AN ALGORITHM TO COMPUTE THE CANONICAL BASIS
OF AN IRREDUCIBLE MODULE
OVER A QUANTIZED ENVELOPING ALGEBRA

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Abstract

The paper describes an algorithm to compute the canonical basis of an irreducible module over a quantized enveloping algebra of a finite-dimensional semisimple Lie algebra. The algorithm works for any module that is constructed as a submodule of a tensor product of modules with known canonical bases.

1. Introduction

In this paper we consider the problem of constructing the canonical basis (see [17]) of an irreducible module over a quantized enveloping algebra. There are several possible ways to approach this problem, and they may depend on how the module is constructed. In [4] an algorithm is described that works for any module, provided that we have a method for computing the action of elements of the algebra. In [11] and [12] the irreducible module is first constructed as a submodule of a tensor product of other modules. Then, using the known canonical bases of these other modules, an algorithm is described for constructing the canonical basis of the submodule.

Since constructing irreducible modules as submodules of tensor products can be quite efficient (see [5]), it would be worthwhile to have an algorithm tailored to this situation. Therefore, in this paper we take the second approach above. In fact, we describe an algorithm that is very similar to those in [11, 12]. The main difference is that we do not assume that the root system is of a certain type. The algorithm given here works for all types, assuming that somehow we know the canonical bases of the fundamental modules. These can, for instance, be constructed using the algorithm of [4].

This paper is organised as follows. In Section 2 the theoretical concepts and the notation used in the paper are introduced. Then in Section 3 a result is described concerning the form of the elements of the canonical basis of a tensor product. In Section 4 this is used, along with the description of a monomial basis of an irreducible module (from [10]), to give an algorithm for constructing the canonical basis. Next, in Section 5 this algorithm is compared to the algorithm from [11] in the A_n-case. It is shown that in this case the two algorithms are very similar (but not exactly the same). In the final section, some examples of practical experiences with the algorithm are reported.

2. Preliminaries

In this section we briefly sketch the concepts and notation that we will be using. Our main reference is [6].

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. By Φ we denote the root system of $\mathfrak{g}$, and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ will denote a fixed set of simple roots of Φ. Let W denote the Weyl group of Φ, which is generated by the simple reflections $s_i = s_{\alpha_i}$ for $1 \leqslant i \leqslant l$. Let $\mathbb{R}\Phi$ be the vector space over $\mathbb{R}$ spanned by Φ. On $\mathbb{R}\Phi$ we fix a W-invariant inner product $(,)$ such that $(\alpha, \alpha) = 2$ for short roots α. This means that $(\alpha, \alpha) = 2, 4, 6$ for $\alpha \in \Phi$.

We work over the field $\mathbb{Q}(q)$. For $\alpha \in \Phi$, we set $q_\alpha = q^{(\alpha, \alpha)/2}$. For $n \in \mathbb{Z}$, we set $[n]_\alpha = q_\alpha^{-n+1} + q_\alpha^{-n+3} + \ldots + q_\alpha^{n-1}$. Also $[n]_\alpha! = [n]_\alpha [n-1]_\alpha \ldots [1]_\alpha$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_\alpha = \frac{[n]_\alpha!}{[k]_\alpha![n-k]_\alpha!}.$$

Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be a simple system of Φ. Then the quantized enveloping algebra $U_q = U_q(\mathfrak{g})$ is the associative algebra (with an identity element) over $\mathbb{Q}(q)$ generated by F_α, K_α, K_α^{-1} and E_α for $\alpha \in \Delta$, subject to the following relations:

$$K_\alpha K_\alpha^{-1} = K_\alpha^{-1} K_\alpha = 1,$$
$$K_\alpha K_\beta = K_\beta K_\alpha,$$
$$E_\beta K_\alpha = q^{-(\alpha, \beta)} K_\alpha E_\beta,$$
$$K_\alpha F_\beta = q^{-(\alpha, \beta)} F_\beta K_\alpha,$$
$$E_\alpha F_\beta = F_\beta E_\alpha + \delta_{\alpha, \beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}},$$
$$\sum_{k=0}^{1-\langle \beta, \alpha^\vee \rangle} (-1)^k \begin{bmatrix} 1 - \langle \beta, \alpha^\vee \rangle \\ k \end{bmatrix}_\alpha E_\alpha^{1-\langle \beta, \alpha^\vee \rangle - k} E_\beta E_\alpha^k = 0,$$
$$\sum_{k=0}^{1-\langle \beta, \alpha^\vee \rangle} (-1)^k \begin{bmatrix} 1 - \langle \beta, \alpha^\vee \rangle \\ k \end{bmatrix}_\alpha F_\alpha^{1-\langle \beta, \alpha^\vee \rangle - k} F_\beta F_\alpha^k = 0,$$

where the last two relations hold for all $\alpha \neq \beta$.

Let U^-, U^0 and U^+ be the subalgebras of U_q generated, respectively, by F_α for $\alpha \in \Delta$, $K_\alpha^{\pm 1}$ for $\alpha \in \Delta$, and E_α for $\alpha \in \Delta$. Then, as a vector space, $U_q \cong U^- \otimes U^0 \otimes U^+$ (see [**6**, Theorem 4.21]). Let $\nu = \sum_k a_k \alpha_k$ with $a_k \in \mathbb{Z}_{\geqslant 0}$. Then we let U_ν^+ be the subspace of U^+ spanned by all $E_{\alpha_{i_1}} \ldots E_{\alpha_{i_r}}$ such that $\alpha_{i_1} + \ldots + \alpha_{i_r} = \nu$. Similarly, U_ν^- denotes the subspace of U^- spanned by all $F_{\alpha_{i_1}} \ldots F_{\alpha_{i_r}}$ such that $\alpha_{i_1} + \ldots + \alpha_{i_r} = \nu$.

We denote by $\lambda_1, \ldots, \lambda_l$ the fundamental weights, and by $P = \mathbb{Z}\lambda_1 + \ldots + \mathbb{Z}\lambda_l$, the weight lattice. Also, $P^+ = \mathbb{Z}_{\geqslant 0}\lambda_1 + \ldots + \mathbb{Z}_{\geqslant 0}\lambda_l$ is the set of dominant weights. Now, for every dominant $\lambda \in P^+$, there is an irreducible U_q-module $V(\lambda)$. The module $V(\lambda)$ is spanned by vectors v_μ for $\mu \in P$, with $K_\alpha \cdot v_\mu = q^{(\mu, \alpha)} v_\mu$. These v_μ are called *weight-vectors of weight* μ. Among them there is the vector v_λ (which is unique up to scalar multiples), with $U^+ \cdot v_\lambda = 0$. This v_λ is called the *highest-weight vector*. We find that $V(\lambda) = U^- \cdot v_\lambda$. Furthermore, every finite-dimensional irreducible U_q-module is isomorphic to a $V(\lambda)$; see [**6**, Theorem 5.10].

