Canad. Math. Bull. Vol. **19** (3), 1976

# REPRESENTING HOMOLOGY CLASSES ON SURFACES

BY
JAMES A. SCHAFER

Let $T^2 = S^1 \times S^1$, where $S^1$ is the unit circle, and let $\{\alpha, \beta\}$ be the integral basis of $H_1(T^2)$ induced by the 2 $S^1$-factors. It is well known that $0 \neq X = p\alpha + q\beta$ is represented by a simple closed curve (i.e. the homotopy class $\alpha^p\beta^q$ contains a simple closed curve) if and only if $gcd(p, q) = 1$. It is the purpose of this note to extend this theorem to oriented surfaces of genus $g$.

Let $S_g$ be an oriented surface of genus $g > 0$. The fundamental group $\pi_g$ can be presented as $\{a_1, \ldots, a_g, b_1, \ldots, b_g: \prod_1^g [a_i, b_i]\}$ and $H_1(S_g)$ is a free abelian group with $2g$ canonical generators, $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$, the images of $a_i$, $b_i$ respectively. Recall that a homology class $X \in H_1(S_g)$ is said to be represented by a simple closed curve if and only if there exists a homotopy class $\alpha \in \pi_1(S_g)$ containing a simple closed curve and $h(\alpha) = X$ where $h: \pi_1(S_g) \to \pi_1(S_g)_{ab} = H_1(S_g)$ is the Hurewicz map. It is the purpose of this note to prove the following theorem.

THEOREM. *Let $S_g$ be an oriented surface of genus $g > 0$, and let $\{\alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g\}$ be the canonical basis of $H_1(S_g)$. Then $0 \neq X = \sum_1^g p_i\alpha_i + q_i\beta_i \in H_1(S_g)$ is represented by a simple closed curve if and only if $gcd\{p_1, \ldots, p_g, q_1, \ldots, q_g\} = 1$.*

**Proof.** Suppose $gcd\{p_1, \ldots, q_g\} = 1$, then $X$ is part of an integral basis for $H_1(S_g)$. Now $H_1(S_g)$ supports a non-degenerate skew-symmetric bilinear form (the intersection pairing) and $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ is a symplectic basis for this pairing. Since the form is skew-symmetric, every vector is isotropic, i.e. $X \cdot X = 0$. From the non-degeneracy of the form, choose $Y \neq 0$ so that $X \cdot Y = 1$. It is easy to see that the form restricted to the subspace generated by $X$ and $Y$ is non degenerate. It follows easily that $H_1(S_g) \simeq \{X, Y\} \oplus \{X, Y\}^\perp$. Hence $X$ is part of a symplectic basis for $H_1(S_g)$. We may assume $X = M(\alpha_1)$ where $M$ is a symplectic transformation on $H_1(S_g)$. Now every symplectic automorphism of $H_1(S_g)$ is induced by an automorphism of $\pi_1(S_g)$. This is most easily demonstrated by exhibiting a set of generators for the symplectic automorphisms of $Z^{2g}$ and showing each is induced by an automorphism of $\pi_1(S_g)$ [1]. By Nielsen's theorem [2], every automorphism of $\pi_1(S_g)$ is induced by a homeomorphism of $S_g$. If $h$ is a homeomorphism $h: S_g \to S_g$ such that $h_* = M$ and $\lambda: S^1 \to S_g$ is a simple closed curve representing $\alpha_1$ then $h\lambda$ is a simple closed curve representing $X$.



https://doi.org/10.4153/CMB-1976-058-4 Published online by Cambridge University Press

Conversely, suppose $\lambda \in \pi_1(S_g)$ contains a simple closed curve and $\lambda$ represents $X$ in $H_1(S_g)$. Since $\lambda$ represents $X \neq 0$, $\lambda$ contains a non separating simple closed curve. From results of Zieschang [3], there exists an automorphism $\alpha: \pi_1(S_g) \to \pi_1(S_g)$ mapping $a_1$ to $\lambda$. Then $\alpha_*$ is a symplectic automorphism and maps $\alpha_1$ to $X$. Therefore $X$ is a part of a symplectic basis, hence a basis, so the $gcd\{p_1, \ldots, q_g\}$ must be one.

Let $w = w(a_1, \ldots, b_g) \in \pi_1(S_g)$ and let $N(w, a_1) =$ sum of the exponents (positive and negative) of occurrences of $a_i$ in $w$. Similarly define $N(w, b_i)$.

COROLLARY. *If* $gcd\{N(w, a_1), N(w, a_2), \ldots, N(w, b_g)\} \neq 1$ *then the coset* $w[\pi_g, \pi_g]$ *does not contain a simple closed curve.*

**Proof.** The image of $w$ in $H_1(S_g)$ is $\sum_1^q [N(w, a_i)\alpha_i + N(w, b_i)\beta_i]$ and the coset $w[\pi_g, \pi_g]$ contains all possible representatives of the image of $w$.

## REFERENCES

1. W. Magnus, A. Karrass, and D. Solitar, *Combinatorial Group Theory*, Interscience, New York, London, 1966.

2. J. Nielsen, Untersuchungen zur Topologie des gescholossenen Zweiseitigen Flachen I, Acta Math. **50** (1927), pp. 184–358.

3. H. Zieschang, Algorithmen für einfache Kurven auf Flächen: Math. Scand., **17** (1965), 17–40.

UNIVERSITY OF MARYLAND
COLLEGE PARK, MARYLAND

https://doi.org/10.4153/CMB-1976-058-4 Published online by Cambridge University Press