Can. J. Math., Vol. XXIX, No. 3, 1977, pp. 578–586

QUASI-P-PURE-INJECTIVE GROUPS

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Recently, a great deal of attention has been paid to the concept of quasi-pure injectivity introduced by L. Fuchs as Problem 17 in [**5**]. An abelian group G is said to be *quasi-pure-injective* (q.p.i.) if every homomorphism from a pure subgroup of G to G can be lifted to an endomorphism of G. D. M. Arnold, B. O'Brien and J. D. Reid have succeeded in [**1**] to characterize torsion free q.p.i. of finite rank, whereas in [**2**] we solved the torsion case and in [**3**] we studied certain classes of infinite rank torsion free q.p.i. groups.

While q.p.i. groups have, in general, rather complicated descriptions, it turns out that the class of *quasi-p-pure-injective groups* (q.p.p.i.) is far more tractable. This is due partly to the fact that one can make extensive use of the theory of p-basic subgroups.

In this work we consider mainly p-reduced groups. Our first result shows that the problem naturally divides itself into two cases. First, a "torsion" case which is completely characterized as follows: A p-reduced group H with torsion p-basic subgroup B is q.p.p.i. if and only if it is isomorphic to a pure fully invariant subgroup of the p-adic completion $\hat{B}$ of B. Such subgroups are completely determined by certain dual ideals, in the meet-semilattice of all Ulm sequences of elements of $\hat{B}$ as was shown by A. Mader in [**8**].

Second, a torsion free case for which we have: a p-reduced torsion free group K is q.p.p.i. if and only if it is a torsion free q.p.i. R-module, where R is a unital pure subring of the ring J_p of p-adic integers. (R is said to be a *unital subring* of J_p if the units of R are exactly those units of J_p that are in R).

Also, we examine various situations where $H \oplus K$ is q.p.p.i. when H and K are as in the previous cases. Finally, we show that finite rank q.p.i. R-modules are free and that free R-Modules are q.p.i. if and only if they are of finite rank.

All groups considered here are abelian. The notation follows closely that of [**5**].

I. Generalities. Let p be a prime number, we say that a group G is *quasi-p-pure injective* (abbreviated q.p.p.i.) if every homomorphism from a p-pure subgroup of G to G can be lifted to an endomorphism of G.

Evidently, q.p.p.i. groups are also quasi-pure-injective. However Z is q.p.i. but not q.p.p.i.

Received June 23, 1976 and in revised form, September 28, 1976. The work of the first author was supported by NRC Grant No. A-5591.

https://doi.org/10.4153/CJM-1977-059-5 Published online by Cambridge University Press

We collect in the first proposition, without proof, some elementary but useful facts.

PROPOSITION 1.1. (a) *A direct summand of a q.p.p.i. group is q.p.p.i.* (b) *A p-pure fully invariant subgroup of a q.p.p.i. group is q.p.p.i.* (c) *A torsion group G is q.p.i. if and only if for every prime p, G_p is q.p.p.i.*

In [**2**] we have characterized torsion q.p.i. groups as follows:

THEOREM 1.2. [**2**]. *A torsion group G is q.p.i. if and only if G_p is the direct sum of a divisible group and a torsion complete group, for every prime number p.*

We need also the following:

PROPOSITION 1.3. *Let G be a q.p.p.i. group whose torsion subgroup is p-primary. Then:*

a) *If G is p-divisible, G is divisible.*

b) *If G is reduced, G is p-reduced.*

c) *G is m-divisible for every m such that* $(m, p) = 1$.

Proof. a) Let $(m, p) = 1$. Then mG is p-pure in G, and for $x \in G$ the correspondence $mx \rightarrow x$ defines a homomorphism $f\colon mG \rightarrow G$ which can be lifted to an endomorphism g of G; now $x = f(mx) = g(mx) = mg(x)$, therefore G is divisible.

b) Let H be the maximal p-divisible subgroup of G. Since H is p-pure and fully invariant in G, H is q.p.p.i. By a), H is divisible and therefore $H = 0$.

c) The same argument as in a) shows that $G = mG$ for every m such that $(m, p) = 1$.

Finally, the following observation is useful and easy to prove.

PROPOSITION 1.4. *Let H be a p-pure subgroup of a q.p.p.i. group G. If H is isomorphic to G, then H is a direct summand of G.*

2. *p*-reduced q.p.p.i. groups. If G is a p-reduced q.p.p.i. group, then $G^1 = \bigcap_{n=1}^{\infty} p^n G$ because $mG = G$ for every m such that $(m, p) = 1$. For the main result in this section, we need to establish that $G^1 = 0$ for such groups.

