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The rank of a hypergeometric system

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Abstract

The holonomic rank of the A-hypergeometric system $M_A(\beta)$ is the degree of the toric ideal I_A for generic parameters; in general, this is only a lower bound. To the semigroup ring of A we attach the ranking arrangement and use this algebraic invariant and the exceptional arrangement of non-generic parameters to construct a combinatorial formula for the rank jump of $M_A(\beta)$. As consequences, we obtain a refinement of the stratification of the exceptional arrangement by the rank of $M_A(\beta)$ and show that the Zariski closure of each of its strata is a union of translates of linear subspaces of the parameter space. These results hold for generalized A-hypergeometric systems as well, where the semigroup ring of A is replaced by a non-trivial weakly toric module $M \subseteq \mathbb{C}[\mathbb{Z}A]$. We also provide a direct proof of the main result in [M. Saito, *Isomorphism classes of A-hypergeometric systems*, Compositio Math. **128** (2001), 323–338] regarding the isomorphism classes of $M_A(\beta)$.

Contents

1. Introduction

An A-hypergeometric system $M_A(\beta)$ is a D-module determined by an integral matrix A and a complex parameter vector $\beta \in \mathbb{C}^d$. These systems are also known as *GKZ-systems*, as they were introduced in the late 1980s by Gelfand *et al.* [GGZ87, GZK89]. Their solutions occur naturally in mathematics and physics, including the study of roots of polynomials, Picard–Fuchs equations for the variation of Hodge structure of Calabi–Yau toric hypersurfaces, and generating functions for intersection numbers on moduli spaces of curves, see [BvS95, HLY96, Oko02, Stu00].

The (holonomic) rank of $M_A(\beta)$ coincides with the dimension of its solution space at a non-singular point. In this article, we provide a combinatorial formula for the rank of $M_A(\beta)$ in terms

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of certain lattice translates determined by A and β. For a fixed matrix A, this computation yields a geometric stratification of the parameter space $\mathbb{C}^d$ that refines its stratification by the rank of $M_A(\beta)$.

Notation 1.1. Let $A = [a_1 a_2 \cdots a_n]$ be an integer $(d \times n)$-matrix with integral column span $\mathbb{Z}A = \mathbb{Z}^d$. Assume further that A is *pointed*, meaning that the origin is the only linear subspace of the cone $\mathbb{R}_{\geqslant 0}A = \{\sum_{i=1}^n \gamma_i a_i \mid \gamma_i \in \mathbb{R}_{\geqslant 0}\}$.

A subset F of the column set of A is called a *face* of A, denoted $F \preceq A$, if $\mathbb{R}_{\geqslant 0}F$ is a face of the cone $\mathbb{R}_{\geqslant 0}A$.

Let $x = x_1, \ldots, x_n$ be variables and $\partial = \partial_1, \ldots, \partial_n$ their associated partial differentiation operators. In the polynomial ring $R = \mathbb{C}[\partial]$, let

$$I_A = \langle \partial^u - \partial^v \mid Au = Av,\, u, v \in \mathbb{N}^n \rangle \subseteq R$$

denote the toric ideal associated to A, and let $S_A = R/I_A$ be its quotient ring. Note that S_A is isomorphic to the semigroup ring of A, which is

$$S_A \cong \mathbb{C}[\mathbb{N}A] := \bigoplus_{a \in \mathbb{N}A} \mathbb{C} \cdot t^a \tag{1.1}$$

with multiplication given by semigroup addition of exponents. The Weyl algebra

$$D = \mathbb{C}\langle x, \partial \mid [\partial_i, x_j] = \delta_{ij}, [x_i, x_j] = 0 = [\partial_i, \partial_j] \rangle$$

is the ring of $\mathbb{C}$-linear differential operators on $\mathbb{C}[x]$.

DEFINITION 1.2. The *A-hypergeometric system* with parameter $\beta \in \mathbb{C}^d$ is the left D-module

$$M_A(\beta) = D/D \cdot (I_A, \{E_i - \beta_i\}_{i=1}^d),$$

where $E_i - \beta_i = \sum_{j=1}^n a_{ij} x_j \partial_j - \beta_i$ are *Euler operators* associated to A.

The *rank* of a left D-module M is

$$\operatorname{rank} M = \dim_{\mathbb{C}(x)} \mathbb{C}(x) \otimes_{\mathbb{C}[x]} M.$$

The rank of a holonomic D-module is finite and equal to the dimension of its solution space of germs of holomorphic functions at a generic non-singular point [Kas83].

1.1 The exceptional arrangement of a hypergeometric system

In [GZK89], Gelfand *et al.* showed that when S_A is Cohen–Macaulay and standard $\mathbb{Z}$-graded, the A-hypergeometric system $M_A(\beta)$ is holonomic of rank $\operatorname{vol}(A)$ for all parameters β, where $\operatorname{vol}(A)$ is $d!$ times the Euclidean volume of the convex hull of A and the origin. Adolphson established further that $M_A(\beta)$ is holonomic for all choices of A and β and that the holonomic rank of $M_A(\beta)$ is generically given by $\operatorname{vol}(A)$ [Ado94]. However, an example found by Sturmfels and Takayama showed that equality need not hold in general [ST98] (see also [SST00]). At the same time, Cattani *et al.* produced an infinite family of such examples through a complete investigation of the rank of $M_A(\beta)$ when I_A defines a projective monomial curve [CDD99].

The relationship between $\operatorname{vol}(A)$ and the rank of $M_A(\beta)$ was made precise by Matusevich *et al.* who used Euler–Koszul homology to study the holonomic *generalized A-hypergeometric system* $\mathcal{H}_0(M, \beta)$ associated to a toric module M (see Definition 2.3). The Euler–Koszul homology $\mathcal{H}_\bullet(M, \beta)$ of M with respect to β is the homology of a twisted Koszul complex on $D \otimes_R M$ given by the sequence $E - \beta$. This includes the A-hypergeometric system $M_A(\beta) = \mathcal{H}_0(S_A, \beta)$ as the special case that $M = S_A$. As in this special case, and for the purposes of this article, suppose that the generic rank of $\mathcal{H}_0(M, -)$ is $\mathrm{vol}(A)$.

The matrix A induces a natural $\mathbb{Z}^d$-grading on R; the *quasidegree set* of a finitely generated $\mathbb{Z}^d$-graded R-module N is defined to be the Zariski closure in $\mathbb{C}^d$ of the set of vectors α for which the graded piece N_α is non-zero. In [MMW05], an explicit description of the *exceptional arrangement*

$$\mathcal{E}_A(M) = \{\beta \in \mathbb{C}^d \mid \mathrm{rank}\, \mathcal{H}_0(M, \beta) \neq \mathrm{vol}(A)\}$$

associated to M is given in terms of the quasidegrees of certain Ext modules involving M (see (4.2)). This description shows that $\mathcal{E}_A(M)$ is a subspace arrangement in $\mathbb{C}^d$ given by translates of linear subspaces that are generated by the faces of the cone $\mathbb{R}_{\geqslant 0}A$, and that $\mathcal{E}_A(M)$ is empty exactly when $M \neq 0$ is a maximal Cohen–Macaulay S_A-module. It is also shown in [MMW05] that the rank of $\mathcal{H}_\bullet(M, \beta)$ is upper semi-continuous as a function of β. Thus the exceptional arrangement $\mathcal{E}_A(M)$ is the nested union over $i \geqslant 0$ of the Zariski closed sets

$$\mathcal{E}_A^i(M) = \{\beta \in \mathbb{C}^d \mid \mathrm{rank}\, \mathcal{H}_0(M, \beta) - \mathrm{vol}(A) > i\}.$$

In particular, the rank of $\mathcal{H}_0(M, \beta)$ induces a stratification of $\mathcal{E}_A(M)$, which we call its *rank stratification*.

1.2 A homological study of rank jumps

The present article is a study of the rank stratification of $\mathcal{E}_A(M)$ when $M \subseteq S_A[\partial_A^{-1}]$ is $\mathbb{Z}^d$-graded such that the degree set $\mathbb{M} = \deg(M)$ of M is a non-trivial $\mathbb{N}A$-monoid. In particular, M is weakly toric (see Definition 2.5). If $\mathbb{M} = \mathbb{N}A$, then M is the semigroup ring S_A from (1.1) and $\mathcal{H}_0(M, \beta) = M_A(\beta)$ is the A-hypergeometric system at β. The module M could also be a localization of S_A along a subset of faces of A. As M will be fixed throughout this article, we will often not indicate dependence on M in the notation.

Examination of the long exact sequence in Euler–Koszul homology induced by the short exact sequence of weakly toric modules

$$0 \to M \to S_A[\partial_A^{-1}] \to Q \to 0$$

reveals that the *rank jump* of M at β,

$$j(\beta) = \mathrm{rank}\, \mathcal{H}_0(M, \beta) - \mathrm{vol}(A),$$

can be calculated in terms of Q by

$$j(\beta) = \mathrm{rank}\, \mathcal{H}_1(Q, \beta) - \mathrm{rank}\, \mathcal{H}_0(Q, \beta). \tag{1.2}$$

We define the *ranking arrangement* $\mathcal{R}_A(M)$ of M to be the quasidegrees of Q. Vanishing properties of Euler–Koszul homology imply that the exceptional arrangement $\mathcal{E}_A(M)$ is contained in the ranking arrangement $\mathcal{R}_A(M)$. We show in Theorem 4.3 that $\mathcal{R}_A(M)$ is the union of $\mathcal{E}_A(M)$ and an explicit collection of hyperplanes.

