TOPOLOGIES ON BOOLEAN ALGEBRAS DEFINED BY IDEALS AND DUAL IDEALS

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Introduction. In the paper [5], Rema used the well-known fact that in a Boolean algebra $\mathscr{B} = \langle B; \vee, \wedge, '; 0, 1\rangle$ the binary operation $d: B \times B \to B$ defined by $d(a, b) = (a \wedge b') \vee (b \wedge a')$ is a " metric " operation to show that, if D is any dual ideal of $\mathscr{B}$, then the sets $U_p = \{(x, y): d(x, y) \leqq p\}$, where $p \in D$, form a base for a uniformity of $\mathscr{B}$, the resulting topological space $\langle B; T[D]\rangle$ being called an auto-topologized Boolean algebra. Recently, Kent and Atherton [1, 4] exhibited a family of topologies on an arbitrary lattice $\mathscr{L}$ defined in terms of ideals and dual ideals. More specifically, if I and D are respectively an ideal and a dual ideal of $\mathscr{L}$, then the $T[I:D]$ topology on $\mathscr{L}$ is the topology defined by taking the sets of the form $a^* \cap b^+$, where $a \in I$, $b \in D$, $a^* = \{x \in \mathscr{L} : x \geqq a\}$ and $b^+ = \{x \in \mathscr{L} : x \leqq b\}$, as sub-base for the open sets. It is these topologies that are studied in this paper.

It is first shown that a $T[I:D]$ topology on a Boolean algebra $\mathscr{B}$ is an auto-topology if and only if I is the " Boolean-complement " of D. The property of a topology on $\mathscr{B}$ being a $T[I:D]$ (auto-)topology is shown to be " productive " as well as being " c-hereditary " in that, if S is a complete subalgebra of a Boolean algebra endowed with a $T[I:D]$ (auto-)topology, then the subspace topology on S is a $T[I:D]$ (auto-)topology. Necessary and sufficient conditions are then established for a $T[I:D]$ topology to be Hausdorff and employed to show that a Hausdorff $T[I:D]$ topology is totally disconnected whereas an auto-topology is Hausdorff if and only if it is totally disconnected. Various connectedness properties of $T[I:D]$ topologies are studied in some detail and it is shown, in particular, that such a topology is connected if and only if I is contained in the " lower section " of D and that an auto-topology $T[D]$ is locally connected if and only if D is a principal dual ideal. Finally, we show that a Boolean algebra admits a compact, Hausdorff $T[I:D]$ topology if and only if it is complete and atomic.

Notation and terminology. The topological concepts and results referred to throughout the paper can be found in [3], while the lattice-theoretic results are to be found in [2]. If S is a nonempty subset of a Boolean algebra $\mathscr{B}$, then we denote the set $\{a'; a \in S\}$ by S' and refer to it as the *Boolean complement* of S. The usual partial ordering of $\mathscr{B}$ will be denoted by $\leqq$ and $[a, b]$ will denote the interval $\{x \in \mathscr{B} : a \leqq x \leqq b\}$. For the sake of brevity we frequently write $a.b$ instead of $a \wedge b$ for the lattice meet of a and b and $a \vee b$ for the lattice join. The symbols $\subseteq$, $\cup$, $\cap$ will be reserved for set inclusion, union and intersection respectively.

1. THEOREM 1.1. *$T[I:D]$ is an auto-topology if and only if $D = I'$ and, when this condition is satisfied, $T[I:D] = T[D]$.*

Proof. Suppose that $T[I:D]$ coincides with the auto-topology $T[F]$ defined by the dual ideal F of $\mathscr{B}$; then the set $\{U_f[a]: f \in F\}$, where $U_f[a] = \{x : d(x, a) \leqq f\}$, forms a base for

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the $T[F]$ neighbourhood system of the point $a\in\mathscr{B}$, and the set $\{[a\wedge q, a\vee p]: p\in D, q\in I\}$ forms a base for the $T[I:D]$ neighbourhood system of $a\in\mathscr{B}$. Now $d(x,a)\leqq f\leftrightarrow a\wedge f'\leqq x\leqq a\vee f$, so that $U_f[a]=[a\wedge f', a\vee f]$, and it follows that $\forall a\in\mathscr{B}, \forall p\in D, \forall q\in I, \exists f\in F$ such that $[a\wedge f', a\vee f]\subseteq[a\wedge q, a\vee p]$. On taking $a=0$, we deduce that every element in D contains some element in F and this implies that $D\subseteq F$. Furthermore, on taking $a=1$, it follows that $\forall q\in I, \exists f\in F$ such that $q\leqq f'$, or equivalently $f\leqq q'$, and so $q'\in F$, which implies that $I\subseteq F'$. Similarly $\forall a\in\mathscr{B}, \forall f\in F, \exists p\in D$ and $q\in I$ such that $[a\wedge q, a\vee p]\subseteq[a\wedge f', a\vee f]$. Taking $a=0$, we have that $\forall f\in F, \exists p\in D$ such that $p\leqq f$ and so $f\in D$, which shows that $F\subseteq D$. Again, taking $a=1$, we have that $\forall f\in F, \exists q\in I$ such that $f'\leqq q$ and this implies that $F'\subseteq I$. In summary then, $D=F$ and $I=F'$; whence $D=I'$.