LEMMA 2.1. *If* $mG = G$ *whenever* $(m, p) = 1$ *and if* $mx \in G^1$ *then* $x \in G^1$.

PROPOSITION 2.2. *Let G be a group satisfying the following conditions*:

(a) *The torsion subgroup T of G is a torsion-complete p-group.*

(b) $G = mG$ *whenever* $(m, p) = 1$ *and* G/T *is divisible.*

Then G^1 *is divisible.*

Proof. Since T is torsion-complete, $T \cap G^1 = 0$ and $G^1 \simeq (G^1 \oplus T)/T$. Let K/T be the smallest pure subgroup of G/T containing $(G^1 \oplus T)/T$. Then K/pG^1 is a p-group. Indeed, let $x \in K$, there exists $n \in \mathbf{N}, g \in G^1$ and $t \in T$ such that $nx = g + t$. We may write $n = p^r m$, where $(m, p) = 1$. Since

$mG = G$, we have:

$$mp^r x = mg' + mt'$$

where $t' \in T$ and in view of Lemma 2.1., $g' \in G^1$. Since T is a p-group we see that

$$p^r x = g' + t'.$$

Let $O(t') = p^s$, then $p^{r+s+1}x = p^{s+1}g' \in pG^1$. It can easily be shown that $(T \oplus pG^1)/pG^1$ is a pure subgroup of K/pG^1. But T is torsion-complete and therefore $K/pG^1 = ((T \oplus pG^1)/pG^1) \oplus (R/pG^1)$ for some R containing pG^1. Thus, $K = R \oplus T$, and since K/T is divisible, $G^1 = K^1 = R$ is divisible.

Definition. We denote by $I(G)$ the family of subgroups of G for which any homomorphism into G can be lifted to an endomorphism of G. So, a group G is q.p.p.i. if and only if $I(G)$ contains all p-pure subgroups of G.

As is mentioned in the introduction, p-basic subgroups play an important role in this study. This is due partly to the following:

PROPOSITION 2.3. *A p-reduced group G is q.p.p.i. if and only if $I(G)$ contains all p-basic subgroups of G.*

Proof. The statement is a simple consequence of the fact that every p-pure subgroup of G possesses a p-basic subgroup which can be extended, by a direct summand to a p-basic subgroup of G.

Definition. We say that a group G possesses the property (*) if every idempotent endomorphism of a p-basic subgroup of G is induced by an endomorphism of G.

Such an endomorphism of G is also idempotent if G is p-reduced. Therefore, a p-reduced group G possesses the property (*) if every direct decomposition of a p-basic subgroup $B = B_1 \oplus B_2$ of G induces a decomposition of $G = G_1 \oplus G_2$ where $B_i \subset G_i$, $i = 1, 2$. Clearly, B_i is a p-basic subgroup of G_i and the G_i's are uniquely determined by the B_i's. In fact G_i is the closure of B_i in the p-adic topology of G.

PROPOSITION 2.4. *A p-reduced q.p.p.i. group possesses the* (*) *property.*

We are now ready for the main result of this section.

THEOREM 2.5. *Let G be a p-reduced q.p.p.i. group. Then*
a) *$mG = G$ whenever $(m, p) = 1$,*
b) *$G = H \oplus K$ where H is the closure of $T(G)$ in the p-adic topology of G, K is torsion-free and both are q.p.p.i.*

Proof. The first statement has already been proved. If B is a p-basic subgroup of G, $B = B_1 \oplus B_2$ where B_1 is a p-group and B_2 is free. Since G possesses the property (*), $G = H \oplus K$ where $B_1 \subseteq H$ and $B_2 \subseteq K$. As direct summands of a q.p.p.i. group, H and K are both q.p.p.i. Obviously K is torsion-free.

Therefore $T(G) \subseteq H$. Since K is torsion-free and reduced, $K^1 = 0$. Therefore H is closed in the p-adic topology of G and it is the closure of $T(G)$.

COROLLARY. *Let G be a p-reduced q.p.p.i. group, H and K as in Theorem 2.5. Then H is a fully invariant subgroup of G and if H' is a pure fully invariant subgroup of G containing $T(G)$, then $H' \subseteq H$ or $H' = G$. Furthermore $G^1 = 0$.*

Proof. $T(G)$ is a fully invariant subgroup of G and $H/T(G)$ is a fully invariant subgroup of $G/T(G)$. Therefore H is a fully invariant subgroup of G.