For a fixed $\beta \in \mathcal{E}_A(M)$, we then proceed to compute $j(\beta)$. In § 5, we combinatorially construct a *finitely generated* $\mathbb{Z}^d$-graded *ranking toric module* P^β with $\mathcal{H}_\bullet(Q, \beta) \cong \mathcal{H}_\bullet(P^\beta, \beta)$. Since $j(\beta)$ is determined by the Euler–Koszul homology of Q by (1.2), we see that P^β contains the information essential to computing the rank jump $j(\beta)$. To outline the construction of the module P^β, let

$$\mathcal{F}(\beta) = \{F \preceq A \mid \beta + \mathbb{C}F \subseteq \mathcal{R}_A(M)\}$$

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be the polyhedral complex of faces of A determined by the components of the ranking arrangement $\mathcal{R}_A(M)$ that contain β. We call the collection of integral points

$$\mathbb{E}^\beta = \mathbb{Z}^d \cap \bigcup_{F \in \mathcal{F}(\beta)} (\beta + \mathbb{C}F) \backslash (\mathbb{M} + \mathbb{Z}F)$$

the *ranking lattices* $\mathbb{E}^\beta$ of M at β. This set is a union of translates of lattices generated by faces of A, where the vectors in these lattice translates of $\mathbb{Z}F$ in $\mathbb{E}^\beta$ are precisely the degrees of Q which cause $\beta + \mathbb{C}F$ to lie in the ranking arrangement. Since it contains full lattice translates, $\mathbb{E}^\beta$ cannot be the degree set of a finitely generated S_A-module. Thus, to complete the construction of the degree set $\mathbb{P}^\beta$ of P^β, we intersect $\mathbb{E}^\beta$ with an appropriate half space (see Definition 5.6). To give a flavor of our approach for $\beta \in \mathbb{R}^d$, this is equivalent to intersecting $\mathbb{E}^\beta$ with $\mathcal{C}_A(\beta) = \mathbb{Z}^d \cap [\beta + \mathbb{R}_{\geqslant 0} A]$. By setting up the proper module structure, $\mathbb{P}^\beta = \mathcal{C}_A(\beta) \cap \mathbb{E}^\beta$ gives the $\mathbb{Z}^d$-graded degree set of the desired toric module P^β with $j(\beta) = \operatorname{rank} \mathcal{H}_1(P^\beta, \beta) - \operatorname{rank} \mathcal{H}_0(P^\beta, \beta)$. After translating the computation of the rank jump $j(\beta)$ to P^β, we obtain a generalization of the formula given by Okuyama in the case $d = 3$ [Oku06].

THEOREM 1.3. *The rank jump $j(\beta)$ of M at β can be computed from the combinatorics of the ranking lattices $\mathbb{E}^\beta$ of M at β.*

In particular, the rank of the hypergeometric system is the same at parameters which share the same ranking lattices. The proof of Theorem 1.3 can be found in § 6.4 as a special case of our main result, Theorem 6.6.

1.3 The ranking slab stratification of the exceptional arrangement

Let X and Y be subspace arrangements in $\mathbb{C}^d$. We say that a stratification $\mathcal{S}$ of X *respects* Y if, for each irreducible component Z of Y and each stratum $S \in \mathcal{S}$, either $S \cap Z = \varnothing$ or $S \subseteq Z$. A *ranking slab* of M is a stratum in the coarsest stratification of $\mathcal{E}_A(M)$ that respects the arrangements $\mathcal{R}_A(M)$ and the negatives of the quasidegrees of each of the Ext modules that determine $\mathcal{E}_A(M)$ (see Definition 4.7).

Proposition 5.4 states that the parameters $\beta, \beta' \in \mathbb{C}^d$ belong to the same ranking slab of M exactly when their ranking lattices coincide, that is, $\mathbb{E}^\beta = \mathbb{E}^{\beta'}$. Combining this with Theorem 1.3, we see that the rank of $\mathcal{H}_0(M, -)$ is constant on each ranking slab.

COROLLARY 1.4. *The function $j(-)$ is constant on each ranking slab. In particular, the stratification of the exceptional arrangement $\mathcal{E}_A(M)$ by ranking slabs refines its rank stratification.*

Hence, like $\mathcal{E}_A(M)$, each set $\mathcal{E}_A^i(M)$ is a union of translated linear subspaces of $\mathbb{C}^d$ which are generated by faces of $\mathbb{R}_{\geqslant 0} A$. In order for the ranking slab stratification of $\mathcal{E}_A(M)$ to refine its rank stratification, it must respect each of the arrangements appearing in its definition; this can be seen from Examples 4.5, 6.24, and 6.25. In particular, as $\mathcal{R}_A(M) \supsetneq \mathcal{E}_A(M)$, the exceptional arrangement $\mathcal{E}_A(M)$ does not generally contain enough information to determine its rank stratification.

1.4 A connection to the isomorphism classes of hypergeometric systems

When $M = S_A$, the ranking lattices $\mathbb{E}^\beta$ are directly related to the combinatorial sets $E_\tau(\beta)$ defined by Saito, which determine the isomorphism classes of $M_A(\beta)$. In [Sai01, ST01], various b-functions arising from an analysis of the symmetry algebra of A-hypergeometric systems are used to link these isomorphism classes to the sets $E_\tau(\beta)$. We conclude this paper with a shorter proof, replacing the use of b-functions with Euler–Koszul homology.

1.5 Outline

The following is a brief outline of this article. In § 2, we summarize definitions and results on weakly toric modules and Euler–Koszul homology, following [MMW05, SW09]. Section 3 is a study of the structure of the Euler–Koszul complex of maximal Cohen–Macaulay toric face modules. The relationship between the exceptional and ranking arrangements of M is made precise in § 4. In § 5, we define the class of ranking toric modules, which play a pivotal role in calculating the rank jump $j(\beta)$. Section 6 contains our main theorem, Theorem 6.6, which results in the computation of $j(\beta)$ for a fixed parameter β. We close with a discussion on the isomorphism classes of A-hypergeometric systems in § 7.

2. The language of Euler–Koszul homology

In this section, we summarize definitions found in the literature and set notation. Most important are the definitions of a weakly toric module [SW09] and Euler–Koszul homology [MMW05].

Let $a_1, a_2, \ldots, a_n$ denote the columns of A. For a face $F \preceq A$, let F^c denote the complement of a face F in the column set of A. If F is any subset of the columns of A, the *codimension* of F is $\operatorname{codim}(F) := \operatorname{codim}_{\mathbb{C}^d}(\mathbb{C}F)$, the codimension of the $\mathbb{C}$-vector space generated by F. The *dimension* of F is $\dim(F) = d - \operatorname{codim}_{\mathbb{C}}(\mathbb{C}F)$.

A face F of A is a *facet* of A if $\dim(F) = d-1$. Recall that the *primitive integral support function* of a facet $F \preceq A$ is the unique linear functional $p_F : \mathbb{C}^d \to \mathbb{C}$ such that:

(i) $p_F(\mathbb{Z}A) = \mathbb{Z}$;

(ii) $p_F(a_i) \geqslant 0$ for all $j = 1, \ldots, n$; and

(iii) $p_F(a_i) = 0$ exactly when $a_i \in F$.

The *volume* of a face F, denoted by $\operatorname{vol}(F)$, is the integer $\dim(F)!$ times the Euclidean volume in $\mathbb{Z}F \otimes_{\mathbb{Z}} \mathbb{R}$ of the convex hull of F and the origin.

Definition 2.1. Let $\mathbb{N}F = \{\sum_{a_i \in F} \gamma_i a_i \mid \gamma_i \in \mathbb{N}\}$ be the semigroup generated by the face F and, as in (1.1),

$$S_F = \mathbb{C}[\mathbb{N}F]$$

is the corresponding semigroup ring, called a *face ring* of A. Let $x_F = \{x_i \mid a_i \in F\}$ and $\partial_F = \{\partial_i \mid a_i \in F\}$. Define

$$R_F = \mathbb{C}[\partial_F]$$

to be the polynomial ring in ∂_F and

$$D_F = \mathbb{C}\langle x_F, \partial_F \mid [x_i, \partial_j] = \delta_{ij}, [x_i, x_j] = 0 = [\partial_i, \partial_j]\rangle$$

to be the *Weyl algebra* associated to F. Note that

$$S_F \cong R_F/(I_F + \langle \partial_{F^c}\rangle) \quad \text{with } I_F = \ker(R_F \to S_F) \text{ and } F^c = A \backslash F.$$

Definition 2.2. Let $t = t_1, \ldots, t_d$ be variables. For a face $F \preceq A$, we say that a subset $\mathbb{S} \subseteq \mathbb{Z}^d$ is an $\mathbb{N}F$*-module* if $\mathbb{S} + \mathbb{N}F \subseteq \mathbb{S}$. Further, we call an $\mathbb{N}F$-module $\mathbb{S}$ an $\mathbb{N}F$*-monoid* if it is closed under addition, that is, for all $s, s' \in \mathbb{S}$, $s + s' \in \mathbb{S}$. Given an $\mathbb{N}F$-module $\mathbb{S}$, define the S_F-module $\mathbb{C}\{\mathbb{S}\} = \bigoplus_{s \in \mathbb{S}} \mathbb{C} \cdot t^s$ as a $\mathbb{C}$-vector space with S_F-action given by $\partial_i \cdot t^s = t^{s+a_i}$. Further, $\mathbb{C}\{\mathbb{S}\}$ is equipped with a multiplicative structure given by $t^s \cdot t^{s'} = t^{s+s'}$ for $s, s' \in \mathbb{S}$ and extended $\mathbb{C}$-linearly. By definition, $\mathbb{N}F$ is an $\mathbb{N}F$-monoid and $S_F \cong \mathbb{C}\{\mathbb{N}F\}$ as rings.

Define a $\mathbb{Z}^d$-grading on $R_F \subseteq D_F$ by setting

$$\deg(\partial_i) = a_i \quad \text{and} \quad \deg(x_i) = -a_i.$$

Then $\mathbb{C}\{\mathbb{S}\}$ is naturally a $\mathbb{Z}^d$-graded S_F-module by setting $\deg(t^s) = s$.

The *saturation* of F in $\mathbb{Z}F$ is the semigroup $\widetilde{\mathbb{N}F} = \mathbb{R}_{\geqslant 0}F \cap \mathbb{Z}F$. The *saturation*, or *normalization*, of S_F is the semigroup ring of the saturation of F in $\mathbb{Z}F$, which is given by $\widetilde{S}_F = \mathbb{C}\{\widetilde{\mathbb{N}F}\}$ as a $\mathbb{Z}^d$-graded S_F-module. Note that $\widetilde{S}_F$ is a Cohen–Macaulay S_F-module [Hoc72].

If N is a $\mathbb{Z}^d$-graded R-module and $v \in \mathbb{Z}^d$, the *degree set* of N, denoted $\deg(N)$, is the set of all $v \in \mathbb{Z}^d$ such that $N_v \neq 0$. Let $N(v)$ denote the $\mathbb{Z}^d$-graded module with v'-graded piece $N(v)_{v'} = N_{v+v'}$.

We now recall the definitions of toric and weakly toric modules and their quasidegree sets, which can be found in [MMW05, Definition 4.5] and [SW09, § 5], respectively.

DEFINITION 2.3. A $\mathbb{Z}^d$-graded R-module is *toric* if it has a filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{\ell-1} \subseteq M_\ell = M$$

such that, for each i, M_i/M_{i-1} is a $\mathbb{Z}^d$-graded translate of S_{F_i} for some face $F_i \preceq A$. Notice that toric modules are necessarily finitely generated R-modules.