The converse has been established by Atherton [1] who showed that, if this condition is satisfied, then $T[I:D]=T[D]$.

A property $\mathscr{P}$ of a topology on a Boolean algebra is said to be *productive* if and only if the product of any family of Boolean algebras, each being endowed with a topology possessing the property $\mathscr{P}$, also possesses $\mathscr{P}$.

THEOREM 1.2. *The property of being a $T[I:D]$ topology is productive.*

Proof. Suppose that $\{\langle\mathscr{B}_\alpha; T[I_\alpha:D_\alpha]\rangle\}_{\alpha\in\Lambda}$ is an arbitrary family of Boolean algebras each endowed with a $T[I:D]$ topology. Let D be the subset of the direct product $\mathscr{B}$ of the $\mathscr{B}_\alpha$'s consisting of all functions $f\in\mathscr{B}$ with the property that $f(\alpha)=1_\alpha$, $\forall\alpha\in\Lambda$, except when α is in some finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of Λ, in which case $f(\alpha_i)\in D_{\alpha_i}$. Similarly, let I be the subset of $\mathscr{B}$ consisting of all functions $f\in\mathscr{B}$ with the property that $f(\alpha)=0_\alpha$, $\forall\alpha\in\Lambda$, except when α is in some finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of Λ, in which case $f(\alpha_i)\in I_{\alpha_i}$. Then it is easily shown that D is a dual ideal and I an ideal of $\mathscr{B}$, and we prove that the product topology $\prod_{\alpha\in\Lambda} T[I_\alpha:D_\alpha]$ on $\mathscr{B}$ coincides with the topology $T[I:D]$.

To this end, let $f\in U\in\prod_{\alpha\in\Lambda} T[I_\alpha:D_\alpha]$; then, by definition of the product topology, there exist open sets $U_{\alpha_j}\in T[I_{\alpha_j}:D_{\alpha_j}]$ ($\alpha_j\in\Lambda$, $1\leqq j\leqq m$) such that the corresponding sub-basic open sets $U^*_{\alpha_j}=\{f\in\mathscr{B}: f(\alpha_j)\in U_{\alpha_j}\}$ in $\prod_{\alpha\in\Lambda} T[I_\alpha:D_\alpha]$ satisfy $f\in\bigcap_{j=1}^m U^*_{\alpha_j}\subseteq U$. Now, since $f(\alpha_j)\in U_{\alpha_j}$ and $U_{\alpha_j}\in T[I_{\alpha_j}:D_{\alpha_j}]$, it follows that $\exists p_{\alpha_j}\in D_{\alpha_j}$ and $q_{\alpha_j}\in I_{\alpha_j}$ such that $[f(\alpha_j)\wedge p_{\alpha_j}, f(\alpha_j)\vee q_{\alpha_j}]\subseteq U_{\alpha_j}(1\leqq j\leqq n)$. Let $q\in\mathscr{B}$ be defined by $q(\alpha)=0_\alpha$, $\forall\alpha\in\Lambda$ except where $\alpha=\alpha_j$, when $q(\alpha_j)=q_{\alpha_j}(1\leqq j\leqq m)$, and let $p\in\mathscr{B}$ be defined by $p(\alpha)=1_\alpha$, $\forall\alpha\in\Lambda$ except where $\alpha=\alpha_j$, when $p(\alpha_j)=p_{\alpha_j}(1\leqq j\leqq m)$. Then $p\in D, q\in I$ and $[f\wedge q, f\vee p]$ is a $T[I:D]$-open neighbourhood of f which is contained in $\bigcap_{j=1}^m U^*_{\alpha_j}$; for if $g\in[f\wedge q, f\vee p]$, then, in particular, $f(\alpha_j)\wedge q(\alpha_j)\leqq g(\alpha_j)\leqq f(\alpha_j)\vee p(\alpha_j)$, so that $g(\alpha_j)\in[f(\alpha_j)\wedge q_{\alpha_j}, f(\alpha_j)\vee p_{\alpha_j}]$ $(1\leqq j\leqq m)$, whence $g\in\bigcap_{j=1}^m U^*_{\alpha_j}$. Thus $\prod_{\alpha\in\Lambda} T[I_\alpha:D_\alpha]\subseteq T[I:D]$.