Consider H' a pure fully invariant subgroup of G containing $T(G)$. $H' = (H \cap H') \oplus (K \cap H')$. If $K \cap H' = 0$, $H' \subseteq H$. If $K \cap H' \neq 0$, we can find $x \in K \cap H'$ such that $\langle x \rangle$ is a p-pure subgroup of $K \cap H'$, and hence of G, and for every $a \in G$, we can define a homomorphism f_a: $\langle x \rangle \to G$ such that $f_a(x) = a$. Each f_a is induced by an endomorphism of G so that $H' = G$. Now, $G^1 = H^1 \oplus K^1$. Since K is torsion-free and reduced, $K^1 = 0$. $T(G)$ is torsion-complete by Theorem 1.2, so H satisfies all the conditions of proposition 2.2. Therefore $H^1 = K^1 = G^1 = 0$.

We close this section with a theorem which justifies our interest in p-reduced q.p.p.i. groups.

THEOREM 2.6. *Let $G = D \oplus R$ where D is divisible and R p-reduced. G is q.p.p.i. if and only if R is q.p.p.i.*

Proof. If G is q.p.p.i. then R is q.p.p.i. by Proposition 1.1. Conversely, let K be a p-pure subgroup of G and f: $K \to G$ a homomorphism. Then it is easy to see that $K + D$ is p-pure in G and $(K + D)/D$ is p-pure in G/D. Now, $f(K \cap D) \subseteq G^1 = D$, since $R^1 = 0$ by the corollary to Theorem 2.5. Therefore f induces a homomorphism $\bar{f}$: $(K + D)/D \to G/D$ defined by $\bar{f}(k + D) = f(k) + D$ for every $k \in K$. Since G/D is q.p.p.i. there exists a φ: $R \to R$ such that if $k = d + r \in K$ where $d \in D$ and $r \in R$, $\varphi(r) = \pi_R(f(k))$. Thus there exists φ': $K \to R$ with $\varphi'(k) = \varphi(r)$ and $f(k) - \varphi'(k) \in D$. Since D is injective, there is ψ: $G \to D$ such that $\psi|_K = f - \varphi'$. Then $(\bar{\varphi} + \psi)(k) = \varphi'(k) + \psi(k) = f(k)$, where $\bar{\varphi} = 0_D \oplus \varphi$ and $\bar{\varphi} + \psi$ is the desired extension of f.

3. The "torsion" case. From now on, following A. Mader [8], we refer to groups with torsion p-basic subgroups as "torsion" groups.

THEOREM 3.1. *Let G be a p-reduced "torsion" group and B a p-basic subgroup of G. Then G is q.p.p.i. if and only if it is isomorphic to a p-pure fully invariant subgroup of the p-adic completion $\hat{B}$ of B.*

Proof. Suppose that G is isomorphic to a p-pure fully invariant subgroup of $\hat{B}$. $\hat{B}$ is p-pure-injective and therefore q.p.p.i., so that G is q.p.p.i. by Proposition 1.1.

Conversely, suppose that G is q.p.p.i. In view of the corollary to Theorem 2.5, we have $G^1 = 0$. Then it is a well-known fact that G is isomorphic to a

https://doi.org/10.4153/CJM-1977-059-5 Published online by Cambridge University Press

p-pure subgroup of $\hat{B}$. Because T, the torsion subgroup of G, is torsion-complete, $T = T(\hat{B})$. Consider an endomorphism f: $\hat{B} \to \hat{B}$. $f(B) \subseteq T \subseteq G$. Since G is q.p.p.i. there exists $\bar{f}$: $G \to G$ such that f and $\bar{f}$ are equal when both are restricted to B. Therefore $\bar{f} = f|_G$ and G is isomorphic to a p-pure fully invariant subgroup of $\hat{B}$.

Such subgroups of the p-adic completion of a direct sum of cyclic p-groups have been completely characterized by A. Mader [**8**]. Thus, the "torsion" case is completely solved for p-reduced groups.

4. Torsion-free q.p.p.i. Now we turn our attention toward torsion-free q.p.p.i. groups. These groups are p-reduced if and only if they are reduced, as was shown in proposition 1.3. They are also q.p.i. and homogeneous of type $\tau = (0, \infty, \infty, \ldots)$ where the first entry corresponds to p. We have this result.