Let M be a finite-dimensional U_q-module. Then M has a crystal base $(\mathcal{M}, \mathcal{B})$ as defined in [**6**, 9.4]. Here $\mathcal{M}$ is an A-submodule of M, where A is the subring of $\mathbb{Q}(q)$ consisting of rational functions without pole at 0, and $\mathcal{B}$ is a basis of $\mathcal{M}/q\mathcal{M}$. For $\alpha \in \Delta$, we have the Kashiwara operators $\widetilde{F}_\alpha, \widetilde{E}_\alpha : \mathcal{M} \to \mathcal{M}$, and the induced operators $\widetilde{F}_\alpha, \widetilde{E}_\alpha : \mathcal{B} \to \mathcal{B} \cup \{0\}$; see [**6**, 9.2, 9.4].

There is a $\mathbb{Q}$-algebra isomorphism

$$^{-} : U_q \to U_q, \quad \text{with } \overline{q} = q^{-1}, \ \overline{E_\alpha} = E_\alpha, \ \overline{F_\alpha} = F_\alpha \text{ and } \overline{K_\alpha} = K_\alpha^{-1};$$

see [6, Proposition 11.9]. If $V(\lambda)$ is an irreducible U_q-module with highest weight λ, and fixed highest-weight vector v_λ, then we have an induced map

$$^{-} : V(\lambda) \to V(\lambda) \quad \text{by} \quad \overline{u \cdot v_\lambda} = \overline{u} \cdot v_\lambda.$$

(This is well defined by [6, Proposition 11.9].) The fixed choice for v_λ leads to a fixed crystal base $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ of $V(\lambda)$, where $\mathcal{L}(\lambda)$ is spanned by all $\widetilde{F}_{\alpha_{i_1}} \ldots \widetilde{F}_{\alpha_{i_r}} (v_\lambda)$, for $r \geqslant 0$. Now (for example, by [16, Theorem 1.8] and [6, Theorem 11.10]) there is a unique basis $\{G_\lambda(b) \mid b \in \mathcal{B}(\lambda)\}$ of $\mathcal{L}(\lambda)$, such that:

1. $G_\lambda(b) = b \mod q\mathcal{L}(\lambda)$;
2. $\overline{G_\lambda(b)} = G_\lambda(b)$.

This basis is called the *canonical basis of $V(\lambda)$*.

In the discussion that follows, when we write '*the* crystal base' or '*the* canonical basis' of $V(\lambda)$, we are always assuming that a fixed highest-weight vector v_λ has been chosen, which makes the choice of the crystal base or the canonical basis unique.

The crystal graph Γ_λ of the module $V(\lambda)$ is defined as follows. The points of Γ_λ are the elements of $\mathcal{B}(\lambda)$, and there is an edge $b_1 \xrightarrow{\alpha} b_2$ if $\widetilde{F}_\alpha(b_1) = b_2$. There is a very elegant method of computing the crystal graph, using Littelmann's path method. Let $\mathbb{R}P$ be the vector space over $\mathbb{R}$ spanned by the weights. Let Π be the set of piecewise linear paths $\pi : [0, 1] \to \mathbb{R}P$, such that $\pi(0) = 0$. For $\alpha \in \Delta$, Littelmann defined operators $e_\alpha, f_\alpha : \Pi \to \Pi \cup \{0\}$ (see [13, 14]), with the following property. Let $\lambda \in P^+$ be a dominant weight, and let π_λ be the path given by $\pi_\lambda(t) = \lambda t$ (that is, a straight line from the origin to λ). Let Π_λ be the set of all $f_{\alpha_{i_1}} \ldots f_{\alpha_{i_k}} (\pi_\lambda)$. Then all paths in Π_λ end in an element of P. Furthermore, the number of paths ending in $\mu \in P$ is equal to the dimension of the weight space with weight μ in the irreducible U_q-module $V(\lambda)$.

Now we consider the directed labeled graph with point set Π_λ, and edges $\pi_1 \xrightarrow{\alpha} \pi_2$ if $f_\alpha(\pi_1) = \pi_2$. This graph is isomorphic to the crystal graph of $V(\lambda)$; see [8].

Let M_1 and M_2 be U_q-modules; then $M_1 \otimes M_2$ is a U_q-module via the comultiplication of U_q. There are many possible ways of defining this, and the comultiplication $\Delta : U_q \to U_q \otimes U_q$ that we use is given by

$$\Delta(E_\alpha) = E_\alpha \otimes K_\alpha^{-1} + 1 \otimes E_\alpha;$$
$$\Delta(F_\alpha) = F_\alpha \otimes 1 + K_\alpha \otimes F_\alpha;$$
$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha;$$

see [6, 9.13].

3. Canonical bases of tensor products

Here we give a description of the canonical basis of a tensor product, following [6, Chapter 9] and [17, 27.3].

Let $V(\mu)$ and $V(\mu')$ be two irreducible U_q-modules, with highest weights μ and μ'. Let $C = \{v_1, \ldots, v_m\}$ and $C' = \{v_1', \ldots, v_n'\}$ be fixed canonical bases of $V(\mu)$ and $V(\mu')$. Denote the weights of v_i and v_j' by ν_i and ν_j', respectively. Then $\nu_i = \mu - \sum_k a_{k,i}\alpha_k$ with $a_{k,i} \in \mathbb{Z}_{\geqslant 0}$, and we say that $\sum_k a_{k,i}$ is the *height of v_i*. The height of v_j' is defined similarly.

We assume that the bases C and C' are ordered according to increasing height. So $v_1 = v_\mu$ and $v'_1 = v_{\mu'}$ are the highest-weight vectors.

Let $(\mathcal{L}, \mathcal{B})$ and $(\mathcal{L}', \mathcal{B}')$ be crystal bases of $V(\mu)$ and $V(\mu')$ respectively. Here $\mathcal{L}$ and $\mathcal{L}'$ are spanned by C and C', respectively. Furthermore, $\mathcal{B}$ and $\mathcal{B}'$ consist of the cosets $v_i \bmod q\mathcal{L}$ and $v'_j \bmod q\mathcal{L}'$. Now, by [6, Theorem 9.17], $(\mathcal{L} \otimes \mathcal{L}', \mathcal{B} \otimes \mathcal{B}')$ is a crystal base of $V(\mu) \otimes V(\mu')$.

We let Θ be the element from [17, 4.1], and then

$$P : U_q \otimes U_q \to U_q \otimes U_q$$

is the algebra homomorphism defined by $P(a \otimes b) = b \otimes a$. We set $\Theta^0 = P(\overline{\Theta})$; then $\Theta^0 = \sum_{\eta \geqslant 0} \Theta^0_\eta$, where the sum runs over all $\eta = \sum_k b_k \alpha_k$ with $b_k \in \mathbb{Z}_{\geqslant 0}$. Furthermore,

$$\Theta^0_\eta \in U^+_\eta \otimes U^-_\eta \quad \text{and} \quad \Theta^0_0 = 1 \otimes 1.$$

Now $\Psi_0 : V(\mu) \otimes V(\mu') \to V(\mu) \otimes V(\mu')$ is the map defined by $\Psi_0(v \otimes v') = \Theta^0(\overline{v} \otimes \overline{v}')$.