Conversely, suppose that $f\in U\in T[I:D]$; then $\exists p\in D$, $q\in I$ such that $f\in[f\wedge q, f\vee p]\subseteq U$. Now suppose that $p(\alpha)=1_\alpha$ $\forall\alpha\in\Lambda-J$, $p(\alpha_j)\in D_{\alpha_j}$ $\forall\alpha_j\in J$, where J is a finite subset of Λ, and $q(\beta)=0_\beta$ $\forall\beta\in\Lambda-K$, $q(\beta_k)\in I_{\beta_k}$ $\forall\beta_k\in K$, where K is a finite subset of Λ. Let $L=J\cup K$ and, for

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each $\gamma \in L$, consider the sub-basic $\prod_{\alpha \in \Lambda} T[I_\alpha : D_\alpha]$-open set $U_\gamma^* = \{b \in \mathscr{B} : b(\gamma) \in U_\gamma\}$, where U_γ is the basic $T[I_\gamma : D_\gamma]$-open set $[f(\gamma) \wedge q(\gamma), f(\gamma) \vee p(\gamma)]$. Now $f \in \bigcap_{\gamma \in L} U_\gamma^* \subseteq [f \wedge q, f \vee p] \subseteq U$; for, if $g \in \bigcap_{\gamma \in L} U_\gamma^*$, then $g(\gamma) \in U_\gamma$, $\forall \gamma \in L$, or, equivalently, $f(\gamma) \wedge q(\gamma) \leqq g(\gamma) \leqq f(\gamma) \vee p(\gamma)$, $\forall \gamma \in L$, and, if $\alpha \in \Lambda - L$, so that $\alpha \in \Lambda - J$ and $\alpha \in \Lambda - K$, then $q(\alpha) = 0_\alpha$ and $p(\alpha) = 1_\alpha$, which implies that $f(\alpha) \wedge q(\alpha) \leqq g(\alpha) \leqq f(\alpha) \vee p(\alpha)$, $\forall \alpha \in \Lambda$, i.e., $g \in [f \wedge q, f \vee p]$. Hence $T[I : D] \subseteq \prod_{\alpha \in \Lambda} T[I_\alpha : D_\alpha]$ and therefore equality holds.

COROLLARY 1.3. *The property of being an auto-topology is productive.*

Proof. If each of the topologies $T[I_\alpha : D_\alpha]$ in the theorem is an auto-topology, then, by Theorem 1.1, $D_\alpha = I'_\alpha$ $\forall \alpha \in \Lambda$, and it is easily shown that the associated ideal I and dual idea D of $\mathscr{B}$ satisfy $D = I'$. Hence the product topology on $\mathscr{B}$ is an auto-topology.

A property $\mathscr{P}$ of a topology on a Boolean algebra is said to be *c-hereditary* if and only if the subspace topology on any complete subalgebra of a Boolean algebra endowed with a topology possessing the property $\mathscr{P}$ also possesses $\mathscr{P}$.

THEOREM 1.4. *The property of being a $T[I : D]$ topology is c-hereditary.*

Proof. Let S be a complete subalgebra of the Boolean algebra $\mathscr{B}$; for each $p \in D$, let $t_p = \bigvee(p^+ \cap S)$ and form the dual ideal D_s in S generated by the set $T_D = \{t_p : p \in D\}$. Observe that, since T_D is closed under finite meets, $D_s = \{s \in S : s \geqq t_p$ for some $p \in D\}$. For each $q \in I$, let $t_q = \bigwedge(q^* \cap S)$, form the ideal I_s in S generated by the set $T_I = \{t_q : q \in I\}$ and, once again, observe that $I_s = \{s \in S : s \leqq t_q$ for some $q \in I\}$. We show that the subspace topology $T[I : D]/S$ on S is identical with $T[I_s : D_s]$. Let $a \in U \in T[I : D]/S$; then $\exists p \in D, q \in I$ such that $a \in [a \wedge q, a \vee p] \cap S \subseteq U$. Consider the interval $[a \wedge t_q, a \vee t_p]_s$ in S, i.e., $\{s \in S : a \wedge t_q \leqq s \leqq a \vee t_p\}$; then, since $t_p \leqq p$ and $t_q \geqq q$, it follows that $[a \wedge t_q, a \vee t_p]_s = [a \wedge q, a \vee p] \cap S$. Hence $T[I : D]/S \subseteq T[I_s : D_s]$.