THEOREM 4.1. *Let G be a reduced torsion-free group. G is q.p.p.i. if and only if G is q.p.i. and homogeneous of type* $\tau = (0, \infty, \infty, \ldots)$.

Proof. We need only prove that the condition is sufficient. Let A be a p-pure subgroup of G, f: $A \to G$ a homomorphism and $\langle A \rangle_*$ the smallest pure subgroup of G containing A. Consider $x \in \langle A \rangle_*$. There exists $n \in \mathbf{N}$ such that $nx = a \in A$. Because A is p-pure in G and G is torsion-free, we may suppose that $(n, p) = 1$. $\chi_G(f(a)) = (k, \infty, \infty, \ldots)$, so there is a unique $y \in G$ such that $f(a) = ny$. Define $\bar{f}$: $\langle A \rangle_* \to G$ by $\bar{f}(x) = y$. One can routinely verify that $\bar{f}$ is a well-defined homomorphism which is an extension of f: $A \to G$. Since G is q.p.i. there is an endomorphism φ: $G \to G$ which induces $\bar{f}$ on $\langle A \rangle_*$ and therefore f on A.

A reduced torsion-free q.p.p.i. group G possesses the property (*). A p-basic subgroup B of G is free. If $B = \langle x \rangle \oplus B'$, $G = H \oplus G'$, where $\langle x \rangle \subseteq H$ and $B' \subseteq G'$. H is a torsion-free reduced q.p.p.i. group with a p-basic subgroup isomorphic to Z. We turn our attention to such groups and find that certain subrings of the ring J_p of the p-adic integers play an important role in the problem.

Definition. A subring A of a ring R is said to be a *unital* subring if $x \in A$ and $x^{-1} \in R \Rightarrow x^{-1} \in A$. In other words, the set of units of A is the intersection of A and the set of units of R.

PROPOSITION 4.2. *Let G be a torsion-free reduced group with p-basic subgroup of rank one. Then G is q.p.p.i. if and only if G is isomorphic to a unital p-pure subring of the ring* J_p *of p-adic integers.*

Proof. We first suppose that G is q.p.p.i. $G^1 = 0$ and G is isomorphic to a p-pure subgroup of J_p, containing 1 of J_p. Let $x \in G$ and define f: $\langle 1 \rangle \to G$ by $f(1) = x$. f induces an endomorphism φ of G, which in turn induces an endomorphism ψ of J_p. However every endomorphism of J_p is given by the multiplication by a p-adic integer [**7**, p. 98]. In this case ψ must be the multiplication by x. If $y \in G$, $xy = \varphi(y) \in G$, so G is a subring of J_p.

Let $x \in G$ and $x^{-1} \in J_p$. Then $x \notin pJ_p$ and $\langle x \rangle$ is a p-pure subgroup of G. Define f: $\langle x \rangle \to G$ by $f(x) = 1$. As above, there are endomorphisms φ and ψ of G and J_p respectively such that $\psi|_G = \varphi$ and $\varphi|_{\langle x \rangle} = f$. ψ must be the multiplication by x^{-1}, so $\varphi(1) = x^{-1} \in G$.

Conversely, suppose that G is a p-pure subring of J_p such that, if $x \in G$ and $x^{-1} \in J_p$, $x^{-1} \in G$. Consider A a p-pure subgroup of G and a homomorphism f: $A \to G$. Since J_p is q.p.p.i., f induces an endomorphism ψ of J_p, which is the multiplication by an element of J_p, say y. Now, A being p-pure, it contains an inversible element a of J_p and $f(a) = ay \in G$, therefore $y = a^{-1}f(x) \in G$. Then $\psi(G) \subseteq G$ and $\psi|_G$ is an endomorphism of G which extends f.

We should notice that, in the last part of the proof of Proposition 4.2, we have shown that any endomorphism of a p-pure unital subring R of J_p is the multiplication by an element of R.

We know that any cyclic direct summand $\langle x \rangle$ of a p-basic subgroup B of a reduced torsion-free q.p.p.i. group G can be embedded in a p-pure unital subring R_x of J_p. But a question arises: if $B = \langle x \rangle \oplus \langle y \rangle \oplus B'$, how do R_x and R_y compare with each other?