DEFINITION 2.4. If N is a finitely generated $\mathbb{Z}^d$-graded R-module, a vector $v \in \mathbb{C}^d$ is a *quasidegree* of N, written $v \in \operatorname{qdeg}(N)$, if v lies in the Zariski closure of $\deg(N)$ under the natural embedding $\mathbb{Z}^d \hookrightarrow \mathbb{C}^d$. Notice that if N is toric, then $\operatorname{qdeg}(N)$ is a finite subspace arrangement in $\mathbb{C}^d$, consisting of translated subspaces generated by faces of A, see [DMM10, Lemma 2.5].

A partially ordered set $(\mathfrak{S}, \leqslant)$ is *filtered* if for each $s', s'' \in \mathfrak{S}$ there exists $s \in \mathfrak{S}$ with $s' \leqslant s$ and $s'' \leqslant s$.

DEFINITION 2.5. We say that a $\mathbb{Z}^d$-graded R-module M is *weakly toric* if there is a filtered partially ordered set $(\mathfrak{S}, \leqslant)$ and a $\mathbb{Z}^d$-graded direct limit

$$\phi_s : M_s \to \varinjlim_{s \in \mathfrak{S}} M_s = M$$

where M_s is a toric R-module for each $s \in \mathfrak{S}$. We then define the *quasidegree set* of M to be

$$\operatorname{qdeg}(M) = \bigcup_{s \in \mathfrak{S}} \operatorname{qdeg}(\phi_s(M_s)),$$

where each $\operatorname{qdeg}(\phi_s(M_s))$ is defined by Definition 2.4.

Example 2.6. If $\mathbb{M} \subseteq \mathbb{Z}^d$ is an $\mathbb{N}A$-module, then $M = \mathbb{C}\{\mathbb{M}\}$ is weakly toric because it is a direct limit over $b \in \mathbb{M}$ of $S_A(-b)$ under the natural A-homogeneous inclusion $S_A(-b) \subseteq S_A[\partial_A^{-1}] \cong \mathbb{C}[\mathbb{Z}^d]$.

Example 2.7. Consider the matrix $A = \left[\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{smallmatrix}\right]$ with face $F = \left[\begin{smallmatrix} 1 \\ 4 \end{smallmatrix}\right]$. The module $S_F[\partial_F^{-1}]$ is weakly toric with quasidegree set

$$\operatorname{qdeg}(S_F[\partial_F^{-1}]) = \mathbb{C}F$$

because it is a filtered direct limit over $b \in \mathbb{Z}F$ of $S_F(-b)$. Similarly, the module $S_A[\partial_F^{-1}]$ is weakly toric with $\operatorname{qdeg}(S_A[\partial_F^{-1}]) = \mathbb{C}^2$. The quotient $S_A[\partial_F^{-1}]/S_A$ is also weakly toric. Its quasidegree set consists of the point $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ and the union of lines in $\{t_2 = k \mid k \in \mathbb{Z}_{<0}\}$, where (t_1, t_2) are the coordinates of $\mathbb{C}^2$.

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We now recall the definition of the Euler–Koszul complex of a weakly toric module M with respect to a parameter $\beta \in \mathbb{C}^d$. For $1 \leqslant i \leqslant d$, each *Euler operator* $E_i - \beta_i = \sum_{j=1}^n a_{ij} x_j \partial_j - \beta_i$ determines a $\mathbb{Z}^d$-graded D-linear endomorphism of $D \otimes_R M$, defined on a homogeneous $y \in D \otimes_R M$ by

$$(E_i - \beta_i) \circ y = (E_i - \beta_i + \deg_i(y))y$$

and extended $\mathbb{C}$-linearly. This sequence $E - \beta$ of commuting endomorphisms determines a Koszul complex $\mathcal{K}_\bullet^A(M, \beta) = \mathcal{K}_\bullet(M, \beta)$ on the left D-module $D \otimes_R M$, called the *Euler–Koszul complex* of M with parameter β. The ith *Euler–Koszul homology module* of M is $\mathcal{H}_i^A(M, \beta) = \mathcal{H}_i(M, \beta) = H_i(\mathcal{K}_\bullet(M, \beta))$. Our object of study will be the *generalized A-hypergeometric system* $\mathcal{H}_0(M, \beta)$ associated to M.

The Euler–Koszul complex defines an exact functor from the category of weakly toric modules with degree-preserving morphisms to the category of bounded complexes of $\mathbb{Z}^d$-graded left D-modules with degree-preserving morphisms, so short exact sequences of weakly toric modules yield long exact sequences of Euler–Koszul homology. Notice also that Euler–Koszul homology behaves well under $\mathbb{Z}^d$-graded translations: for $b \in \mathbb{Z}^d$,

$$\mathcal{H}_q(M(b), \beta) \cong \mathcal{H}_q(M, \beta - b)(b). \tag{2.1}$$

We close this section by recording two important vanishing results for Euler–Koszul homology.

Proposition 2.8. *For a weakly toric module M, the following are equivalent:*

(i) $\mathcal{H}_i(M, \beta) = 0$ *for all* $i \geqslant 0$*;*

(ii) $\mathcal{H}_0(M, \beta) = 0$*;*

(iii) $\beta \notin \operatorname{qdeg}(M)$.

Proof. See [SW09, Theorem 5.4]. □

Theorem 2.9. *Let M be a weakly toric module. Then $\mathcal{H}_i(M, \beta) = 0$ for all $i > 0$ and for all $\beta \in \mathbb{C}^d$ if and only if M is a maximal Cohen–Macaulay S_A-module.*

Proof. See [MMW05, Theorem 6.6] for the toric case. The extension to the weakly toric case can be found in [SW09]. □

3. Euler–Koszul homology and toric face modules

Theorem 2.9 provides a criterion for higher vanishing of Euler–Koszul homology via maximal Cohen–Macaulay S_A-modules. In this section, we provide a description of the Euler–Koszul homology modules of maximal Cohen–Macaulay S_F-modules for a face $F \preceq A$ and use it to understand the images of maps between such modules.

Throughout this section, N is a toric S_F-module for a face $F \preceq A$. Recall the definitions of toric, S_F, D_F, and R_F from Definition 2.1, and let

$$x_{F^c} = \{x_i \mid a_i \in F^c\}.$$

Notation 3.1. Let $\mathcal{I}_F$ be the lexicographically first subset of $\{1, 2, \ldots, d\}$ of cardinality $\dim(F)$ such that $\{E_i - \beta_i\}_{i \in \mathcal{I}_F}$ is a set of linearly independent Euler operators on $D \otimes_R N$. The existence of $\mathcal{I}_F$ follows from the fact that the matrix A has full rank. We use $\mathcal{K}_\bullet^F(N, \beta)$ to denote the Euler–Koszul complex on $D_F \otimes_{R_F} N$ given by the operators $\{E_i - \beta_i\}_{i \in \mathcal{I}_F}$, and set

$$\mathcal{H}_i^F(N, \beta) = H_i(\mathcal{K}_\bullet^F(N, \beta)).$$

Using the standard basis of $\mathbb{Z}A = \mathbb{Z}^d$, let

$$\mathbb{Z}F^{\perp} = \left\{ v \in \mathbb{Z}^d \,\middle|\, \sum_{i=1}^{d} v_i a_{ij} = 0 \ \forall a_j \in F \right\},$$

and let $\bigwedge^{\bullet} \mathbb{Z}F^{\perp}$ denote a complex with trivial differentials. We show now that when $\beta \in \mathbb{C}F$, $\mathcal{K}_{\bullet}(N,\beta)$ is quasi-isomorphic to a complex involving $\mathcal{K}^F_{\bullet}(N,\beta)$ and $\bigwedge^{\bullet} \mathbb{Z}F^{\perp}$.

PROPOSITION 3.2. *Let $F \preceq A$ and N be a toric S_F-module. If $\beta \in \mathbb{C}F$, then there is a quasi-isomorphism of complexes*

$$\mathcal{K}_{\bullet}(N,\beta) \simeq_{\mathrm{qis}} \mathbb{C}[x_{F^c}] \otimes_{\mathbb{C}} \mathcal{K}^F_{\bullet}(N,\beta) \otimes_{\mathbb{Z}} (\textstyle\bigwedge^{\bullet} \mathbb{Z}F^{\perp}). \tag{3.1}$$

In particular, if N is maximal Cohen–Macaulay as an S_F-module, there is a decomposition

$$\mathcal{H}_{\bullet}(N,\beta) = \mathbb{C}[x_{F^c}] \otimes_{\mathbb{C}} \mathcal{H}^F_0(N,\beta) \otimes_{\mathbb{Z}} (\textstyle\bigwedge^{\bullet} \mathbb{Z}F^{\perp}). \tag{3.2}$$

Under the hypotheses of Proposition 3.2,

$$\mathcal{H}_i(N,\beta) = \mathbb{C}[x_{F^c}] \otimes_{\mathbb{C}} \mathcal{H}^F_0(N,\beta^F)^{\binom{\mathrm{codim}(F)}{i}},$$

for $i > 0$, as shown in [Oku06]. In particular,

$$\operatorname{rank} \mathcal{H}_i(N,\beta) = \binom{\mathrm{codim}(F)}{i} \cdot \operatorname{rank} \mathcal{H}_0(N,\beta). \tag{3.3}$$

We show in Proposition 3.6 that surjections of maximal Cohen–Macaulay toric modules for nested faces yield induced maps on Euler–Koszul homology that respect the decompositions of (3.2). The additional information stored in $\bigwedge^{\bullet} \mathbb{Z}F^{\perp}$ of (3.1) shows how images of collections of such surjections overlap, which will be vital to our calculation of $j(\beta)$ in § 6.

Proof of Proposition 3.2. Fix a matrix $g_F \in \mathrm{GL}_d(\mathbb{Z})$ such that the entries of each row of $g_F F$ not corresponding to $\mathcal{I}_F$ are zero and the rows of A that do correspond to $\mathcal{I}_F$ are identical in A and $g_F A$. Setting $A' = g_F A$, S_A and $S_{A'}$ are isomorphic rings, and the matrix g_F gives a bijection of their degree sets, sending $\mathbb{N}A$ to $\mathbb{N}A'$. This identification makes N a $\mathbb{Z}A'$-graded $S_{F'}$-module, where $F' = g_F F$, and there is a quasi-isomorphism of complexes

$$\mathcal{K}_{\bullet}(N,\beta) \simeq_{\mathrm{qis}} \mathcal{K}^{A'}_{\bullet}(N, g_F\beta).$$

Let $F' = g_F F$ and $\beta' = g_F \beta$, and recall that $A' = g_F A$. By the definition of g_F, $\beta'_i = 0$ for $i \notin \mathcal{I}_F$ because $\beta' \in \mathbb{C}F'$. Let $D_{A'}$ and $R_{A'}$ denote the Weyl algebra and the polynomial ring $\mathbb{C}[\partial]$ with an A'-grading. Since N is an $S_{F'}$-module, $0 = \partial_{F'^c} \otimes N \subseteq D_{A'} \otimes_{R_{A'}} N$, and so there is an isomorphism $D_{A'} \otimes_{R_{A'}} N \cong \mathbb{C}[x_{F'^c}] \otimes_{\mathbb{C}} D_{F'} \otimes_{R_{F'}} N$. Hence the action of each element in $\{\sum_{j=1}^{n} a'_{ij} x_j \partial_j\}_{i \notin \mathcal{I}_F}$ on $D_{A'} \otimes_{R_{A'}} N$ is 0.