Conversely, let $a \in U \in T[I_s : D_s]$; then $\exists p_1 \in D_s, q_1 \in I_s$ such that $[a \wedge q_1, a \vee p_1]_s \subseteq U$. But $p_1 \in D_s \leftrightarrow p_1 \geqq t_p$ for some $p \in D$, and $q_1 \in I_s \leftrightarrow q_1 \leqq t_q$ for some $q \in I$. We show that $[a \wedge q, a \vee p] \cap S \subseteq [a \wedge q_1, a \vee p_1]_s$. Let $s \in [a \wedge q, a \vee p] \cap S$, so that $s \in S$ and $s \wedge a' \leqq p$; then it follows that $s \wedge a' \leqq t_p \leqq p_1$, or, equivalently, $s \leqq a \vee p_1$. Similarly, since $a' \vee s \geqq q$ and $a' \vee s \in S$, it follows that $a' \vee s \geqq t_q \geqq q_1$, or, equivalently, $a \wedge q_1 \leqq s$. Hence $s \in [a \wedge q_1, a \vee p_1]_s$ and therefore $T[I_s : D_s] \subseteq T[I : D]/S$.

COROLLARY 1.5. *The property of being an auto-topology is c-hereditary.*

Proof. If the topology $T[I : D]$ of the theorem is an auto-topology, then $D = I'$ and it suffices to show that $D_s = I'_s$. To this end let $s \in D_s$; then $\exists p \in D$ such that $s \geqq t_p = \bigvee(p^+ \cap S)$. Now, since $p = q'$ for some $q \in I$, $t'_p = \bigwedge(p^+ \cap S)' = \bigwedge(q^* \cap S) = t_q$ and so $s' \leqq t_q$, which implies that $s' \in I_s$, or, equivalently, $s \in I'_s$. Hence $D_s \subseteq I'_s$. Similarly, if $s \in I'_s$, so that $s = r'$ for some $r \in I_s$, then $\exists q \in I$ such that $t'_q \leqq s$. Now $q = p'$ for some $p \in D$, and so $t'_q = [\bigwedge(q^* \cap S)]' = \bigvee(q^* \cap S)' = \bigvee(p^+ \cap S) = t_p$, which implies that $t_p \leqq s$ and therefore $s \in D_s$. Hence $I'_s \subseteq D_s$, completing the proof.

2. Connectedness properties. Prior to establishing necessary and sufficient conditions for $T[I:D]$ to be Hausdorff, we recall that an ideal (dual ideal) in the pseudo-complemented lattice $\mathscr{L}$ of all ideals (dual ideals) of a Boolean algebra $\mathscr{B}$ is said to be (algebraically) *dense* if and only if its pseudo-complement is the zero element of $\mathscr{L}$. We remark that an ideal I of $\mathscr{B}$ is dense if and only if its *upper section* $I^* = \{x \in \mathscr{B}: x \geqq q, \forall q \in I\}$ contains only the element 1, while a dual ideal D is dense if and only if its *lower section* $D^+ = \{x \in \mathscr{B}: x \leqq p, \forall p \in D\}$ contains only the element 0.

THEOREM 2.1. *The topology $T[I: D]$ is Hausdorff if and only if both I and D are dense.*

Proof. Suppose that $T[I:D]$ is Hausdorff and $x \in I^*$ but $x \neq 1$; then $\exists p_1 \in D$ and $\exists q_1, q_2 \in I$ such that $[q_2, 1] \cap [x \wedge q_1, x \vee p_1] = \emptyset$, which gives a contradiction on observing that the element $q = q_1 \vee q_2 \in I$ satisfies $q_2 \leqq q$ and $x \wedge q_1 = q_1 \leqq q \leqq x \leqq x \vee p_1$ and therefore lies in the intersection. Hence $I^* = \{1\}$. Similarly, suppose that $x \in D^+$ but $x \neq 0$; then $\exists p_1, p_2 \in D$ and $\exists q_1 \in I$ such that $[0, p_2] \cap [x \wedge q_1, x \vee p_1] = \emptyset$, which, on observing that $p = p_1 \wedge p_2 \in D$ satisfies $p \leqq p_2$ and $x \wedge q_1 \leqq x \leqq p \leqq x \vee p_1$, gives a contradiction. Hence $D^+ = \{0\}$.