LEMMA 1. *We have* $\Psi_0(u \cdot v \otimes v') = \overline{u} \cdot \Psi_0(v \otimes v')$ *for all* $u \in U^-$. *Furthermore,* $\Psi^2_0(v \otimes v') = v \otimes v'$ *for all* $v \in V(\mu)$, $v' \in V(\mu')$.

Proof. This is the same as the proof of the corresponding results in [17, 27.3.1]. The difference is that we use a different comultiplication. Denoting the comultiplication used in [17] by Δ_L, we have $\Delta_L(F_\alpha) = F_\alpha \otimes K^{-1}_\alpha + 1 \otimes F_\alpha$. This means that for $u \in U^-$, we have
$$\Delta(u) = P(\overline{\Delta}_L(u)), \quad \text{where} \quad \overline{\Delta}_L(u) = \overline{\Delta_L(\overline{u})}.$$

The property that $\Delta_L(u)\Theta = \Theta\overline{\Delta}_L(u)$ (see [17, Theorem 4.1.2]) now translates to

$$\Delta(u)\Theta^0 = \Theta^0\overline{\Delta}(u),$$

where $\overline{\Delta}$ is defined similarly to $\overline{\Delta}_L$. From this, the first statement follows. The second follows from $\Theta^0\overline{\Theta}^0 = 1 \otimes 1$; see [17, Corollary 4.1.3]. $\qquad\square$

We define a partial order on the $v_i \otimes v'_j$. We set $v_i \otimes v'_j < v_k \otimes v'_l$ if and only if $i < k$, $j > l$, and $v_i + v'_j = v_k + v'_l$.

PROPOSITION 2. *There are unique elements* $w_{ij} \in V(\mu) \otimes V(\mu')$ *such that*

(i) $\Psi_0(w_{ij}) = w_{ij}$;

(ii) $w_{ij} = v_i \otimes v'_j + \sum_k \zeta_k v_{i_k} \otimes v'_{j_k}$, *with* $\zeta_k \in q\mathbb{Z}[q]$, *and* $v_{i_k} + v'_{j_k} = v_i + v'_j$.

Also, $v_{i_k} \otimes v'_{j_k} < v_i \otimes v'_j$ *for all* k. *The elements* w_{ij} *form a basis of* $V(\mu) \otimes V(\mu')$.

Proof. This proceeds in the same way as the proof of [17, Theorem 27.3.2]. Note that

$$\Psi_0(v_i \otimes v_j) = v_i \otimes v_j + \sum_k \xi_k v_{i_k} \otimes v'_{j_k}, \qquad (1)$$

with $\xi_k \in \mathbb{Z}[q, q^{-1}]$. From $\Theta^0_\eta \in U^+_\eta \otimes U^-_\eta$ and the assumption on the ordering of C and C', it follows that $v_{i_k} \otimes v'_{j_k} < v_i \otimes v'_j$ for all k. Let X be the set of all (i, j) with $v_i + v'_j = v$, for a certain v. Order the elements of X in such a way that $v_i \otimes v'_j < v_k \otimes v'_l$ implies that (i, j) appears before (k, l). Let (i, j) be the smallest element of X. Then, by (1), we see that $\Psi_0(v_i \otimes v'_j) = v_i \otimes v'_j$. So in this case we set $w_{ij} = v_i \otimes v'_j$. Now choose a $(k, l) \in X$, and suppose that $w_{r,s}$ exist for all $(r, s) \in X$ appearing before (k, l). Then, using (1) and

the triangular form of the $w_{r,s}$, we can write $\Psi_0(v_k \otimes v_l') - v_k \otimes v_l' = \sum_{r,s} \zeta_{r,s} w_{r,s}$, where $v_r \otimes v_s' < v_k \otimes v_l'$. After taking images under Ψ_0, and using the fact that Ψ_0 is an involution, we see that the $\zeta_{r,s} \in \mathbb{Z}[q, q^{-1}]$ satisfy $\bar{\zeta}_{r,s} = -\zeta_{r,s}$. This implies that there are unique $\delta_{r,s} \in q\mathbb{Z}[q]$ with $\zeta_{r,s} = \delta_{r,s} - \bar{\delta}_{r,s}$. Now set $w_{k,l} = v_k \otimes v_l' + \sum_{r,s} \delta_{r,s} w_{r,s}$. For the uniqueness, suppose that there are $w_{ij}' \in V(\mu) \otimes V(\mu')$ satisfying conditions (i) and (ii). Then we write w_{ij}' as a linear combination of w_{ij}. By condition (ii), the coefficients are in $\mathbb{Z}[q]$. Then condition (i) implies that they are in $\mathbb{Z}$. Finally, from condition (ii) we see that one coefficient is 1, and the others are 0. $\qquad\square$

Let $V(\lambda)$ denote the U_q-submodule of $V(\mu) \otimes V(\mu')$ generated by $v_\mu \otimes v_{\mu'} = v_1 \otimes v_1'$. So $V(\lambda)$ is the irreducible U_q-module with highest weight $\lambda = \mu + \mu'$. Set $\mathcal{L}(\lambda) = (\mathcal{L} \otimes \mathcal{L}') \cap V(\lambda)$, and $\mathcal{B}(\lambda) = (\mathcal{B} \otimes \mathcal{B}') \cap \mathcal{L}(\lambda)/q\mathcal{L}(\lambda)$. Then, by [**6**, Proposition 9.10], $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ is a crystal base of $V(\lambda)$ (the hypotheses of this proposition are satisfied by [**6**, Proposition 9.23 and Lemma 9.26]).

THEOREM 3. *The elements of the canonical basis of $V(\lambda)$ have the form*

$$v_i \otimes v_j' + \sum_k \zeta_k v_{i_k} \otimes v_{j_k}',$$

with $\zeta_k \in q\mathbb{Z}[q]$, and $v_{i_k} \otimes v_{j_k}' < v_i \otimes v_j'$ for all k.