If $\{e_1, \ldots, e_d\}$ denotes the standard basis of $\mathbb{Z}^d = \mathbb{Z}A'$, then the set $\{g_F^{-1} e_i\}_{i \notin \mathcal{I}_F}$ generates $\mathbb{Z}F^{\perp}$ by choice of g_F. Applying the isomorphism $D_F \otimes_{R_F} N \cong D_{F'} \otimes_{R_{F'}} N$ in the reverse direction, we obtain (3.1). Finally, if N is maximal Cohen–Macaulay as an S_F-module, $\mathcal{H}^F_i(N,\beta^F) = 0$ for all $i > 0$ by Theorem 2.9. $\square$

Remark 3.3. Let δ and κ_F respectively denote the differentials of the Euler–Koszul complexes $\mathcal{K}^{g_F A}_{\bullet}(N, g_F\beta)$ and $\mathcal{K}^F_{\bullet}(N,\beta)$. Under the hypotheses of Proposition 3.2, if $i + j = q$ and $f \otimes a \otimes b \in \mathbb{C}[x_{F^c}] \otimes_{\mathbb{C}} \mathcal{K}^F_i(N,\beta) \otimes_{\mathbb{Z}} (\bigwedge^j \mathbb{Z}F^{\perp})$, then

$$\delta(f \otimes a \otimes b) = f \otimes \kappa_F(a) \otimes b$$

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is an element of

$$\mathbb{C}[x_{F^c}] \otimes_{\mathbb{C}} \mathcal{K}^F_{i-1}(N,\beta) \otimes_{\mathbb{C}} \left(\textstyle\bigwedge^j \mathbb{Z}F^{\perp}\right) \subseteq \mathcal{K}^{A'}_{q-1}(N,\beta).$$

Example 3.4. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Set $\beta = \left[\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right] \in \mathbb{C}^3$, and let e_1, e_2, e_3 denote the standard basis vectors in $\mathbb{Z}A = \mathbb{Z}^3$. Notice that every face ring of A is Cohen–Macaulay because the semigroup generated by each face of A is saturated. For the face $\varnothing \preceq A$, we choose $g_{\varnothing}$ to be the identity matrix. The proof of Proposition 3.2 shows that there is an isomorphism of complexes

$$\mathcal{K}_{\bullet}(S_{\varnothing},\beta) \cong \bigotimes_{i=1}^{3} (\mathbb{C}[x]\cdot e_i \xrightarrow{0\cdot} \mathbb{C}[x]),$$

so the Euler–Koszul homology of $S_{\varnothing}$ at β is

$$\mathcal{H}_{\bullet}(S_{\varnothing},\beta) = \wedge^{\bullet}\left(\bigoplus_{i=1}^{3} \mathbb{C}[x]\cdot e_i\right).$$

For the face $F = [a_1\ a_2]$ of A, (A,β) is again already in the desired form, so take g_F to be the identity matrix and write $\mathbb{C}[x]\langle\partial_1,\partial_2\rangle$ in place of $\mathbb{C}[x_{F^c}] \otimes_{\mathbb{C}} D_F \otimes_{R_F} S_F$. Then Proposition 3.2 implies that

$$\mathcal{H}_q(S_F,\beta) = \begin{cases} \mathbb{C}[x]\langle\partial_1,\partial_2\rangle & \text{if } q=0, \\ \mathbb{C}[x]\langle\partial_1,\partial_2\rangle\cdot e_3 & \text{if } q=1, \\ 0 & \text{otherwise.} \end{cases}$$

For $G_1 = [a_1]$ and g_{G_1} as the identity matrix,

$$\mathcal{H}_q(S_{G_1},\beta) = \begin{cases} \mathbb{C}[x]\langle\partial_1\rangle & \text{if } q=0, \\ \mathbb{C}[x]\langle\partial_1\rangle\cdot e_2 \oplus \mathbb{C}[x]\langle\partial_1\rangle\cdot e_3 & \text{if } q=1, \\ \mathbb{C}[x]\langle\partial_1\rangle\cdot e_2\wedge e_3 & \text{if } q=2, \\ 0 & \text{otherwise.} \end{cases}$$

For the face $G_2 = [a_2]$, setting $g_{G_2} = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$ yields the decomposition

$$\mathcal{H}_q(S_{G_2},\beta) = \begin{cases} \mathbb{C}[x]\langle\partial_2\rangle & \text{if } q=0, \\ \mathbb{C}[x]\langle\partial_2\rangle\cdot(e_1-e_2) \oplus \mathbb{C}[x]\langle\partial_1\rangle\cdot e_3 & \text{if } q=1, \\ \mathbb{C}[x]\langle\partial_2\rangle\cdot(e_1-e_2)\wedge e_3 & \text{if } q=2, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.5. *Let $G \preceq F$ be faces of A, N be a toric S_F-module, and L be a toric S_G-module. Regard N and L as toric S_A-modules via the natural maps $S_A \twoheadrightarrow S_F \twoheadrightarrow S_G$. Let $\pi : N \to L$ be a*

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morphism of S_A-modules. Then there is a commutative diagram

$$\begin{tikzcd} \mathcal{K}_\bullet(N,\beta) \arrow[r, "\mathcal{K}_\bullet(\pi,\beta)"] \arrow[d] & \mathcal{K}_\bullet(L,\beta) \arrow[d] \\ \mathbb{C}[x_{F^c}] \otimes_{\mathbb{C}} \mathcal{K}^F_\bullet(N,\beta) \otimes_{\mathbb{Z}} (\bigwedge^\bullet \mathbb{Z}F^\perp) \arrow[r] & \mathbb{C}[x_{G^c}] \otimes_{\mathbb{C}} \mathcal{K}^G_\bullet(L,\beta) \otimes_{\mathbb{Z}} (\bigwedge^\bullet \mathbb{Z}G^\perp) \end{tikzcd}$$

with vertical maps as in (3.1).

Proof. By choice of $\mathcal{I}_F$ and $\mathcal{I}_G$ in Notation 3.1 and the corresponding g_F and g_G, the diagram

$$\begin{tikzcd} \mathcal{K}_\bullet(N,\beta) \arrow[rr, "\mathcal{K}_\bullet(\pi,\beta)"] \arrow[d] & & \mathcal{K}_\bullet(L,\beta) \arrow[d] \\ \mathcal{K}^{g_F A}_\bullet(N,\beta) \arrow[r] & \mathcal{K}^{g_F A}_\bullet(L,\beta) \arrow[r] & \mathcal{K}^{g_G A}_\bullet(L,\beta) \end{tikzcd}$$

commutes. Hence the result follows from the proof of Proposition 3.2. $\square$

Proposition 3.6. *Let $G \preceq F$ be faces of A, N be a maximal Cohen–Macaulay toric S_F-module, and L be a maximal Cohen–Macaulay toric S_G-module. Regard N and L as toric S_A-modules via the natural maps $S_A \twoheadrightarrow S_F \twoheadrightarrow S_G$. Let $\pi : N \twoheadrightarrow L$ be a surjection of S_A-modules. If $\beta \in \mathbb{C}G$, then*

$$\operatorname{image} \mathcal{H}_\bullet(\pi,\beta) = \mathbb{C}[x_{G^c}] \otimes_{\mathbb{C}} \mathcal{H}^G_0(L,\beta) \otimes_{\mathbb{C}} (\textstyle\bigwedge^\bullet \mathbb{Z}F^\perp)$$

as a submodule of

$$\mathbb{C}[x_{G^c}] \otimes_{\mathbb{C}} \mathcal{H}^G_0(L,\beta) \otimes_{\mathbb{C}} (\textstyle\bigwedge^\bullet \mathbb{Z}G^\perp).$$

Example 3.7 (Continuation of Example 3.4). The surjection of face rings given by $\pi : S_F \twoheadrightarrow S_{G_1}$ induces the following image in Euler–Koszul homology:

$$\operatorname{image} \mathcal{H}_q(\pi,\beta) = \begin{cases} \mathbb{C}[x]\langle\partial_1\rangle & \text{if } q=0, \\ \mathbb{C}[x]\langle\partial_1\rangle \cdot e_3 & \text{if } q=1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Proposition 3.6. With $A' = g_F A$, the image of $\mathcal{H}_\bullet(\pi,\beta)$ is isomorphic to the image of $\mathcal{H}^{A'}_\bullet(\pi,\beta)$. By Proposition 3.2, there are decompositions

$$\mathcal{H}^{A'}_\bullet(N,\beta) = \mathbb{C}[x_{F^c}] \otimes_{\mathbb{C}} \mathcal{H}^F_0(N,\beta) \otimes_{\mathbb{Z}} (\textstyle\bigwedge^\bullet \mathbb{Z}F^\perp)$$

and

$$\mathcal{H}^{A'}_\bullet(L,\beta) = \mathbb{C}[x_{F^c}] \otimes_{\mathbb{C}} \mathcal{H}^F_\bullet(L,\beta) \otimes_{\mathbb{Z}} (\textstyle\bigwedge^\bullet \mathbb{Z}F^\perp),$$

so it is enough to find the image of $\mathcal{H}^F_0(N,\beta)$ as a submodule of $\mathcal{H}^F_\bullet(L,\beta)$. The result now follows because the sequence

$$\mathcal{H}^F_0(N,\beta) \xrightarrow{\mathcal{H}^F_0(\pi,\beta)} \mathcal{H}^F_0(L,\beta) \longrightarrow 0$$

is exact, $\mathcal{H}^F_0(L,\beta) = \mathbb{C}[x_{F\setminus G}] \otimes_{\mathbb{C}} \mathcal{H}^G_0(L,\beta)$, and $\mathcal{H}^F_i(N,\beta) = 0$ for $i > 0$. $\square$