Conversely, suppose that both I and D are dense, but $T[I:D]$ is not Hausdorff; then there exist distinct points $a, b \in \mathscr{B}$ such that every open neighbourhood of a meets every open neighbourhood of b. Hence $[a \wedge q, a \vee p] \cap [b \wedge q, b \vee p] \neq \emptyset, \forall p \in D, \forall q \in I$. But $\exists x \in \mathscr{B}$ satisfying

$$\begin{aligned} x \in [a \wedge q, a \vee p] \cap [b \wedge q, b \vee p] &\leftrightarrow aq \vee bq \leqq x \leqq (a \vee p)(b \vee p) \\ &\leftrightarrow (a' \vee b')p'x \vee (a \vee b)qx' = 0 \\ &\leftrightarrow (a' \vee b')(a \vee b)p'q = 0 \\ &\leftrightarrow d(a, b)p'q = 0 \end{aligned}$$

and so it follows that $d(a, b)q \leqq p, \forall p \in D, \forall q \in I$. Whence

$$\begin{aligned} d(a, b)q \in D^+ = \{0\}, \forall q \in I &\leftrightarrow q \leqq d'(a, b), \forall q \in I \\ &\leftrightarrow d'(a, b) \in I^* = \{1\} \\ &\leftrightarrow d(a, b) = 0 \\ &\leftrightarrow a = b, \end{aligned}$$

giving a contradiction and therefore proving that $T[I:D]$ is Hausdorff.

COROLLARY 2.2. *An auto-topology $T[D]$ is Hausdorff if and only if D is a dense dual ideal.*

THEOREM 2.3. *If $T[I: D]$ is Hausdorff, then it is totally disconnected.*

Proof. It is, of course, well known that a Hausdorff, zero-dimensional space is totally disconnected and so, in proving the theorem, it suffices to show that each basic open set $[a \wedge q, a \vee p]$ $(p \in D, q \in I)$ is clopen. Now $x \in \operatorname{Cl}[a \wedge q_1, a \vee p_1]$, the closure of $[a \wedge q_1, a \vee p_1]$,

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if and only if every neighbourhood of x meets $[a \wedge q_1, a \vee p_1]$ or, equivalently, $\mathscr{I} = [x \wedge q, x \vee p] \cap [a \wedge q_1, a \vee p_1] \neq \emptyset, \forall p \in D, \forall q \in I$. But

$$\begin{aligned}
\exists y \in \mathscr{I} &\leftrightarrow xq \vee aq_1 \leqq y \leqq (x \vee p)(a \vee p_1) \\
&\leftrightarrow (x'p' \vee a'p'_1)y \vee (xq \vee aq_1)y' = 0 \\
&\leftrightarrow (x'p' \vee a'p'_1)(xq \vee aq_1) = 0 \\
&\leftrightarrow aq_1 x'p' = 0 \quad \text{and} \quad a'p'_1 xq = 0 \\
&\leftrightarrow ax'q_1 \leqq p \quad \text{and} \quad q \leqq a \vee x' \vee p_1.
\end{aligned}$$

Hence $\mathscr{I} \neq \emptyset, \forall p \in D, \forall q \in I \leftrightarrow ax'q_1 \leqq p, \forall p \in D$ and $q \leqq a \vee x' \vee p_1, \forall q \in I \leftrightarrow ax'q_1 \in D^+ = \{0\}$ and $a \vee x' \vee p_1 \in I^* = \{1\} \leftrightarrow ax'q_1 = 0$ and $a'p'_1 x = 0 \leftrightarrow a \wedge q_1 \leqq x \leqq a \vee p_1$. It follows now that $[a \wedge q_1, a \vee p_1]$ is clopen and the theorem is proved.

COROLLARY 2.4. *An auto-topology is Hausdorff if and only if it is totally disconnected.*

Proof. It is well known that cmp(a), the component of a, is contained in the intersection of all clopen sets containing the point a and so, since the $T[D]$-open sets $[0, p]$ $(p \in D)$ are clopen, it follows that $\text{cmp}(0) \subseteq \bigcap_{p \in D} [0, p] = D^+$. We show that the subspace D^+ is indiscrete and therefore connected. To this end, let V be an open set containing the element l in the subspace D^+, so that $V = U \cap D^+$ for some $T[D]$-open set U containing l. Then $\exists p \in D$ such that $[l \wedge p', l \vee p] \cap D^+ \subseteq V$. Furthermore, $D^+ \subseteq [l \wedge p', l \vee p]$; for, if $x \in D^+$, so that $x \leqq p, \forall p \in D$, then $d(x, l) \leqq x \vee l \leqq p$ and so $x \in [l \wedge p', l \vee p]$. It follows now that $D^+ = V$ and so the only open sets in the subspace D^+ are itself and the empty set. Hence D^+ is an indiscrete subspace. Now cmp(0) is the largest connected set containing the element 0 and so, by the connectedness of D^+, $\text{cmp}(0) = D^+$. Hence, if $\langle B; T[D] \rangle$ is totally disconnected, $D^+ = \text{cmp}(0) = \{0\}$ and it follows, by Corollary 2.2, that $\langle B; T[D] \rangle$ is Hausdorff.