Proof. We know that $\Psi_0(v_1 \otimes v_1') = v_1 \otimes v_1'$.
So by Lemma 1, Ψ_0 coincides with $\bar{}$ on $V(\lambda)$ (where $\overline{u \cdot v_1 \otimes v_1'} = \bar{u} \cdot v_1 \otimes v_1'$). Hence the elements of the canonical basis of $V(\lambda)$ are invariant under Ψ_0. Also, since the elements of the canonical basis lie in $\mathcal{L}(\lambda)$ and are equal to a $v_i \otimes v_j'$ mod $q\mathcal{L}(\lambda)$, they must be of the form $v_i \otimes v_j + \sum_k \zeta_k v_{i_k} \otimes v_{j_k}'$ with $\zeta_k \in q\mathbb{Z}[q]$. Now Proposition 2 finishes the proof. $\quad\square$

Now let $V(\mu_1), \ldots, V(\mu_r)$ be irreducible U_q-modules with canonical bases $C_i = \{v_1^i, \ldots, v_{m_i}^i\}$, ordered according to increasing height. We consider the tensor product $V = V(\mu_1) \otimes \ldots \otimes V(\mu_r)$. We write

$$v_{i_1}^1 \otimes \ldots \otimes v_{i_r}^r <_{\text{lex}} v_{j_1}^1 \otimes \ldots \otimes v_{j_r}^r$$

if there is a k with $i_1 = j_1, \ldots, i_k = j_k$ and $i_{k+1} < j_{k+1}$. Set $\lambda = \mu_1 + \ldots + \mu_r$ and let $V(\lambda)$ be the U_q-submodule of V generated by $v_1^1 \otimes \ldots \otimes v_1^r$.

COROLLARY 4. *The elements of the canonical basis of $V(\lambda)$ have the form*

$$v_{i_1}^1 \otimes \ldots \otimes v_{i_r}^r + \sum_k \zeta_k x_k,$$

where $\zeta_k \in q\mathbb{Z}[q]$, $x_k \in C_1 \otimes \ldots \otimes C_r$, and $x_k <_{\text{lex}} v_{i_1}^1 \otimes \ldots \otimes v_{i_r}^r$.

Proof. The case $r = 2$ is covered by Theorem 3, so suppose that $r > 2$. Let W be the U_q-submodule of $V(\mu_2) \otimes \ldots \otimes V(\mu_r)$ generated by $v_1^2 \otimes \ldots \otimes v_1^r$. Then W is the irreducible U_q-module with highest weight $\mu_2 + \ldots + \mu_r$. Let $\{w_1, \ldots, w_s\}$ be the canonical basis of W. Then by Theorem 3 the elements of the canonical basis of $V(\lambda)$ have the form

$$v_{i_1}^1 \otimes w_{j_1} + \sum_{k \geqslant 2} \zeta_k v_{i_k}^1 \otimes w_{j_k},$$

with $i_k < i_1$ for all $k \geqslant 2$, and $\zeta_k \in q\mathbb{Z}[q]$. We obtain the result by writing all w_{j_k} for $k \geqslant 1$ as linear combinations of elements of $C_2 \otimes \ldots \otimes C_r$, and then using induction. $\quad\square$

4. *A monomial basis of $V(\lambda)$*

In this section we first describe a basis of $V(\lambda)$, following [**10**]. Then, using this, we derive an algorithm for constructing the canonical basis of $V(\lambda)$, when $V(\lambda)$ is viewed as a submodule of a tensor product.

Let $\pi \in \Pi_\lambda$. Then the first direction of π is $w(\lambda)$ for some $w \in W/W_\lambda$ (see [**13**, 5.2]), where W_λ is the stabilizer of λ. Set $\phi(\pi) = w$. Let $s_{i_1} \ldots s_{i_r}$ be the reduced expression for $\phi(\pi)$, which is lexicographically the smallest. (Here, $s_{i_1} \ldots s_{i_r}$ is lexicographically smaller than $s_{j_1} \ldots s_{j_r}$ if there is a $k > 0$ such that $i_1 = j_1, \ldots, i_{k-1} = j_{k-1}$ and $i_k < j_k$.) Then we define integers $n_1, \ldots, n_r$, and paths $\pi_0, \pi_1, \ldots, \pi_r$ in the following way. First, $\pi_0 = \pi$. We let n_k be maximal such that $e_{\alpha_{i_k}}^{n_k} \pi_{k-1} \neq 0$, and we set $\pi_k = e_{\alpha_{i_k}}^{n_k} \pi_{k-1}$. Set

$$\eta_\pi = (n_1, \ldots, n_r) \quad \text{and} \quad F_\pi = F_{\alpha_{i_1}}^{(n_1)} \ldots F_{\alpha_{i_r}}^{(n_r)}.$$

Let $b_\lambda \in \mathcal{B}(\lambda)$ denote the unique element of weight λ (it is the coset of v_λ modulo $q\mathcal{L}(\lambda)$). Set

$$b_\pi = \widetilde{F}_{\alpha_{i_1}}^{n_1} \ldots \widetilde{F}_{\alpha_{i_r}}^{n_r}(b_\lambda);$$

then $\mathcal{B}(\lambda) = \{b_\pi \mid \pi \in \Pi_\lambda\}$ (this follows from [**8**]). In the discussion that follows, we let $<_B$ denote the Bruhat order on the Weyl group W. The lexicographical order on sequences of length r is defined by $(m_1, \ldots, m_r) <_{\text{lex}} (n_1, \ldots, n_r)$ if there is a k such that $m_1 = n_1, \ldots, m_{k-1} = n_{k-1}$ and $m_k < n_k$. We now define a partial order on Π_λ as follows. First of all, $\pi < \sigma$ if $\phi(\pi) <_B \phi(\sigma)$. Secondly, if $\phi(\pi) = \phi(\sigma)$, then $\pi < \sigma$ if $\eta_\pi >_{\text{lex}} \eta_\sigma$. For the proof of the folowing theorem we refer to [**10**].

THEOREM 5.

$$F_\pi \cdot v_\lambda = G_\lambda(b_\pi) + \sum_{\sigma < \pi} \zeta_{\pi,\sigma} G_\lambda(b_\sigma),$$

where $\zeta_{\pi,\sigma} \in \mathbb{Z}[q, q^{-1}]$.

COROLLARY 6. *The set* $\{F_\pi \cdot v_\lambda \mid \pi \in \Pi_\lambda\}$ *is a basis of* $V(\lambda)$.

Let $\pi \in \Pi_\lambda$, and $F_\pi = F_{\alpha_{i_1}}^{(n_1)} \ldots F_{\alpha_{i_r}}^{(n_r)}$. Then we say that π is *of weight* $\nu = \sum_k n_k \alpha_{i_k}$. We note that this means that $F_\pi \cdot v_\lambda$ is a weight vector in $V(\lambda)$ of weight $\lambda - \nu$. By $\Pi_{\lambda,\nu}$, we denote the set of all $\pi \in \Pi_\lambda$ of weight ν.

Suppose that $\lambda = \mu_1 + \ldots + \mu_r$, where the μ_i are dominant weights. Also suppose that we are given the modules $V(\mu_i)$ with canonical bases $C_i = \{v_1^i, \ldots, v_{m_i}^i\}$, ordered according to increasing height. We identify $V(\lambda)$ with the U_q-submodule of $V(\mu_1) \otimes \ldots \otimes V(\mu_r)$ generated by $v_\lambda = v_1^1 \otimes \ldots \otimes v_r^1$. Set $C = C_1 \otimes \ldots \otimes C_r$, which is a basis of $V(\mu_1) \otimes \ldots \otimes V(\mu_r)$, ordered with respect to $<_{\text{lex}}$ (see the previous section).