Example 3.8 (Continuation of Example 3.7). Let $\pi_i : S_{G_i} \twoheadrightarrow S_{\varnothing}$ for $i = 1, 2$. Then

$$\operatorname{image} \mathcal{H}_q(\pi_1, \beta) = \begin{cases} \mathbb{C}[x] & \text{if } q = 0, \\ \mathbb{C}[x] \cdot e_2 \oplus \mathbb{C}[x] \cdot e_3 & \text{if } q = 1, \\ \mathbb{C}[x] \cdot e_2 \wedge e_3 & \text{if } q = 2, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\operatorname{image} \mathcal{H}_q(\pi_2, \beta) = \begin{cases} \mathbb{C}[x] & \text{if } q = 0, \\ \mathbb{C}[x] \cdot (e_2 - e_1) \oplus \mathbb{C}[x] \cdot e_3 & \text{if } q = 1, \\ \mathbb{C}[x] \cdot (e_2 - e_1) \wedge e_3 & \text{if } q = 2, \\ 0 & \text{otherwise.} \end{cases}$$

The intersection of the images of Euler–Koszul homology at β applied to π_1 and π_2 is

$$[\operatorname{image} \mathcal{H}_q(\pi_1, \beta)] \cap [\operatorname{image} \mathcal{H}_q(\pi_2, \beta)] = \begin{cases} \mathbb{C}[x] & \text{if } q = 0, \\ \mathbb{C}[x] \cdot e_3 & \text{if } q = 1, \\ 0 & \text{otherwise} \end{cases}$$

because $\mathbb{Z}G_1^{\perp} \cap \mathbb{Z}G_2^{\perp} = \mathbb{Z} \cdot e_3$.

We close this section with an observation that is vital to our rank jumps computations. For faces $F_1, F_2 \preceq A$, set $G = F_1 \cap F_2$. Let N_i be maximal Cohen–Macaulay toric S_{F_i}-modules, L be a maximal Cohen–Macaulay toric S_G-module, and $\pi_i : N_i \twoheadrightarrow L$ be S_A-module surjections. Suppose that $\beta \in \mathbb{C}G$. Using the equality $\mathbb{Z}F^{\perp} \cap \mathbb{Z}G^{\perp} = \mathbb{Z}[F \cup G]^{\perp}$, Proposition 3.6 implies that

$$\operatorname{image} \mathcal{H}_i(\pi_1, \beta) \cap \operatorname{image} \mathcal{H}_i(\pi_2, \beta) = \mathbb{C}[x_{G^c}] \otimes_{\mathbb{C}} \mathcal{H}_0^G(L, \beta) \otimes_{\mathbb{C}} \big(\textstyle\bigwedge^i \mathbb{Z}[F \cup G]^{\perp}\big),$$

which has rank

$$\binom{\operatorname{codim}_{\mathbb{C}^d}(\mathbb{C}F_1 + \mathbb{C}F_2)}{i} \cdot \operatorname{rank} \mathcal{H}_0^G(L, \beta).$$

4. Stratifications of the exceptional arrangement

Let $\mathbb{M} \subseteq \mathbb{Z}^d$ be a non-empty $\mathbb{N}A$-monoid (see Definition 2.2), so that the non-trivial module $M = \mathbb{C}\{\mathbb{M}\} \subseteq \mathbb{C}\{\mathbb{Z}^d\} \cong S_A[\partial_A^{-1}]$ is weakly toric (see Example 2.6) and

$$\mathbb{M} = \deg(M). \tag{4.1}$$

Since $\mathbb{M}$ is an $\mathbb{N}A$-monoid, the generic rank of $\mathcal{H}_0(M, -)$ is $\operatorname{vol}(A)$. The *rank jump* of M at β is the non-negative integer

$$j(\beta) = \operatorname{rank} \mathcal{H}_0(M, \beta) - \operatorname{vol}(A),$$

and the *exceptional arrangement* associated to M is the set

$$\mathcal{E}_A(M) = \{\beta \in \mathbb{C}^d \mid j(\beta) > 0\}$$

of parameters with non-zero rank jump. By [MMW05, SW09], the exceptional arrangement can be described in terms of certain Ext modules involving M, namely

$$\mathcal{E}_A(M) = -\bigcup_{i=0}^{d-1} \operatorname{qdeg}(\operatorname{Ext}_R^{n-i}(M, R)(-\varepsilon_A)), \tag{4.2}$$

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where $\varepsilon_A = \sum_{i=1}^n a_i$. It follows that $\mathcal{E}_A(M)$ is a union of translates of linear subspaces spanned by the faces of A, see [MMW05, Corollary 9.3].

We begin our study of $j(\beta)$ with the short exact sequence

$$0 \to M \to S_A[\partial_A^{-1}] \to Q \to 0. \tag{4.3}$$

While Q is not a Noetherian S_A-module, it is a filtered limit of Noetherian $\mathbb{Z}^d$-graded S_A-modules and is therefore weakly toric (see Definition 2.5). Thus the *ranking arrangement* of M

$$\mathcal{R}_A(M) = \operatorname{qdeg}(Q)$$

is an infinite union of translates of linear subspaces of $\mathbb{C}^d$ spanned by proper faces of A. Since $S_A[\partial_A^{-1}]$ is a maximal Cohen–Macaulay S_A-module, Theorem 2.9 implies that $\mathcal{H}_i(S_A[\partial_A^{-1}], \beta) = 0$ for all $i > 0$. Moreover, by [Oku06, Theorem 4.2], $\operatorname{rank} \mathcal{H}_0(S_A[\partial_A^{-1}], \beta) = \operatorname{vol}(A)$ for all β. Examination of the long exact sequence in Euler–Koszul homology from (4.3) reveals that

$$j(\beta) = \operatorname{rank} \mathcal{H}_1(Q, \beta) - \operatorname{rank} \mathcal{H}_0(Q, \beta). \tag{4.4}$$

This implies that, for $\beta \in \mathcal{E}_A(M)$, $\mathcal{H}_1(Q, \beta)$ is non-zero. Therefore there is an inclusion $\mathcal{E}_A(M) \subseteq \mathcal{R}_A(M)$. We make this relationship precise in Theorem 4.3.

Lemma 4.1. *Let $v \in \mathbb{Z}^d$. The number of irreducible components of $\mathcal{R}_A(S_A)$ which intersect $v + \mathbb{R}_{\geqslant 0}A$ is finite.*

Proof. View S_A and its shifted saturation $\widetilde{S}_A(-v)$ as graded submodules of $S_A[\partial_A^{-1}]$. To see that the intersection $\mathcal{R}_A(S_A) \cap (v + \mathbb{R}_{\geqslant 0}A)$ involves only a finite number of irreducible components of $\mathcal{R}_A(S_A)$, it is enough to show that the arrangement given by the quasidegrees of the module $\widetilde{S}_A(-v)/(S_A \cap \widetilde{S}_A(-v))$ has finitely many irreducible components. This follows since $\widetilde{S}_A(-v)$ is toric. □

Recall from (4.1) that $\mathbb{M} = \deg(M)$. For $b \in \mathbb{Z}^d$ and $F \preceq A$, let

$$\nabla(M, b) = \{F \preceq A \mid b \in \mathbb{M} + \mathbb{Z}F\}.$$

Lemma 4.2. *Let $M \subseteq S_A[\partial_A^{-1}]$ be a weakly toric module, $b \in \mathbb{Z}^d$, $F \preceq A$ be a face of codimension at least two, and $\alpha \in \mathbb{Z}^d$ be an interior vector of $\mathbb{N}F$. If F is maximal among faces of A not in $\nabla(M, b)$, then for all sufficiently large positive integers m, the vector $b - m\alpha \in \mathcal{E}_A(M)$ is an exceptional degree of M.*

Proof. This is [MM05, Lemma 14] when $M = S_A$ and A is homogeneous. (The matrix A is called *homogeneous* when the vector $(1, 1, \ldots, 1)$ is in the $\mathbb{Q}$-row span of A.) The same argument yields this generalization by $\mathbb{Z}^d$-graded local duality, see [BH93, § 3.5]. □

Theorem 4.3. *Let $M \subseteq S_A[\partial_A^{-1}]$ be a weakly toric module. The ranking arrangement $\mathcal{R}_A(M)$ contains the exceptional arrangement $\mathcal{E}_A(M)$ and*

$$\mathcal{R}_A(M) = \mathcal{E}_A(M) \cup \mathcal{Z}_A(M),$$

where $\mathcal{Z}_A(M)$ is pure of codimension one.

Proof. We must show that $\mathcal{E}_A(M)$ contains all irreducible components of $\mathcal{R}_A(M)$ of codimension at least two. To this end, let $\beta \in \mathcal{R}_A(M)$ be such that $\beta + \mathbb{C}F \subseteq \mathcal{R}_A(M) = \operatorname{qdeg}(Q)$ is an irreducible component with $\operatorname{codim}(F) \geqslant 2$. Then there are submodules $M', M'' \subseteq Q$ and $b' \in \mathbb{Z}^d$ such that $M'/M'' \cong S_F(b')$ and $b' + \mathbb{C}F = \beta + \mathbb{C}F$. In fact, there is a $b \in \deg(M'/M'')$

with $b+\widetilde{\mathbb{N}F} \subseteq \deg(Q)$ and $b+\mathbb{C}F=\beta+\mathbb{C}F$. We may further choose b so that F is maximal among faces of A that are not in the set $\nabla(M,b)$. To see this, first note that $F \notin \nabla(M,b+r)$ for all $r \in \widetilde{\mathbb{N}F}$. Indeed, for if $(b+\widetilde{\mathbb{N}F}) \cap (\mathbb{M}+\mathbb{Z}F) \neq \varnothing$ then there are $a \in b+\widetilde{\mathbb{N}F}$ and $s \in \mathbb{N}F$ with $a+s \in \mathbb{M} \cap (b+\widetilde{\mathbb{N}F}) \subseteq \mathbb{M} \cap \deg(Q) = \varnothing$, which is a contradiction.