THEOREM 2.5. *The topology $T[I : D]$ is connected if and only if $I \subseteq D^+$.*

Proof. Suppose that $T[I : D]$ is connected. Let I_m be an arbitrary maximal ideal in $\mathscr{B}$ and let $p \in D$, $q \in I$ be given; then the set $\{[a \wedge q, a \vee p] : a \in I_m\}$ forms an open cover of I_m and, by a well-known property of maximal ideals, the set $\{[b \wedge q, b \vee p] : b \in I'_m\}$ forms an open cover of $\mathscr{B} - I_m$. Hence the open sets of $U = \bigcup_{a \in I_m} [a \wedge q, a \vee p]$, $V = \bigcup_{b \in I_m'} [b \wedge q, b \vee p]$ cover $\mathscr{B}$ and therefore cannot be disjoint. This implies that $\exists\, a, c \in I_m$ such that $[a \wedge q, a \vee p] \cap [c' \wedge q, c' \vee p] \neq \emptyset \leftrightarrow \exists\, x \in \mathscr{B}$ such that

$$\begin{aligned}
(a \vee c')q \leqq x \leqq ac' \vee p &\leftrightarrow (a' \vee c)p'x \vee (a \vee c')qx' = 0 \\
&\leftrightarrow p'q(a' \vee c)(a \vee c') = 0 \\
&\leftrightarrow qp'd(a, c') = 0 \leftrightarrow qp' \leqq d(a, c) \in I_m.
\end{aligned}$$

Hence $qp' \in I_m$, so that, since I_m is an arbitrary maximal ideal and the intersection of all maximal ideals of $\mathscr{B}$ contains only the element 0, it follows that $q \leqq p, \forall p \in D, \forall q \in I$. Therefore $I \subseteq D^+$.

Conversely, suppose that $I \subseteq D^+$ and let C be any clopen subset of $\langle B; T[I : D] \rangle$. Then either $C = \emptyset$ or $\exists\, a \in C$. In the latter case suppose that $\exists\, b \in \mathscr{B} - C$. Then $\exists\, p_1 \in D$, $q_1 \in I$ such that $[b \wedge q_1, b \vee p_1] \subseteq \mathscr{B} - C$. Also, since C is open, $\exists p_2 \in D$, $q_2 \in I$ such that $[a \wedge q_2, a \vee p_2] \subseteq C$ and so these intervals are disjoint. But $I \subseteq D^+ \leftrightarrow q \leqq p, \forall p \in D, \forall q \in I$ and we observe that the element $s = bq_1 \vee aq_2$ lies in their intersection, giving a contradiction. Therefore, if C is clopen, then either $C = \emptyset$ or $C = \mathscr{B}$; whence the space is connected.

THEOREM 2.6. *An auto-topology $T[D]$ is locally connected if and only if D is a principal dual ideal.*

Proof. Suppose that $T[D]$ is a locally connected auto-topology on the Boolean algebra $\mathscr{B}$. Then there exists a base σ for $T[D]$ consisting of connected open sets. Let U be any member of σ containing the least element of $\mathscr{B}$. Then $\exists p \in D$ such that $p^+ \subseteq U$, which, since p^+ is a nonempty clopen set and therefore clopen in the subspace U of $T[D]$, implies that $p^+ = U$. Now suppose that D is non-principal. Then $\exists p_1 \in D$ such that $p_1 < p$ and so the clopen set p_1^+ is properly contained in the connected set p^+, giving a contradiction. Hence D is a principal dual ideal of $\mathscr{B}$.

Conversely, suppose that $T[D]$ is induced by the principal dual ideal $D = p^*$ generated by p. Then it is is obvious that the set $\{[a \wedge p', a \vee p] : a \in \mathscr{B}\}$ forms a base for $T[D]$. Furthermore these intervals are connected sets; for otherwise there exists a nonempty clopen set U_a, containing the element a, in the subspace $[a \wedge p', a \vee p]$ and distinct from it, which, since U_a must be $T[D]$-open, implies that $[a \wedge p', a \vee p] \subseteq U_a$. It follows that $T[D]$ is locally connected.