PROPOSITION 4.3. *Let G be a reduced torsion-free q.p.p.i. group. If $G = G_1 \oplus G_2$ where G_1 and G_2 have cyclic p-basic subgroups, then $G_1 \cong G_2$.*

Proof. Let $\langle x \rangle$ and $\langle y \rangle$ be p-basic subgroups of G_1 and G_2 respectively. Then $\langle x \rangle \oplus \langle y \rangle = \langle x + y \rangle \oplus \langle y \rangle$ is a p-basic subgroup of G. Since G is q.p.p.i., $G = H_1 \oplus H_2$ where $\langle x + y \rangle \subseteq H_1$ and $\langle y \rangle \subseteq H_2$. Then $H_2/\langle y \rangle$ is divisible and $G_2/\langle y \rangle$ is the maximal divisible subgroup of $G/\langle y \rangle \cong G_1 \oplus (G_2/\langle y \rangle)$. Therefore $H_2 \subseteq G_2$. Reciprocally $G_2 \subseteq H_2$. So $H_2 = G_2$ and $H_1 \cong G/H_2 \cong G/G_2 \cong G_1$.

Similarly $G = K_1 \oplus K_2$ where $\langle x \rangle \subseteq K_1$ and $\langle x + y \rangle \subseteq K_2$. As above $K_1 = G_1$. Also $H_1 = K_2$ and $K_2 \cong G/K_1 \cong G/G_1 \cong G_2$. Therefore $G_1 \cong G_2$.

PROPOSITION 4.4. *If R_1 and R_2 are p-pure unital subrings of J_p and $R_1 \cong R_2$, then $R_1 = R_2$.*

Proof. The isomorphism φ between R_1 and R_2 induces an endomorphism of J_p, the multiplication by an inversible element x of J_p. $\varphi(1) = x \in R_2$. Then $x^{-1} \in R_2$ and if $y \in R_1$, $\varphi(y) = xy \in R_2$. Therefore $R_1 \subseteq R_2$. Reciprocally $R_2 \subseteq R_1$.

If G is a torsion-free R-module, where R is a p-pure unital subring of J_p, if $B = \bigoplus_{i \in I} \langle x_i \rangle$ is a p-pure subgroup of G and $\langle B \rangle$ is the submodule generated by B, then $\langle B \rangle = \bigoplus_{i \in I} Rx_i$ and is pure in G.

We can now state the main result of this section.

THEOREM 4.5. *A torsion-free reduced group G is q.p.p.i. if and only if it is a torsion-free q.p.i. R-module, where R is a p-pure unital subring of J_p.*

Proof. Consider $B = \bigoplus_{i \in I} \langle x_i \rangle$ a p-basic subgroup of G. For every $i \in I$, $G = G_i \oplus H_i$ where $\langle x_i \rangle \subseteq G_i$ and $G_i \cong R$. G_i is an R-module and if $r \in R$, we can define a homomorphism $f_r : B \to G$ by $f_r(x_i) = rx_i$, for every $i \in I$. f_r induces an endomorphism $\bar{f}_r$ of G. If $r = 1, f_r = 1_G$. If r_1 and r_2 are elements of $R, \bar{f}_{r_1} \circ \bar{f}_{r_2} = \bar{f}_{r_1 r_2}$ and $\bar{f}_{r_1} + \bar{f}_{r_2} = \bar{f}_{r_1 + r_2}$ since $f_{r_1} \circ f_{r_2} = f_{r_1 r_2}$ and $f_{r_1} + f_{r_2} = f_{r_1 + r_2}$. So G is an R-module.

If A is an R-submodule of G which is pure in G as a subgroup then evidently $A \in I(G)$ and G is a q.p.i. R-module.

Conversely, suppose that G is a q.p.i. R-module. By Proposition 2.3, G is a q.p.p.i. group if and only if every p-basic subgroup B of G is an element of $I(G)$. However any homomorphism f: $B \to G$ can be easily extended to $\langle B \rangle$, the R-submodule generated by B, which is a pure subgroup of G. Hence f can be extended to an endomorphism of G and $B \in I(G)$.

COROLLARY. *If G is a reduced torsion-free q.p.p.i. group with a p-basic subgroup of finite rank, then G is a free R-module of finite rank, for a p-pure unital subring R of J_p.*

Proof. This is a consequence of property (*).

5. Free R-modules. Torsion-free reduced q.p.p.i. groups are R-modules, where R is a p-pure unital subring of the ring J_p of p-adic integers. Such R's are principal ideal domains. We consider first the simplest case: suppose that G is a free R-module. If $\langle x \rangle$ is a p-pure subgroup of G, then Rx, the R-submodule generated by x, is also a p-pure subgroup of G. We have the following well-known fact [**4**, p. 88].