Theorem 5 leads to the following algorithm for computing the $G_\lambda(b_\pi)$, for $\pi \in \Pi_{\lambda,\nu}$. Let $\sigma_1, \ldots, \sigma_r$ be the elements from $\Pi_{\lambda,\nu}$ that are smaller than π. We assume that the $G_\lambda(b_{\sigma_i})$ have already been computed. Write $G_\lambda(b_{\sigma_i}) = y_i + \sum_k \zeta_k y_{i,k}$, where $y_i, y_{i,k} \in C$ and $y_i >_{\text{lex}} y_{i,k}$ for all k. We assume that $y_i <_{\text{lex}} y_j$ implies that $i > j$. Then we proceed as follows.

1. Write $X = F_\pi \cdot v_\lambda$ as a linear combination of elements of C.

2. For $i = 1, \ldots, r$, we act as follows.
    Let ζ_i be the coefficient of y_i in X. Let ξ_i be the unique element of $\mathbb{Z}[q, q^{-1}]$ such that $\overline{\xi}_i = \xi_i$ and $\zeta_i + \xi_i \in q\mathbb{Z}[q]$.
    Set $X := X + \xi_i G_\lambda(b_{\sigma_i})$.

PROPOSITION 7. *When the loop in Step 2 terminates, we have* $X = G_\lambda(b_\pi)$.

Proof. Note that by Theorem 5 there are coefficients ξ_i such that

$$G_\lambda(b_\pi) = F_\pi \cdot v_\lambda + \sum_{i=1}^{r} \xi_i G_\lambda(b_{\sigma_i}).$$

This implies that $\overline{\xi}_i = \xi_i$. Also, by Corollary 4, we know that $G_\lambda(b_\pi)$ is of the form $x + \sum_k \omega_k x_k$, where $x, x_k \in C$ and $\omega_k \in q\mathbb{Z}[q]$. Note that by Corollary 4, y_1 does not occur in any $G_\lambda(b_{\sigma_i})$, except for $G_\lambda(b_{\sigma_1})$. Therefore, ξ_1 is uniquely determined by the requirements that it should be invariant under $\overline{}$, and $\zeta_1 + \xi_i \in q\mathbb{Z}[q]$. Then, in the same way, we see that ξ_2 is uniquely determined, and so on. □

EXAMPLE 8. Let Φ be the root system of type G_2. We denote the simple roots of Φ by α and β, where β is long. The fundamental module $V(\lambda_1)$ is seven-dimensional, and the canonical basis is $C_1 = \{v_1, \dots, v_7\}$; these are weight vectors of weights $(1, 0)$, $(-1, 1)$, $(2, -1)$, $(0, 0)$, $(-2, 1)$, $(1, -1)$ and $(-1, 0)$. Here we abbreviate a weight $m\lambda_1 + n\lambda_2$ as (m, n). The fundamental module $V(\lambda_2)$ is fourteen-dimensional and has canonical basis $C_2 = \{w_1, \dots, w_{14}\}$. The w_i are weight vectors of weights $(0, 1)$, $(3, -1)$, $(1, 0)$, $(-1, 1)$, $(-3, 2)$, $(2, -1)$, $(0, 0)$, $(0, 0)$, $(3, -2)$, $(-2, 1)$, $(1, -1)$, $(-1, 0)$, $(-3, 1)$ and $(0, -1)$. A description of the action of the generators of U_q on $V(\lambda_1)$ can, for instance, be found in [7], and the action of U_q on $V(\lambda_2)$ is described in [6, 5A.4]. Alternatively, these modules can be constructed using the GAP4 package QuaGroup [2, 3]. This package has been used to perform many of the calculations used in the rest of this example. Now we set $\lambda = 2\lambda_1 + \lambda_2$. Then $V(\lambda)$ is the submodule of $W = V(\lambda_1) \otimes V(\lambda_1) \otimes V(\lambda_2)$ generated by $v_1 \otimes v_1 \otimes w_1$. We construct the elements of the canonical basis of $V(\lambda)$ that are of weight $\mu = (-2, 2)$. We use the following elements of weight μ:

$x_1 = v_1 \otimes v_2 \otimes w_{10}$;	$x_2 = v_1 \otimes v_4 \otimes w_5$;	$x_3 = v_1 \otimes v_5 \otimes w_4$;
$x_4 = v_2 \otimes v_1 \otimes w_{10}$;	$x_5 = v_2 \otimes v_2 \otimes w_7$;	$x_6 = v_2 \otimes v_2 \otimes w_8$;
$x_7 = v_2 \otimes v_3 \otimes w_5$;	$x_8 = v_2 \otimes v_4 \otimes w_4$;	$x_9 = v_2 \otimes v_5 \otimes w_3$;
$x_{10} = v_2 \otimes v_7 \otimes w_1$;	$x_{11} = v_3 \otimes v_2 \otimes w_5$;	$x_{12} = v_4 \otimes v_1 \otimes w_5$;
$x_{13} = v_4 \otimes v_2 \otimes w_4$;	$x_{14} = v_4 \otimes v_5 \otimes w_1$;	$x_{15} = v_5 \otimes v_1 \otimes w_4$;
$x_{16} = v_5 \otimes v_2 \otimes w_3$;	$x_{17} = v_5 \otimes v_4 \otimes w_1$;	$x_{18} = v_7 \otimes v_2 \otimes w_1$.

They are listed in lexicographical order; that is, $x_1 <_{\text{lex}} x_2 <_{\text{lex}} \dots <_{\text{lex}} x_{18}$. The weight space of weight μ in $V(\lambda)$ is five-dimensional. So we have five paths π_i in the crystal graph. The corresponding words in the Weyl group are

$$\phi(\pi_1) = s_\alpha s_\beta s_\alpha;$$
$$\phi(\pi_2) = s_\beta s_\alpha s_\beta;$$
$$\phi(\pi_3) = s_\alpha s_\beta s_\alpha;$$
$$\phi(\pi_4) = s_\alpha s_\beta s_\alpha s_\beta;$$
$$\phi(\pi_5) = s_\alpha s_\beta s_\alpha s_\beta s_\alpha.$$

Setting $\eta_i = \eta_{\pi_i}$ we have $\eta_1 = (4, 2, 1)$, $\eta_2 = (1, 5, 1)$, $\eta_3 = (3, 2, 2)$, $\eta_4 = (3, 1, 2, 1)$ and $\eta_5 = (2, 1, 2, 1, 1)$. So we see that $\pi_1 < \pi_3 < \pi_4 < \pi_5$ and $\pi_2 < \pi_4$.