The preceding theorem characterizes principal dual ideals of a Boolean algebra $\mathscr{B}$ in terms of a property of the associated auto-topologies, while Corollary 2.2. may be regarded as a characterization of dense dual ideals of $\mathscr{B}$. The following theorem characterizes, in the same way, the maximal dual ideals of $\mathscr{B}$.

THEOREM 2.7. *If $T[D]$ is an auto-topology on $\mathscr{B}$, induced by the dual ideal D, then D is maximal if and only if $\langle B; T[D]\rangle$ is non-discrete and, for all $a \in \mathscr{B}$, either a^* or a^+ is an open set.*

Proof. If D is a maximal dual ideal of $\mathscr{B}$, then it is proper, so that $T[D]$ is non-discrete; and, furthermore, if $a \in \mathscr{B}$, then either $a \in D$ or $a' \in D$. In the first case the set $U_a[a] = a^+$ is open, while in the second the set $U_{a'}[a] = a^*$ is open.

Conversely, suppose that $T[D]$ is a non-discrete auto-topology on $\mathscr{B}$ with the property that, for all $a \in \mathscr{B}$, either a^+ or a^* is open. Then D is proper and, furthermore, if a^+ is open, $\exists p \in D$ such that $U_p[a] = [a \wedge p', a \vee p] \subseteq a^+$, which implies that $a \vee p \leqq a$, or, equivalently, $p \leqq a$, and so $a \in D$. In the event that a^* is open, $\exists p \in D$ such that $U_p[a] = [a \wedge p', a \vee p] \subseteq a^*$, which implies that $a \wedge p' \geqq a$, or, equivalently, $p \leqq a'$, and so $a' \in D$. Hence D is a proper dual ideal possessing the property that, for all $a \in \mathscr{B}$, either $a \in D$ or $a' \in D$ and D is therefore maximal.

3. Compact Hausdorff $T[I : D]$ topologies.

THEOREM 3.1. *If a Boolean algebra admits a compact, Hausdorff $T[I : D]$ topology, then it is complete.*

Proof. Let X be any nonempty subset of a Boolean algebra $\mathscr{B}$ admitting a compact, Hausdorff $T[I:D]$ topology and let $\mathscr{L}_X$ be the set of all lower bounds of X; then $\mathscr{L}_X$ is an ideal of $\mathscr{B}$ and consequently the identity map $n:\mathscr{L}_X \to \mathscr{L}_X$ is a net in $\mathscr{L}_X$ which, since $\langle B; T[I:D]\rangle$ is compact, has a cluster point c. Let $p \in D$, $q \in I$ be given; then, since n is frequently in the open neighbourhood $[c \wedge q, c \vee p]$, it follows that $\forall a \in \mathscr{L}_X$, $\exists b \in \mathscr{L}_X$ such that $b \geqq a$ and $b \in [c \wedge q, c \vee p]$. Whence $c \wedge q < b \leqq c \vee p$, from which it follows that $c \wedge q \in \mathscr{L}_X$ and $a \leqq c \vee p$, $\forall a \in \mathscr{L}_X$. Hence, since p and q were arbitrarily chosen, it follows that

$$c \wedge q \leqq x, \forall x \in X, \forall q \in I \leftrightarrow q \leqq c' \vee x, \forall x \in X, \forall q \in I \leftrightarrow c' \vee x \in I^*, \forall x \in \mathscr{L}_X.$$

Also

$$a \leqq c \vee p, \forall a \in \mathscr{L}_X, \forall p \in D \leftrightarrow a \wedge c' \leqq p, \forall a \in \mathscr{L}_X, \forall p \in D \leftrightarrow a \wedge c' \in D^+.$$

But $T[I:D]$ is Hausdorff, or, equivalently, $I^+ = \{1\}$, $D^* = \{0\}$, and so $c' \vee x = 1$, $\forall x \in X$, and $a \wedge c' = 0$, $\forall a \in \mathscr{L}_X$, i.e., $c \in \mathscr{L}_X$ and $a \leqq c$, $\forall a \in \mathscr{L}_X$, so that c is the greatest lower bound on the set X. It follows that $\mathscr{B}$ is complete.