Since $b+\mathbb{C}F=\beta+\mathbb{C}F$ is an irreducible component of $\operatorname{qdeg}(Q)$, it suffices to show that b can be chosen so that each facet F' of A is in $\nabla(M,b)$. First, if $F \not\preceq F'$, then by Lemma 4.1, there are at most a finite number of translates of $\mathbb{C}F'$ that define components of $\operatorname{qdeg}(Q)$ and intersect $b+\widetilde{\mathbb{N}F}$; write these as $c_1+\mathbb{C}F', \dots, c_k+\mathbb{C}F'$. If necessary, replace b by a vector $b' \in b+\widetilde{\mathbb{N}F}$ such that $(b'+\widetilde{\mathbb{N}F}) \cap (c_i+\mathbb{C}F')=\varnothing$ to assume that F' is in $\nabla(M,b)$. Note that after such a replacement, it is still true that $(b+\widetilde{\mathbb{N}F}) \cap (\mathbb{M}+\mathbb{Z}F) \neq \varnothing$ by the previous paragraph, so $F \notin \nabla(M,b)$. Next, suppose that $F \preceq F'$. If $(b+\widetilde{\mathbb{N}F'}) \cap \mathbb{M}=\varnothing$, then $b+\widetilde{\mathbb{N}F'} \subseteq \deg(Q)$, an impossibility because $b+\mathbb{C}F$ defines an irreducible component of $\operatorname{qdeg}(Q)$. Thus it must be that $(b+\widetilde{\mathbb{N}F'}) \cap \mathbb{M} \neq \varnothing$. In this case, $b \in \mathbb{M}+\mathbb{Z}F'$, so F' is in $\nabla(M,b)$. Hence every facet F' of A is in $\nabla(M,b)$, and the claim on the choice of b is established.

Let $\alpha \in \mathbb{Z}^d$ be an interior vector of $\mathbb{N}F$. Lemma 4.2 implies that for all sufficiently large integers m, the vector $b-m\alpha \in \mathcal{E}_A(M)$. Therefore $\beta+\mathbb{C}F=b+\mathbb{C}F \subseteq \mathcal{E}_A(M)$. □

Notation 4.4. For $\beta \in \mathbb{C}^d$, the *β-components* $\mathcal{R}_A(M,\beta)$ of the ranking arrangement of M are the union of the irreducible components of $\mathcal{R}_A(M)$ which contain β. Since A has a finite number of faces, $\mathcal{R}_A(M,\beta)$ has finitely many irreducible components.

By [MMW05, Porism 9.5], the exceptional arrangement $\mathcal{E}_A(S_A)$ of the A-hypergeometric system $\mathcal{H}_0(S_A,\beta)=M_A(\beta)$ has codimension at least two. In the following example we show that there may be components of $\mathcal{E}_A(S_A)$ which are embedded in codimension-one components of the ranking arrangement $\mathcal{R}_A(S_A)$. In particular, the Zariski closure of $\mathcal{E}_A(S_A)\backslash\mathcal{Z}_A(S_A)$ may not agree with $\mathcal{E}_A(S_A)$.

Example 4.5. Let

$$A=\begin{bmatrix} 2 & 3 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

with $\operatorname{vol}(A)=15$, and label the faces $F=[a_1\ a_2\ a_3]$ and $G=[a_3]$. With $\beta=\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]$, the exceptional arrangement of $M=S_A$ is properly contained in a hyperplane component of the ranking arrangement:

$$\mathcal{E}_A(S_A)=\beta+\mathbb{C}G \subsetneq \beta+\mathbb{C}F=\mathcal{R}_A(S_A,\beta) \subsetneq \mathcal{R}_A(S_A).$$

We will discuss the rank jumps of $M_A(\beta)$ in Examples 5.7 and 6.3.

One goal of § 6 is to understand the structure of the sets $\mathcal{E}_A^i(M)=\{\beta \in \mathbb{C}^d \mid j(\beta)>i\}$. We will achieve this by stratifying $\mathcal{E}_A(M)$ by ranking slabs (see Definition 4.7). Another description of ranking slabs (via translates of certain lattices contained in the β-components $\mathcal{R}_A(M,\beta)$) will be given in Proposition 5.4.

Definition 4.6. Let X and Y be subspace arrangements in $\mathbb{C}^d$. We say that a stratification $\mathcal{S}$ of X *respects* Y if for each irreducible component Z of Y and each stratum $S \in \mathcal{S}$, either $S \cap Z=\varnothing$ or $S \subseteq Z$.

Definition 4.7. A *ranking slab* of M is a stratum in the coarsest stratification of $\mathcal{E}_A(M)$ that respects each of the following arrangements: $\mathcal{R}_A(M)$ and $-\mathrm{qdeg}(\mathrm{Ext}_R^{n-i}(M,R)(-\varepsilon_A))$ for $0 \leqslant i < d$.

Since each of the arrangements used in Definition 4.7 is determined by the quasidegrees of a weakly toric module, the closure of each ranking slab of M is the translate of a linear subspace of $\mathbb{C}^d$ that is generated by a face of A. Corollary 1.4 states that $j(-)$ is constant on each ranking slab, so each $\mathcal{E}_A^i(M)$ with $i \geqslant 0$ is a union of translates of linear subspaces of $\mathbb{C}^d$ that are spanned by faces of A. It then follows that the stratification of $\mathcal{E}_A(M)$ by ranking slabs refines its rank stratification. While this is generally a strict refinement, Examples 6.3 and 6.4 show that the two stratifications may coincide for parameters close enough to the cone $\mathbb{R}_{\geqslant 0}A$. We wish to emphasize that the rank jump $j(\beta)$ is not simply determined by holes within the semigroup $\mathbb{N}A$, as can be seen in Example 6.26.

Definition 4.8. A *slab* is a set of parameters in $\mathbb{C}^d$ that lie on a unique irreducible component of the exceptional arrangement $\mathcal{E}_A(M)$ [MM05].

We will show by example that rank need not be constant on a slab. In Example 6.4, this failure results from 'embedded' components of $\mathcal{E}_A(M)$, while, in Example 6.24, it is due to the hyperplanes of $\mathcal{R}_A(M)$ that strictly refine the arrangement stratification of $\mathcal{E}_A(M)$. Together with Example 6.25, these examples show that each of the arrangements listed in Definition 4.7 is necessary to determine such a geometric refinement of the rank stratification of $\mathcal{E}_A(M)$.

5. Ranking toric modules

As in § 4, let $\mathbb{M} \subseteq \mathbb{Z}^d$ be a non-empty $\mathbb{N}A$-monoid (see Definition 2.2), so that $M = \mathbb{C}\{\mathbb{M}\} \subseteq \mathbb{C}\{\mathbb{Z}^d\} \cong S_A[\partial_A^{-1}]$ is a non-trivial weakly toric module (see Example 2.6). For a fixed $\beta \in \mathcal{E}_A(M)$, we know from (4.4) that Q can be used to compute the rank jump $j(\beta)$. However, this module contains a large amount of excess information that does not play a role in $\mathcal{H}_\bullet(Q;\beta)$. To isolate the graded pieces of Q that impact $j(\beta)$, we will define weakly toric modules $S^\beta \subseteq T^\beta$ so that:

$$
\begin{array}{ll}
\text{(i)} & M \subseteq S^\beta \subseteq T^\beta \subseteq S_A[\partial_A^{-1}]; \\[1ex]
\text{(ii)} & \mathcal{R}_A(M,\beta) = \mathrm{qdeg}\left(\dfrac{T^\beta}{S^\beta}\right); \\[1ex]
\text{(iii)} & \beta \notin \mathrm{qdeg}\left(\dfrac{S_A[\partial_A^{-1}]}{T^\beta}\right); \\[1ex]
\text{(iv)} & \beta \notin \mathrm{qdeg}\left(\dfrac{S^\beta}{M}\right); \quad \text{and} \\[1ex]
\text{(v)} & \mathbb{P}^\beta = \deg\left(\dfrac{T^\beta}{S^\beta}\right) \quad \text{is a union of translates of } \widetilde{\mathbb{N}F} \text{ for various } F \preceq A.
\end{array}
\tag{5.1}
$$

In Proposition 5.10, we show that Properties (i)–(iv) of (5.1) allow us to replace Q with $P^\beta = T^\beta/S^\beta$ when calculating $j(\beta)$. To use this module to actually compute $j(\beta)$, we will encounter other toric modules with structure similar to P^β, which are called *ranking toric modules*. Property (v) allows P^β (and similarly, any ranking toric module) to be decomposed into *simple ranking toric modules*. These modules are constructed so that their Euler–Koszul homology modules have easily computable ranks. At the end of § 5.2, we outline more specifically how simple ranking toric modules will play a role in our computation of $j(\beta)$.

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5.1 Combinatorial objects controlling rank

We now construct the class of *ranking toric modules*, which includes the module P^β coming from (5.1). These modules will be constructed via their degree sets, which are unions of $\widetilde{\mathbb{N}F}$-modules for various $F \preceq A$. We begin by isolating the translated lattices contained in $\deg(Q)$ that lie in the β-components $\mathcal{R}_A(M,\beta) \subseteq \operatorname{qdeg}(Q) = \mathcal{R}_A(M)$ of the ranking arrangement of M (see Notation 4.4). The union of these translated lattices will be denoted by $\mathbb{E}^\beta$.

Definition 5.1.

(i) Let

$$\mathcal{F}(\beta) = \{F \preceq A \mid \beta + \mathbb{C}F \subseteq \mathcal{R}_A(M)\}$$

be the set of faces of A corresponding to the β-components $\mathcal{R}_A(M,\beta)$ of the ranking arrangement of M. This set $\mathcal{F}(\beta)$ is a polyhedral cell complex, and $\mathcal{R}_A(M,\beta)$ is the union $\mathcal{R}_A(M,\beta) = \bigcup_{F\in\mathcal{F}(\beta)}(\beta + \mathbb{C}F)$.

(ii) For each $F \in \mathcal{F}(\beta)$, let

$$\mathbb{E}^\beta_F = \mathbb{Z}^d \cap (\beta + \mathbb{C}F) \backslash (\mathbb{M} + \mathbb{Z}F).$$

Lemma 4.2 and Theorem 4.3 together imply that $\mathbb{E}^\beta_F$ is non-empty exactly when there is a containment $\beta + \mathbb{C}F \subseteq \mathcal{R}_A(M,\beta)$.

(iii) Since $\mathbb{M}$ is an $\mathbb{N}A$-monoid, $\mathbb{M} + \mathbb{Z}F$ is closed under addition, so $\mathbb{E}^\beta_F$ is $\mathbb{Z}F$-stable. Thus there is a finite set of $\mathbb{Z}F$-orbit representatives B^β_F such that

$$\mathbb{E}^\beta_F = \bigsqcup_{b\in B^\beta_F} (b + \mathbb{Z}F) \tag{5.2}$$

is partitioned into $\mathbb{Z}F$-orbits as a disjoint union over B^β_F. Notice that $|B^\beta_F| \leqslant [(\mathbb{Z}^d \cap \mathbb{R}F) : \mathbb{Z}F]$.