Therefore we have

$$G_\lambda(b_{\pi_1}) = F_{\pi_1}v_\lambda$$
$$= x_{16} + q^2x_{15} + q^3x_{13} + q^6x_{12} + q^8x_{11} + qx_9 + q^3x_8 + q^7x_7 + q^5x_3 + q^8x_2;$$
$$G_\lambda(b_{\pi_2}) = F_{\pi_2}v_\lambda = x_{11} + q^3x_7 + q^6x_6.$$

Also,

$$F_{\pi_3}v_\lambda = x_{17} + q^2x_{16} + q^2x_{14} + q^3x_{13} + q^6x_{11} + q^3x_9 + q^5x_8 + q^9x_7.$$

All the coefficients, except the first one, are in $q\mathbb{Z}[q]$. Hence $G_\lambda(b_{\pi_3}) = F_{\pi_3}v_\lambda$. Now

$$F_{\pi_4}v_\lambda = (q + q^{-1})x_{16} + (q + q^3)x_{15} + (1 + q^2 + q^4)x_{13} + (q^3 + q^5 + q^7)x_{12}$$
$$+ (q^3 + q^5 + q^7 + q^9)x_{11} + (1 + q^2)x_9 + (2q^2 + q^4)x_8 + (q^4 + 2q^6 + q^8)x_7$$
$$+ q^4x_5 + q^6x_4 + (q^4 + q^6)x_3 + (q^5 + q^7 + q^9)x_2 + q^7x_1.$$

The coefficient of x_{16} is not in $q\mathbb{Z}[q]$. Following the algorithm, we see that $G_\lambda(b_{\pi_4}) = F_{\pi_4}v_\lambda - (q + q^{-1})G_\lambda(b_{\pi_1})$; we obtain

$$G_\lambda(b_{\pi_4}) = x_{13} + q^3x_{12} + (q^3 + q^5)x_{11} + q^2x_8 + (q^4 + q^6)x_7 + q^4x_5 + q^6x_4 + q^5x_2 + q^7x_1.$$

Finally,

$$F_{\pi_5}v_\lambda = x_{18} + (2q + q^{-1})x_{17} + (2q^3 + 2q + q^{-1})x_{16} + (q + q^3)x_{15} + (2q + q^3)x_{14}$$
$$+ (2q^4 + 3q^2 + 1)x_{13} + (q^3 + q^5 + q^7)x_{12} + (q + 2q^3 + 3q^5 + 2q^7 + q^9)x_{11}$$
$$+ q^3x_{10} + (1 + 2q^2 + 2q^4)x_9 + (2q^2 + 3q^4 + q^6)x_8$$
$$+ (2q^4 + 3q^6 + 3q^8 + q^{10})x_7 + (q^4 + q^6)x_5 + q^6x_4$$
$$+ (q^4 + q^6)x_3 + (q^5 + q^7 + q^9)x_2 + q^7x_1.$$

We see that the highest basis vector not having a coefficient in $q\mathbb{Z}[q]$ (apart from x_{18}) is x_{17}. So we look at

$$F_{\pi_5}v_\lambda - (q + q^{-1})G_\lambda(b_{\pi_3}) = x_{18} + (q^3 + q + q^{-1})x_{16}$$
$$+ (q^4 + 2q^2 + 1)x_{13} + (q^4 + q^2 + 1) + \ldots$$

(here, all the coefficients that have not been written lie in $q\mathbb{Z}[q]$). Now x_{16} does not have a coefficient in $q\mathbb{Z}[q]$, so we look at

$$F_{\pi_5}v_\lambda - (q + q^{-1})G_\lambda(b_{\pi_3}) - (q + q^{-1})G_\lambda(b_{\pi_1}) = x_{18} + (q^2 + 1)x_{13} + \ldots.$$

We see that

$$G_\lambda(b_{\pi_5}) = F_{\pi_5}v_\lambda - (q + q^{-1})G_\lambda(b_{\pi_3}) - (q + q^{-1})G_\lambda(b_{\pi_1}) - G_\lambda(b_{\pi_4}).$$

REMARK. Let $\pi \in \Pi_\lambda$, and let $\phi(\pi) = s_{i_1} \ldots s_{i_r}$ be the reduced expression that is the smallest in the lexicographical order. Let

$$F_\pi = F_{\alpha_{i_1}}^{(n_1)} \ldots F_{\alpha_{i_r}}^{(n_r)}.$$

Write $\alpha = \alpha_{i_1}$. If $n_1 > 1$, then by [13, Lemma 5.3(b)], $\phi(e_\alpha\pi) = \phi(\pi)$, and hence

$$F_{e_\alpha\pi} = F_{\alpha_{i_1}}^{(n_1-1)} \ldots F_{\alpha_{i_r}}^{(n_r)}.$$

On the other hand, if $n_1 = 1$, then by [13, Lemma 5.3(a)] we see that $s_\alpha\phi(e_\alpha\pi) \not\prec_B \phi(e_\alpha\pi)$. So by [13, Lemma 5.3(b)], $\phi(\pi) = s_\alpha\phi(e_\alpha\pi)$. Therefore $\phi(e_\alpha\pi) = s_{i_2} \ldots s_{i_r}$, which is

the smallest reduced expression (in the lexicographical order) for $\phi(e_\alpha \pi)$. Hence

$$F_{e_\alpha \pi} = F_{\alpha_{i_2}}^{(n_2)} \ldots F_{\alpha_{i_r}}^{(n_r)}.$$

The conclusion is that

$$F_\pi \cdot v_\lambda = \frac{1}{[n_1]_\alpha} F_\alpha \cdot (F_{e_\alpha \pi} \cdot v_\lambda).$$

So, in order to compute $F_\pi \cdot v_\lambda$, we have only to act with F_α on a vector that we have already computed.

REMARK. Instead of the algorithm described here for getting the monomials F_π, one can also follow the procedure outlined in [15] for constructing so-called 'adapted strings'. Instead of $\phi(\pi)$, this procedure uses the longest element in the Weyl group. However, the monomials that one finds in that case are generally different from the ones that we obtain. Moreover, they do not in general have the nice property described in the previous remark.

5. The A_n-case

In this section we assume that the root system Φ is of type A_n. We use results from [9] to show that in this case our algorithm is very much like the algorithm described in [11]. The simple roots are $\alpha_1, \ldots, \alpha_n$, where we use the usual ordering of the nodes of the Dynkin diagram (see [1]).

