_{-p,i}(K, Q)\right)^{-\frac{p}{n+p-i}} + \mu\left(n\widetilde{W}_{-p,i}(L, Q)\right)^{-\frac{p}{n+p-i}}\right]\right. \\
&\qquad \left. \times \left[\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}}\right]^{-\frac{p}{n+p-i}} : Q \in \mathcal{K}_c^n\right\} \\
&\quad \ge \inf\left\{\lambda\left[n\widetilde{W}_{-p,i}(K, Q)\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}}\right]^{-\frac{p}{n+p-i}} : Q \in \mathcal{K}_c^n\right\} \\
&\qquad + \inf\left\{\mu\left[n\widetilde{W}_{-p,i}(L, Q)\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}}\right]^{-\frac{p}{n+p-i}} : Q \in \mathcal{K}_c^n\right\} \\
&\quad = \lambda\left[\omega_n^{-\frac{p}{n-i}}\widetilde{G}_{-p,i}(K)\right]^{-\frac{p}{n+p-i}} + \mu\left[\omega_n^{-\frac{p}{n-i}}\widetilde{G}_{-p,i}(L)\right]^{-\frac{p}{n+p-i}}.
\end{aligned}
$$

This yields

$$
\widetilde{G}_{-p,i}(\lambda \cdot K +_{-p} \mu \cdot L)^{-\frac{p}{n+p-i}} \ge \lambda\widetilde{G}_{-p,i}(K)^{-\frac{p}{n+p-i}} + \mu\widetilde{G}_{-p,i}(L)^{-\frac{p}{n+p-i}}. \tag{4.2}
$$

By the equality conditions of (4.1), we know that equality holds in (4.2) if and only if K and L are dilates. □

For $K, L \in \mathcal{S}_o^n$, $p \ge 1$ and $\lambda, \mu \ge 0$ (not both zero), the L_p-radial combination, $\lambda \circ K \tilde{+}_p \mu \circ L \in \mathcal{S}_o^n$, of K and L is defined by (see [**4**])

$$
\rho(\lambda \circ K \tilde{+}_p \mu \circ L, \cdot)^p = \lambda\rho(K, \cdot)^p + \mu\rho(L, \cdot)^p. \tag{4.3}
$$

Based on definition (4.3), Wang and Qi [**24**] obtained a Brunn–Minkowski-type inequality for L_p-dual geominimal surface area.

THEOREM 4A. *If $K, L \in \mathcal{K}_{os}^n$, $p \ge 1$ and $\lambda, \mu \ge 0$ (not both zero), then*

$$
\widetilde{G}_{-p}(\lambda \circ K \tilde{+}_{n+p} \mu \circ L) \ge \lambda\widetilde{G}_{-p}(K) + \mu\widetilde{G}_{-p}(L)
$$

with equality if and only if K and L are dilates.

Here we extend Theorem 4A and establish the following Brunn–Minkowski-type inequality for L_p-dual mixed geominimal surface areas. Using definition (4.3), our result can be stated as follows.

THEOREM 4.1. *If $K, L \in \mathcal{S}_o^n$, $p \ge 1$, $\lambda, \mu \ge 0$ (not both zero) and $0 \le i < n$, then*

$$
\widetilde{G}_{-p,i}(\lambda \circ K \tilde{+}_{n+p-i} \mu \circ L) \ge \lambda\widetilde{G}_{-p,i}(K) + \mu\widetilde{G}_{-p,i}(L)
$$

with equality if and only if K and L are dilates.

Proof. From definitions (1.3) and (4.3), we have for $0 \le i < n$

$$
\begin{aligned}
&\omega_n^{-\frac{p}{n-i}} \widetilde{G}_{-p,i}(\lambda \circ K \tilde{+}_{n+p-i} \mu \circ L) \\
&\quad = \inf\left\{ n\widetilde{W}_{-p,i}(\lambda \circ K \tilde{+}_{n+p-i} \mu \circ L, Q)\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n \right\} \\
&\quad = \inf\left\{ n\left[\lambda \widetilde{W}_{-p,i}(K, Q) + \mu \widetilde{W}_{-p,i}(L, Q)\right] \widetilde{W}_i(Q^*)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n \right\} \\
&\quad = \inf\left\{ n\lambda \widetilde{W}_{-p,i}(K, Q)\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}} + n\mu \widetilde{W}_{-p,i}(L, Q)\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n \right\} \\
&\quad \ge \inf\left\{ n\lambda \widetilde{W}_{-p,i}(K, Q)\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n \right\} \\
&\qquad + \inf\left\{ n\mu \widetilde{W}_{-p,i}(L, Q)\widetilde{W}_i(Q^*)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n \right\} \\
&\quad = \omega_n^{-\frac{p}{n-i}} \lambda \widetilde{G}_{-p,i}(K) + \omega_n^{-\frac{p}{n-i}} \mu \widetilde{G}_{-p,i}(L).
\end{aligned}
$$

Thus,

$$
\widetilde{G}_{-p,i}(\lambda \circ K \tilde{+}_{n+p-i} \mu \circ L) \ge \lambda \widetilde{G}_{-p,i}(K) + \mu \widetilde{G}_{-p,i}(L). \tag{4.4}
$$

Equality holds if and only if $\lambda \circ K \tilde{+}_{n+p-i} \mu \circ L$ is a dilate of both K and L. This means that equality holds in (4.4) if and only if K and L are dilates. □

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