(iv) The $\mathbb{Z}F$-orbits $b + \mathbb{Z}F$ in (5.2) are the translated lattices that we will use to construct ranking toric modules. Each is determined by the pair (F, b). We denote the collection of such pairs by

$$\mathcal{J}(\beta) = \{(F,b) \in \mathcal{F}(\beta) \times B^\beta_F \mid (b + \mathbb{Z}F) \subseteq \mathbb{E}^\beta_F\}.$$

(v) For a subset $J \subseteq \mathcal{J}(\beta)$, let

$$\mathbb{E}^\beta_J = \bigcup_{(F,b)\in J} (b + \mathbb{Z}F).$$

The maximal case determines the *ranking lattices* of M at β:

$$\mathbb{E}^\beta := \mathbb{E}^\beta_{\mathcal{J}(\beta)} = \bigcup_{(F,b)\in\mathcal{J}(\beta)} (b + \mathbb{Z}F).$$

Notation 5.2. Many of the objects we define in this section are dependent upon a subset $J \subseteq \mathcal{J}(\beta)$, and this dependence is indicated by the subscript J. Whenever we omit this subscript, it is understood that $J = \mathcal{J}(\beta)$.

By the upcoming Proposition 5.4, two parameters $\beta, \beta' \in \mathbb{C}^d$ belong to the same ranking slab exactly when $\mathbb{E}^\beta = \mathbb{E}^{\beta'}$. This is what will be used to show that the rank jump $j(\beta)$ is constant on ranking slabs.

Lemma 5.3. *The Zariski closure of the ranking lattices $\mathbb{E}^\beta$ of M at β coincides with the β-components $\mathcal{R}_A(M,\beta)$ of the ranking arrangement.*

Proof. It is clear from the definitions that $\mathbb{E}^\beta \subseteq \mathcal{R}_A(M, \beta)$. For the reverse containment, if $\beta + \mathbb{C}F \subseteq \mathcal{R}_A(M, \beta)$, then there exists a vector $b \in \beta + \mathbb{C}F$ such that $b + \widetilde{\mathbb{N}F} \subseteq \mathbb{Z}A \backslash \mathbb{M}$. This implies that $(b + \mathbb{N}F) \cap (\mathbb{M} + \mathbb{Z}F)$ is empty, so the claim now follows from the definition of quasidegree sets in Definitions 2.4 and 2.5. □

PROPOSITION 5.4. *The parameters $\beta, \beta' \in \mathbb{C}^d$ belong to the same ranking slab if and only if the ranking lattices of M at β and β' coincide, that is, if $\mathbb{E}^\beta = \mathbb{E}^{\beta'}$.*

Proof. This is a consequence of Lemmas 5.3, 4.2, and Theorem 4.3. □

Notation 5.5. In light of Proposition 5.4, use equality of ranking lattices to extend the ranking slab stratification of $\mathcal{E}_A(M)$ to the parameter space $\mathbb{C}^d$.

One might try making the sets $\mathbb{E}^\beta_J$ in Definition 5.1 the degree sets of ranking toric modules. However, while the natural map $\mathbb{E}^\beta_F \to \mathbb{E}^\beta_G$ given by faces $G \preceq F$ induces a vector space map $\mathbb{C}\{\mathbb{E}^\beta_F\} \to \mathbb{C}\{\mathbb{E}^\beta_G\}$, this induced map is not a morphism of S_F-modules because it sends units to zero. To overcome this, we introduce the lattice points in a certain polyhedron, denoted by $\mathcal{C}_A(\beta)$, and intersect it with $\mathbb{E}^\beta_J$ to produce the degree set of a ranking toric module.

DEFINITION 5.6.

(i) Recall the primitive integral support functions p_F from the beginning of § 2. In order to construct a ranking toric module from $\mathbb{E}^\beta_J$, (and achieve the various quasidegree sets proposed in (5.1)), set

$$\mathcal{C}_A(\beta) = \left\{ v \in \mathbb{Z}^d \;\middle|\; \text{for each facet } F \text{ of } A, \; \begin{array}{ll} p_F(v) \geqslant p_F(\beta) & \text{if } p_F(\beta) \in \mathbb{R}, \\ p_F(v) \geqslant 0 & \text{else} \end{array} \right\}.$$

For $\beta \in \mathbb{R}^d$, notice that $\mathcal{C}_A(\beta) = \mathbb{Z}^d \cap (\beta + \mathbb{R}_{\geqslant 0}A)$ is simply the integral points in the cone $\mathbb{R}_{\geqslant 0}A$ after translation by β.

(ii) For a pair $(F, b) \in \mathcal{J}(\beta)$, let

$$\mathbb{P}^\beta_{F,b} = \mathcal{C}_A(\beta) \cap [b + \mathbb{Z}F].$$

The degree sets of ranking toric modules are of the form

$$\mathbb{P}^\beta_J = \bigcup_{(F,b)\in J} \mathbb{P}^\beta_{F,b} = \mathcal{C}_A(\beta) \cap \mathbb{E}^\beta_J \tag{5.3}$$

for $J \subseteq \mathcal{J}(\beta)$. The largest of these is

$$\mathbb{P}^\beta := \mathbb{P}^\beta_{\mathcal{J}(\beta)} = \mathcal{C}_A(\beta) \cap \mathbb{E}^\beta, \tag{5.4}$$

the degree set appearing in (5.1).

Example 5.7 (Continuation of Example 4.5). With $\beta = \begin{bmatrix}1\\0\\0\end{bmatrix} \in \mathcal{E}_A(S_A)$ and $b = \begin{bmatrix}1\\1\\0\end{bmatrix}$, the sets of Definitions 5.1 and 5.8 are

$$\mathcal{F}(\beta) = \{\varnothing, G, F\},$$
$$\mathcal{J}(\beta) = \{(\varnothing, \beta), (G, b), (F, \beta)\},$$
$$\mathbb{E}^\beta = [\beta + \mathbb{Z}F] \sqcup [b + \mathbb{Z}G] \quad \text{and} \quad \mathbb{P}^\beta = [\beta + \mathbb{N}F] \sqcup [b + \mathbb{N}G].$$

Having defined the degree sets of ranking toric modules in (5.3), we now construct the modules themselves. Along the way, we meet the modules that satisfy the requirements of (5.1).

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Definition 5.8.

(i) Each ranking toric module will be a quotient of the module T^β (for some $\beta \in \mathbb{C}^d$), where

$$\mathbb{T}^\beta = \mathbb{M} \cup \left[\bigcup_{b \in \mathbb{P}^\beta} (b + \widetilde{\mathbb{N}A}) \right] \quad \text{and} \quad T^\beta = \mathbb{C}\{\mathbb{T}^\beta\}.$$

Notice that if $\beta \in \mathbb{Z}^d$, then $\mathbb{T}^\beta = \mathbb{M} \cup (\beta + \widetilde{\mathbb{N}A})$. The simplest case occurs when $\mathbb{M} = \mathbb{N}A$ and $\beta \in \widetilde{\mathbb{N}A}$, so that $\mathbb{T}^\beta = \widetilde{\mathbb{N}A}$.

(ii) For $J \subseteq \mathcal{J}(\beta)$, let

$$\mathbb{S}^\beta_J = \mathbb{T}^\beta \backslash \mathbb{P}^\beta_J \quad \text{and} \quad S^\beta_J = \mathbb{C}\{\mathbb{S}^\beta_J\}.$$

We show in Proposition 5.9 that T^β and S^β_J are indeed weakly toric modules (see Definition 2.5). When $J = \mathcal{J}(\beta)$, these modules satisfy the properties (5.1). By Notation 5.2, $S^\beta = S^\beta_{\mathcal{J}(\beta)}$.

(iii) For a subset $J \subseteq \mathcal{J}(\beta)$, the quotient

$$P^\beta_J = \frac{T^\beta}{S^\beta_J}$$

has degree set $\deg(P^\beta_J) = \mathbb{P}^\beta_J$, as recorded in Proposition 5.9. In Proposition 5.10, we show that Q in (4.4) can be replaced by $P^\beta = P^\beta_{\mathcal{J}(\beta)}$ when computing $j(\beta)$.

(iv) If a toric module N is isomorphic to P^β_J for a pair (M, β) and a subset $J \subseteq \mathcal{J}(\beta)$, we say that N is a *ranking toric module* determined by J.

Proposition 5.9. *Let $J \subseteq \mathcal{J}(\beta)$. There are containments of weakly toric modules:*

$$M \subseteq S^\beta_J \subseteq T^\beta \subseteq S_A[\partial_A^{-1}]. \tag{5.5}$$

In particular, $P^\beta_J = T^\beta / S^\beta_J$ is a ranking toric module with degree set $\mathbb{P}^\beta_J$.

Proof. By construction, we have the containment $\mathbb{P}^\beta_J \subseteq \mathbb{E}^\beta$. Since the intersection of $\mathbb{E}^\beta$ and $\mathbb{M}$ is empty, $\mathbb{P}^\beta_J \cap \mathbb{M}$ is empty as well. Hence $M \subseteq S^\beta_J$. The other containments in (5.5) are obvious. It is clear from the definitions that the degree sets of all modules in question are closed under addition with elements of $\mathbb{N}A$, so they are all weakly toric modules. For the second statement, since $\mathbb{P}^\beta_J \subseteq \mathcal{C}_A(\beta)$, P^β_J is a finitely generated S_A-module and therefore P^β_J is toric. □

By Lemma 5.3 and the definition (5.4) of $\mathbb{P}^\beta$, the arrangement $\operatorname{qdeg}(P^\beta)$ coincides with the β-components $\mathcal{R}_A(M, \beta)$ of M at β. Further, the construction of P^β in Definition 5.8 is such that P^β can replace Q in (4.4) when calculating $j(\beta)$.

Proposition 5.10. *The Euler–Koszul complexes $\mathcal{K}_\bullet(Q, \beta)$ and $\mathcal{K}_\bullet(P^\beta, \beta)$ are quasi-isomorphic. In particular,*

$$j(\beta) = \operatorname{rank} \mathcal{H}_1(P^\beta, \beta) - \operatorname{rank} \mathcal{H}_0(P^\beta, \beta). \tag{5.6}$$

Proof. Consider the short exact sequences

$$0 \to \frac{T^\beta}{M} \to \frac{S_A[\partial_A^{-1}]}{M} \to \frac{S_A[\partial_A^{-1}]}{T^\beta} \to 0 \quad \text{and} \quad 0 \to \frac{S^\beta}{M} \to \frac{T^\beta}{M} \to \frac{T^\beta}{S^\beta} \to 0.$$

The definition of $\mathcal{C}_A(\beta)$ ensures that β is not a quasidegree of either S^β/M or $S_A[\partial_A^{-1}]/T^\beta$. Thus we obtain the result from long exact sequences in Euler–Koszul homology and Proposition 2.8. □

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DEFINITION 5.11. The tth *partial Euler–Koszul characteristic* of a weakly toric module N is the (non-negative) integer

$$\chi_t(N,\beta) = \sum_{q=t}^{d} (-1)^{q-t} \cdot \operatorname{rank} \mathcal{H}_q(N,\beta).$$

The main result of this article, Theorem 6.6, states that the partial Euler–Koszul characteristics of ranking toric modules are determined by the combinatorics of the ranking lattices $\mathbb{E}^\beta$ of M at β. Lemma 5.12 describes $j(\beta)$ as the second partial Euler–Koszul characteristic of P^β, so our results regarding the combinatorics of rank jumps are a consequence of Theorem 6.6.