Since the fundamental weights are all minuscule, the corresponding irreducible U_q-modules are easy to construct (see [6, Chapter 5A]). For $V(\lambda_k)$, we consider the set of sequences $S = \{(i_1, \ldots, i_k) \mid 1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n+1\}$. Let V be the vectorspace over $\mathbb{Q}(q)$ spanned by v_s for $s \in S$. Let $s \in S$. If i occurs in s, but $i+1$ does not, then we let s^{i-} be the sequence obtained from s by replacing i by $i+1$, and we set $v_s^{i-} = v_{s^{i-}}$. Otherwise, $v_s^{i-} = 0$. Also, if $i+1$ occurs in s, but i does not, then we let s^{i+} be the sequence obtained from s by replacing $i+1$ by i, and we set $v_s^{i+} = v_{s^{i+}}$. Otherwise, $v_s^{i+} = 0$. Now a U_q-action on V is defined by $F_{\alpha_i} \cdot v_s = v_s^{i-}$, $E_\alpha \cdot v_s = v_s^{i+}$, and

$$K_{\alpha_i} \cdot v_s = \begin{cases} q v_s, & \text{if } i \in S \text{ and } i+1 \notin S, \\ q^{-1} v_s, & \text{if } i \notin S \text{ and } i+1 \in S, \\ v_s, & \text{otherwise.} \end{cases}$$

Then the U_q-module V is isomorphic to $V(\lambda_k)$. To see this, we note that v_s is a weight vector of weight $\mu_s = a_1 \lambda_1 + \ldots + a_n \lambda_n$, where $a_i = 1$ if $i \in S, i+1 \notin S$; $a_i = -1$ if $i \notin S, i+1 \in S$; and $a_i = 0$ otherwise. Set $s_{\lambda_k} = (1, 2, \ldots, k)$; then $\mu_{s_{\lambda_k}} = \lambda_k$. If $i \in S$ and $i+1 \notin S$, then $s_{\alpha_i}(\mu_s) = \mu_{s^{i-}}$. Since all elements of S can be obtained from s_{λ_k} by a sequence of 'moves' $s \to s^{i-}$ (see [9, Proposition 3.3.1]), we find that $\{\mu_s \mid s \in S\} = W \cdot \lambda_k$. Finally, we compare this with [6, 5A.1].

Let $\mathcal{L}(\lambda_k)$ be the A-submodule of V spanned by the v_s, and let $\mathcal{B}(\lambda_k)$ be the set of all $v_s \bmod q\mathcal{L}(\lambda_k)$. Then $(\mathcal{L}(\lambda_k), \mathcal{B}(\lambda_k))$ is a crystal base of V (see [6, Lemma 9.6]). Furthermore, $C_k = \{v_s \mid s \in S\}$ is the canonical basis of V. (Indeed, the v_s are certainly invariant under $\bar{}$ because they are of the form $F_{\alpha_{i_1}} \ldots F_{\alpha_{i_r}} \cdot v_{s_{\lambda_k}}$. Secondly, $\{v_s \bmod q\mathcal{L}(\lambda_k) \mid s \in S\} = \mathcal{B}(\lambda_k)$.)

So the elements of $\mathcal{B}(\lambda_k)$ are labeled by the elements of S. From [9] we obtain the action of the Kashiwara operators as follows: $\widetilde{F}_{\alpha_i}(v_s) = v_s^{i-} \bmod q\mathcal{L}(\lambda_k)$, and $\widetilde{E}_{\alpha_i}(v_s) = v_s^{i+} \bmod q\mathcal{L}(\lambda_k)$.

Computing canonical bases

We write a sequence $s = (i_1, \ldots, i_k)$ as a diagram with one column of length k and the elements $i_1, \ldots, i_k$ from top to bottom. For example, the sequence $(1, 4, 5)$ is

$$\begin{array}{|c|} \hline 1 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}.$$

Now let $\lambda = a_1\lambda_1 + \ldots + a_n\lambda_n$ be a dominant weight, and consider the tensor product $W = V(\lambda_1)^{\otimes a_1} \otimes \ldots \otimes V(\lambda_n)^{\otimes a_n}$. The basis elements are labelled by tableaux with $a_n + a_{n-1} + \ldots + a_1$ columns. The first a_n columns have length n, the following a_{n-1} columns have length $n-1$, and so on. The tableaux are filled with elements of $\{1, 2, \ldots, n+1\}$, such that every column is strictly increasing. Then every column of length k determines a basis element of $V(\lambda_k)$. Tensored together, they form a basis element of W; for example,

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 4 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} = \boxed{4} \otimes \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}.$$

Then the highest-weight vector v_λ, of weight λ in W, is labelled by the tableau T_λ, where the ith row contains only the number i. Let $V(\lambda)$ denote the submodule of W generated by v_λ. Let $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ be the crystal base of $V(\lambda)$. Then, by [9], the elements of $\mathcal{B}(\lambda)$ are labelled by tableaux with non-decreasing rows. In particular, these tableaux label the points in the crystal graph. From [9], we obtain the following algorithm for computing $\widetilde{F}_{\alpha_i}(T)$ and $\widetilde{E}_{\alpha_i}(T)$, where T is such a tableau.

1. Write the numbers in the tableau as a sequence, starting from the top right, and going along the columns from right to left, top to bottom. Below each number, write a '+' if it is equal to i, a '−' if it is $i+1$, and a blank otherwise.

2. If there is a + followed by a − (maybe separated by blanks), then replace them by blanks. Continue until this operation is no longer possible.

3. (a) If there is no + left, then $\widetilde{F}_{\alpha_i}(T) = 0$. Otherwise, change the i corresponding to the leftmost + to an $i+1$. Rebuild the tableau, and the result is $\widetilde{F}_{\alpha_i}(T)$.
   (b) If there is no − left, then $\widetilde{E}_{\alpha_i}(T) = 0$. Otherwise, change the $i+1$ corresponding to the rightmost − into an i. Rebuild the tableau; the result is $\widetilde{E}_{\alpha_i}(T)$.

EXAMPLE 9. Let the root system be of type A_3, and set

$$T = \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}.$$

Then the sequence that we obtain is 3, 1, 2, 1, 2, 3. If $i = 2$, this corresponds to $-o + o + -$ (where we represent a blank by 'o'). After the operation of step 2, this becomes $-o + ooo$. We see that

$$\widetilde{F}_{\alpha_2}(T) = \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \quad \text{and} \quad \widetilde{E}_{\alpha_i}(T) = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}.$$

Now, from a tableau T we obtain a monomial $F_T = F_{\alpha_{i_1}}^{(m_1)} \ldots F_{\alpha_{i_p}}^{(m_p)}$ in the following way. Let π be the path corresponding to T (since the paths in Π_λ label the points of the crystal graph, and so do the tableaux, there is a natural correpondence between the two).

Let $\phi(\pi) = s_{i_1} \ldots s_{i_p}$ be the lexicographically smallest reduced expression for $\phi(\pi)$. We obtain the exponents $m_1, \ldots, m_p$ as follows. We let m_1 be maximal, such that

$$\widetilde{E}_{\alpha_{i_1}}^{m_1}(T) \neq 0;$$

then m_2 is maximal, such that

$$\widetilde{E}_{\alpha_{i_2}}^{m_2} \widetilde{E}_{\alpha_{i_1}}^{m_1}(T) \neq 0,$$

and so on. Once we have the monomials F_T for all the tableaux T, we proceed with the algorithm of Section 4.

The algorithm described in [11] for computing the canonical basis of $V(\lambda)$ has the same steps as our algorithm. First, for every tableau T, a monomial

$$F_T = F_{\alpha_{i_1}}^{(n_1)} \ldots F_{\alpha_{i_t}}^{(r_t)}$$

is computed. Secondly, from the vectors $F_T \cdot v_\lambda$ the canonical basis is computed using a triangular algorithm similar to the one that we use. Therefore, the main difference between the algorithms lies in the first step. We investigate this step a little further.