THEOREM 5.1. *Let R be a principal ideal domain, G a free R-module and M a submodule of G of finite rank n. Then there exists a basis B of G, n elements e_i of B and n non-zero elements α_i of R such that*

a) $\{\alpha_i e_i\}_{i=1}^n$ *is a basis of M.*

b) α_i *divides* α_{i+1} *for* $1 \leqq i \leqq n - 1$.

In our case, if M is p-pure, then the α_i are units of R and M is a direct summand of G. So Rx is a direct summand of G and therefore $Rx \in I(G)$. As a matter of fact, one can easily show that

$$Rx = \bigcap_{n=1}^{\infty} (\langle x \rangle + p^n G).$$

Also, if $A = \bigoplus_{i \in I} \langle x_i \rangle$ is a p-pure subgroup of G, then

$$\sum_{i \in I} Rx_i = \bigoplus_{i \in I} Rx_i,$$

https://doi.org/10.4153/CJM-1977-059-5 Published online by Cambridge University Press

and it is a p-pure subgroup of G. Evidently, any homomorphism $f: A \to G$ induces a homomorphism $\bar{f}: \bigoplus_{i \in I} Rx_i \to G$.

If G is a free R-module of finite rank and if $B = \bigoplus_{i=1}^{n} \langle x_i \rangle$ is a p-basic subgroup of G, then $G = \bigoplus_{i=1}^{n} Rx_i$ and G is a q.p.p.i. group. In fact the converse is true.

THEOREM 5.2. *If R is a p-pure unital subring of the ring J_p of p-adic integers, then a free R-module is q.p.p.i. if and only if it is of finite rank.*

Proof. Consider $G = \bigoplus_{i \in I} Rx_i$ a free R-module of infinite rank. Let K be the field of fractions of R. K is an R-module generated by $\{1/p^n\}_{n=1}^{\infty}$ and it is a divisible additive group. There exists an epimorphism $f: G \to K$ whose kernel M is a free R-module, pure in G and isomorphic to G. By Proposition 1.4, $G = M \oplus K$, which is a contradiction.

6. Some results on the mixed case.

In Theorem 2.5, we have shown that a p-reduced q.p.p.i. group G is the direct sum of the largest fully invariant subgroup of G containing $T(G)$ and a torsion-free subgroup. Unfortunately, we have not been able to establish the full converse of this theorem. However, we have partial results.

LEMMA 6.1. *Consider a p-reduced group $G = H \oplus K$ where H is a "torsion" group and K is torsion-free. If A is a p-pure subgroup of H and $f: A \to G$ is a homomorphism, then $f(A) \subseteq H$.*

THEOREM 6.2. *Consider $G = H \oplus K$, a p-reduced group, where H is a "torsion" p-pure-injective group and K is a q.p.p.i. torsion-free group. Then G is q.p.p.i.*

Proof. Let A be a p-pure subgroup of G and $f: A \to G$ be a homomorphism. One can easily verify that $(A + H)/H$ is p-pure in G/H and that $(A \cap H)$ is a p-pure subgroup of H. By Lemma 6.1, $f(A \cap H) \subseteq H$, and therefore f induces a homomorphism $\bar{f}: (A + H)/H \to G/H \cong K$ by $\bar{f}(a + H) = f(a) + H$. Since K is q.p.p.i., we have homomorphisms $\varphi: K \to K$ and $\varphi': A \to K$ such that $\varphi(a) = \varphi'(\pi_K(f(a)) = \pi_K(f(a))$. So $f(a) - \varphi(a) \in H$. Since H is p-pure-injective, there is a homomorphism $\psi: G \to H$ such that $\psi|_A = f - \varphi'$. If $\bar{\varphi} = 0_H \oplus \varphi$, then $\bar{\varphi} + \psi$ coincides with f on A.

On the other hand one can easily show that:

THEOREM 6.3. *If H is a p-reduced "torsion" q.p.p.i. group and if K is a finite rank free R-module, where R is a p-pure unital subring of the ring of p-adic integers, then $G = H \oplus K$ is q.p.p.i.*

The study of q.p.p.i. groups is by no means finished, as should be obvious to the reader. We have, for the sake of reasonable length, omitted from this work our results on the p-divisible reduced q.p.p.i.'s, as well as some further results on unital subrings of J_p.

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https://doi.org/10.4153/CJM-1977-059-5 Published online by Cambridge University Press