THEOREM 3.2. *A Boolean algebra admits a compact, Hausdorff $T[I:D]$ topology if and only if it is complete and atomic.*

Proof. Let $\mathscr{B}$ be a Boolean algebra and suppose that $\langle B; T[I:D]\rangle$ is compact and Hausdorff. Then, by the preceding theorem, $\mathscr{B}$ is complete and it remains only to show that $\mathscr{B}$ is atomic. To this end, let p be an arbitrary element in the dual ideal D distinct from the element 1. Then $\exists q \in I$ such that $q \nleqq p$; otherwise $p \geqq q$, $\forall q \in I$, so that $p \in I^* = \{1\}$, whence $p = 1$. Let I_p be any prime ideal of $\mathscr{B}$ such that $p \in I_p$ but $q \notin I_p$, the existence of such an ideal being well known. Now

$$\mathscr{C} = \{[a \wedge q, a \vee p], [b \wedge q, b \vee p] : a \in I_p, b \in \mathscr{B} - I_p\}$$

is an open cover of $\mathscr{B}$ and so, since $T[I:D]$ is compact, $\exists$ a finite sub-cover

$$\mathscr{C}^* = \{[a_i \wedge q, a_i \vee p], [b_j \wedge q, b_j \vee p] : 1 \leqq i \leqq m, 1 \leqq j \leqq n\}$$

of $\mathscr{B}$. We assert that $\mathscr{C}^{**} = \{[a_i \wedge q, a_i \vee p] : 1 \leqq i \leqq m\}$ is an open cover of I_p; for, if not, $\exists a \in I_p$ such that $a \in [b_j \wedge q, b_j \vee p]$ for some j. But $a \in I_p$ and $b_j \wedge q \leqq a$ implies that $b_j \wedge q \in I_p$, which, since I_p is prime, implies that either $b_j \in I_p$ or $q \in I_p$, both of which give a contradiction. Hence $I_p \subseteq \bigcup_{i=1}^{n} [a_i \wedge q, a_i \vee p]$ so that $x \in I_p \to x \leqq \bigvee_{i=1}^{n} (a_i \vee p) = p \vee \bigvee_{i=1}^{n} a_i \in I_p$. Therefore I_p is a principal ideal of $\mathscr{B}$ generated by m, say. But an ideal in $\mathscr{B}$ is prime if and only if it is maximal and so it follows that m is a maximal element in $\mathscr{B}$. Furthermore, since the complement of a maximal element in $\mathscr{B}$ is an atom, we have shown that $\forall p \in D$ $(p \neq 1)$, $\exists$ an atom $a \leqq p'$.

Let a_p be the join of all atoms contained in p', which exists since $\mathscr{B}$ is complete; we show that $p' = a_p$. For, if $p' > a_p > 0$, let x be the relative complement of a_p in the Boolean interval $[0, p']$, so that $0 < x < p'$ and $a_p \wedge x = 0$. Then $p < x'$, which implies that $x' \in D$ and $x' \neq 1$. Therefore $\exists$ an atom $b \leqq x'' = x$, whence $b < p'$, so that b is an atom contained in p', which implies that $b \leqq a_p$. Then $0 < b \leqq a_p \wedge x = 0$, so that $b = 0$, giving a contradiction. Hence $\forall p \in D$, p' is the join of all atoms it contains. Now we show that every element of $\mathscr{B}$ contains

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an atom. Since $T[I:D]$ is Hausdorff, $D^+ = \{0\}$ and therefore, since $\bigwedge_{p \in D} p \in D^+$, it follows that $\bigwedge_{p \in D} p = 0$, which implies that $\bigvee_{p \in D} p' = 1$. Each p' is, as we have shown, a join of atoms of $\mathscr{B}$ and therefore the element 1 is the join of all atoms of $\mathscr{B}$. Let $\mathscr{A}$ be the set of all atoms of $\mathscr{B}$ and suppose that some element $x \in \mathscr{B}$ contains no member of $\mathscr{A}$. Then $a \wedge x = 0$, $\forall a \in \mathscr{A}$ and so $0 = \bigvee_{a \in \mathscr{A}} (a \wedge x) = x \wedge \bigvee_{a \in \mathscr{A}} a = x \wedge 1 = x$. Therefore every nonzero element of $\mathscr{B}$ contains an atom and so $\mathscr{B}$ is atomic.

Conversely, if $\mathscr{B}$ is complete and atomic, or, equivalently, $\mathscr{B} = 2^N$ for some cardinal N, then, since each two-element Boolean algebra endowed with the discrete topology is a $T[I:D]$ topologized Boolean algebra and the property of being such a topology is productive, it follows that $\mathscr{B}$ admits a compact, Hausdorff $T[I:D]$ topology.

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