In [11, 4.1], the authors describe the following algorithm for obtaining a monomial F_T from a tableau T. Let i_1 be the smallest index such that $i_1 + 1$ occurs in an mth row of T, with $m \leqslant i_1$. Furthermore, r_1 is the number of occurrences of $i_1 + 1$ on an mth row with $m \leqslant i$. Then T_1 is obtained from T by replacing these r_1 occurrences of $i_1 + 1$ by i_1. Continuing with T_1 instead of T, we eventually arrive at the tableau T_λ, at which point the algorithm stops. We have obtained the sequences $i_1, \ldots, i_t$ and $r_1, \ldots, r_t$, and the monomial is

$$F_T = F_{\alpha_{i_1}}^{(n_1)} \ldots F_{\alpha_{i_t}}^{(r_t)}.$$

We note that applying $\widetilde{E}_{\alpha_i}$ amounts to replacing an $i + 1$ by i. Since this $i + 1$ was put there by a series of applications of $\widetilde{F}_{\alpha_{i_k}}$, starting with T_λ, we see that this $i + 1$ must occur on the mth row with $m \leqslant i$. By induction on the number of columns of T, it can be shown that if i_1 is minimal such that $i_1 + 1$ occurs in an mth row of T with $m \leqslant i_1$, then $\widetilde{E}_{\alpha_{i_1}}(T) \neq 0$. So our algorithm for obtaining the monomials F_T is quite similar to that of [11]. However, in our algorithm we follow the lexicographically smallest reduced expression of a word in the Weyl group, in order to find the sequence $i_1, \ldots$. This means that sometimes we obtain a different monomial from that found with the algorithm from [11], as the following example shows.

EXAMPLE 10. Set

$$T = \begin{array}{|c|c|c|} \hline 1 & 1 & 4 \\ \hline 2 & 3 \\ \hline 3 \\ \hline \end{array} .$$

Then the monomial obtained by the algorithm of [11] is $F_{\alpha_2} F_{\alpha_3} F_{\alpha_2} F_{\alpha_1}$. Let π be the corresponding path; then $\phi(\pi) = s_{\alpha_3} s_{\alpha_2} s_{\alpha_1}$. This means that the monomial that we obtain is $F_{\alpha_3} F_{\alpha_2}^{(2)} F_{\alpha_1}$.

We conclude that in the A_n-case our algorithm is very similar to, but not the same as, the algorithm described in [11].

6. *Practical experiences*

In this section we discuss some practical experiences with an implementation of the algorithm inside the package QuaGroup [3], which was written in the computer algebra system GAP4 [2].

Let $\lambda = n_1\lambda_1 + \ldots + n_l\lambda_l$ be a dominant weight. The input to the algorithm is λ, along with the modules $V(\lambda_i)$, for $1 \leqslant i \leqslant l$. These modules are given by their canonical bases. Furthermore, for a generator F_α and an element v of the canonical basis of a $V(\lambda_i)$, we can compute $F_\alpha \cdot v$ by a table lookup.

We recall the notation used in Section 4: $\lambda = \mu_1 + \ldots + \mu_r$, where $\mu_i = \lambda_{j_i}$. We work inside the tensor product $W(\lambda) = V(\mu_1) \otimes \ldots \otimes V(\mu_r)$, with basis $C = C_1 \otimes \ldots \otimes C_r$. We set $v_\lambda = v_{\mu_1} \otimes \ldots \otimes v_{\mu_r}$, where v_{μ_i} is a highest-weight vector of $V(\mu_i)$. Furthermore, the *length* of an element of $w \in W(\lambda)$ is the number of elements of C that occur in the expression of w as a linear combination of elements of C.

The algorithm consists of two steps. In the first step, we express $F_\pi \cdot v_\lambda$ as a linear combination of elements of C. As explained at the end of Section 4, $F_\pi \cdot v_\lambda = \zeta F_\alpha \cdot (F_{\pi'} \cdot v_\lambda)$, where $\zeta \in \mathbb{Q}(q)$ and $F_{\pi'} \cdot v_\lambda$ is a vector that we also have to compute. It follows that for each $F_\pi \cdot v_\lambda$ we have to compute one image $F_\alpha \cdot v$. Hence in the first step we compute $\dim V(\lambda)$ times an image $F_\alpha \cdot v$. In the second step we add some elements of the canonical basis (which we have already computed) to the element $F_\pi \cdot v_\lambda$. The number of these additions for each $F_\pi \cdot v_\lambda$ is clearly bounded by $\dim V(\lambda)$. Hence the total number of additions is bounded by $(\dim V(\lambda))^2$.

So the total number of operations (computing an image $F_\alpha \cdot v$, adding one element of $W(\lambda)$ to another) is bounded by a polynomial in $\dim V(\lambda)$. The cost of these operations is polynomial in the length of the elements that occur. Now the dimension of $W(\lambda)$ is exponential in the n_i. Therefore, an element of $W(\lambda)$ may have a length that is exponential in the n_i. So the question is whether elements of that kind occur during the execution of the algorithm.

Table 1 contains experimental data obtained using the algorithm. The computations were done on a Linux system with a 600MHz Pentium III processor and 64MB of working memory for GAP. We see that the algorithm is efficient enough to be able to compute canonical bases of modules of dimensions into the thousands. We also see that a jump in the maximal length of the elements that occur in the algorithm corresponds to a jump in the running time of the algorithm. This confirms the conclusion of the arguments given above, namely that the length of the elements that occur is an important factor in the running time.

The data given here do not lead to a conclusion as to whether the algorithm has a polynomial- or an exponential-time complexity. However, they do indicate that the maximal length of an element – and the running time – can increase rather rapidly if $\dim V(\lambda)$ increases. This is seen most spectacularly in the case of G_2.

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Table 1: Running times for the computation of the canonical basis. The second column contains the highest weight, where we abbreviate $n_1\lambda_1 + \ldots + n_l\lambda_l$ by $(n_1, \ldots, n_l)$. The same ordering of the fundamental weights as in [**1**] is used. The third column contains the dimension of the corresponding module. The fourth column displays the running time in seconds. The last column contains the maximal length of an element of the canonical basis.

type	weight λ	dim $V(\lambda)$	time (s)	max. length
B_4	(1, 0, 1, 0)	594	27	6
B_4	(0, 1, 1, 0)	1650	146	18
B_4	(1, 1, 0, 1)	2560	495	29
B_4	(1, 0, 1, 1)	5040	1571	78
C_3	(2, 0, 1)	216	26	16
C_3	(1, 1, 1)	512	95	32
C_3	(0, 2, 1)	616	222	63
C_3	(2, 0, 2)	1078	1013	158
G_2	(1, 1)	64	3	5
G_2	(2, 1)	189	58	38
G_2	(1, 2)	286	125	53
G_2	(2, 2)	729	2610	439

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