LEMMA 5.12. *The zeroth partial Euler–Koszul characteristic of every non-trivial ranking toric module for a pair (M,β) is 0. In particular, $j(\beta) = \chi_2(P^\beta, \beta)$.*

Proof. This lemma follows from [Oku06, Theorem 4.2] and (5.6). □

5.2 Simple ranking toric modules

The polyhedral structure of the degree sets of ranking toric modules plays an important role in our computation of their partial Euler–Koszul characteristics. We will use the fact that each face $F \preceq A$ determines an S_F-module that is the quotient of a ranking toric module. Modules of this type are called *simple ranking toric modules*.

DEFINITION 5.13.

(i) For a subset $J \subseteq \mathcal{J}(\beta)$, the ranking toric module P_J^β is *simple* if there is a unique $F \in \mathcal{F}(\beta)$ such that all pairs in J are of the form (F,b).

(ii) For each $F \in \mathcal{F}(\beta)$ and $J \subseteq \mathcal{J}(\beta)$, denote by $P_{F,J}^\beta$ the simple ranking toric module determined by the set $\{(G,b) \in J \mid F = G\}$. The degree set of this module is denoted by $\mathbb{P}_{F,J}^\beta$.

(iii) Call the parameter β *simple* for M if P^β is a simple ranking toric module or, equivalently, if there is an $F \in \mathcal{F}(\beta)$ such that $P^\beta = P_F^\beta$ (see Notation 5.2).

We show in Theorem 6.1 that for $F \preceq A$, each simple ranking toric module $P_{F,J}^\beta$ is a maximal Cohen–Macaulay toric S_F-module. Thus the results of § 3 can be applied to compute the rank of their Euler–Koszul homology modules.

Notice that by setting

$$B_{F,J}^\beta = \{b \in B_F^\beta \mid (F,b) \in J\} \quad \text{and} \quad \mathbb{E}_{F,J}^\beta = \bigsqcup_{b \in B_{F,J}^\beta} (b + \mathbb{Z}F),$$

it follows from Definition 5.13 that $\mathbb{P}_{F,J}^\beta = \mathcal{C}_A(\beta) \cap \mathbb{E}_{F,J}^\beta$. In particular, when $J = \mathcal{J}(\beta)$, $\mathbb{P}_F^\beta = \mathcal{C}_A(\beta) \cap \mathbb{E}_F^\beta$.

PROPOSITION 5.14. *For $F \in \mathcal{F}(\beta)$ and $J \subseteq \mathcal{J}(\beta)$, the simple ranking toric module $P_{F,J}^\beta$ of Definition 5.13 admits an S_F-module structure that is compatible with its S_A-module structure.*

Proof. By construction, $\mathbb{P}_{F,J}^\beta$ is closed under addition with elements of $\mathbb{N}F$. □

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Definition 5.15. We define a partial order $\trianglelefteq$ on $J \subseteq \mathcal{J}(\beta)$ by $(F, b) \trianglelefteq (F', b')$ if and only if $b + \mathbb{Z}F \subseteq b' + \mathbb{Z}F'$ for pairs $(F, b), (F', b') \in J$. We let $\max(J)$ denote the subset of J consisting of maximal elements with respect to $\trianglelefteq$.

For $J \subseteq \mathcal{J}(\beta)$, $\max(J)$ is the smallest subset of J that determines a direct sum of simple ranking toric modules into which P_J^β embeds.

The calculation of the partial Euler–Koszul characteristics of a ranking toric module P_J^β will be achieved by homologically replacing it by an acyclic complex $I_J^\bullet$ composed of simple ranking toric modules. We then examine the spectral sequences determined by the double complex $\mathcal{K}_\bullet(I_J^\bullet, \beta)$ to obtain a formula for the partial Euler–Koszul characteristics of P_J^β.

5.3 A reduction useful for computations

We now define an equivalence relation on the union of the various $\mathbb{Z}F$-orbit representatives of (5.2). We show in Proposition 5.17 that, for $J \subseteq \mathcal{J}(\beta)$, the ranking toric module P_J^β splits as direct sum over the equivalence classes of this relation. Thus, by additivity of rank, (5.6) can be expressed as a sum involving simpler ranking toric modules.

Definition 5.16.

- (i) Let $B^\beta = \bigcup_{F \in \mathcal{F}(\beta)} B_F^\beta$ be the collection of all $\mathbb{Z}F$-orbit representatives from (5.2).
- (ii) Let $\doteqdot$ be the equivalence relation on the elements of B^β that is generated by the relations $b \doteqdot b'$ if there exist $(F, b), (F', b') \in \mathcal{J}(\beta)$ such that $(b + \mathbb{Z}F) \cap (b' + \mathbb{Z}F') \neq \varnothing$.
- (iii) Let $\widehat{B}^\beta$ denote the set of equivalence classes of $\doteqdot$.
- (iv) For $\widehat{b} \in \widehat{B}^\beta$ and $J \subseteq \mathcal{J}(\beta)$, let
$$J(\widehat{b}) = \{(F, b') \in J \mid b' \in \widehat{b}\}.$$
Hence, for $J \subseteq \mathcal{J}(\beta)$, there is a partition of $\mathbb{P}_J^\beta$ over $\widehat{B}^\beta$, namely, $\mathbb{P}_J^\beta = \bigsqcup_{\widehat{b} \in \widehat{B}^\beta} \mathbb{P}^\beta_{J(\widehat{b})}$.

Proposition 5.17. *For $J \subseteq \mathcal{J}(\beta)$, there is a decomposition $P_J^\beta = \bigoplus_{\widehat{b} \in \widehat{B}^\beta} P^\beta_{J(\widehat{b})}$.*

Proof. For distinct $\widehat{b}, \widehat{b'} \in B^\beta$, the sets $\mathbb{P}^\beta_{J(\widehat{b})}$ and $\mathbb{P}^\beta_{J(\widehat{b'})}$ are disjoint by definition of $\doteqdot$. Thus there is a decomposition $P_J^\beta = T^\beta / S_J^\beta = \bigoplus_{\widehat{b} \in B^\beta} T^\beta / S^\beta_{J(\widehat{b})}$. ☐

Example 5.18. As a special case of Proposition 5.17, the simple ranking toric module P_F^β can be expressed as the direct sum $\bigoplus_{b \in B_F^\beta} P^\beta_{(F,b)}$ (see (5.2)).

Definition 5.19. For $J = \mathcal{J}(\beta)$, let the *rank jump from* $\widehat{b}$ *of* M *at* β be
$$j_{\widehat{b}}(\beta) = \operatorname{rank} \mathcal{H}_1(P^\beta_{\mathcal{J}(\widehat{b})}, \beta) - \operatorname{rank} \mathcal{H}_0(P^\beta_{\mathcal{J}(\widehat{b})}, \beta).$$

Corollary 5.20. *The rank jump $j(\beta)$ can be expressed as the sum $j(\beta) = \sum_{\widehat{b} \in \widehat{B}^\beta} j_{\widehat{b}}(\beta)$.*

Proof. This follows from (5.6), Proposition 5.17, and the additivity of $j(\beta)$. ☐

As stated in Corollary 5.20, computing $j(\beta)$ is reduced to finding $j_{\widehat{b}}(\beta)$ for each $\widehat{b} \in \widehat{B}^\beta$. When working with examples, it is typically useful to consider each $j_{\widehat{b}}(\beta)$ individually. In contrast, as we continue with the theory, it is more efficient for our notation to study $j(\beta)$ directly. In §§ 6.1 and 6.2, replacing $\mathcal{F}(\beta)$, P^β, and $j(\beta)$ by their corresponding $\widehat{b}$ counterparts calculates $j_{\widehat{b}}(\beta)$.

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6. Partial Euler–Koszul characteristics of ranking toric modules

We retain the notation of § 5. This section contains our main result, Theorem 6.6, which states that for any subset $J \subseteq \mathcal{J}(\beta)$, the partial Euler–Koszul characteristics of the ranking toric module P_J^β are determined by the combinatorics of $\mathbb{E}_J^\beta$ (refer to Definitions 5.11, 5.8, and 5.1). As a special case of this result, we will have computed $j(\beta) = \chi_2(P^\beta, \beta)$ in terms of the ranking lattices $\mathbb{E}^\beta$, resulting in a proof of Theorem 1.3.

We begin by examining the partial Euler–Koszul characteristics of simple ranking toric modules $\mathbb{P}_F^\beta$ from Definition 5.13. We will compute the partial Euler–Koszul characteristics of a ranking toric module P_J^β by homologically approximating it by a cellular resolution (see Definition 6.7) built from simple ranking toric modules.

6.1 The simple case

The next theorem shows that simple ranking toric modules are maximal Cohen–Macaulay toric face modules, which will be useful in the general case. This allows us to compute the rank jump $j(\beta)$ of M at β when β is simple for M, as in [Oku06, Theorem 2.5].

Theorem 6.1. *Fix $\beta \in \mathbb{C}^d$, $F \in \mathcal{F}(\beta)$, and $J \subseteq \mathcal{J}(\beta)$. Then the simple ranking toric module $P_{F,J}^\beta$ is a maximal Cohen–Macaulay toric S_F-module. Further, there is a decomposition*

$$\mathcal{H}_\bullet(P_{F,J}^\beta, \beta) = \mathbb{C}[x_{F^c}] \otimes_{\mathbb{C}} \mathcal{H}_0^F(P_{F,J}^\beta, \beta) \otimes_{\mathbb{Z}} \big(\textstyle\bigwedge^\bullet \mathbb{Z}F^\perp\big), \tag{6.1}$$

and, for all $q \geqslant 0$,

$$\operatorname{rank} \mathcal{H}_q(P_{F,J}^\beta, \beta) = |B_{F,J}^\beta| \cdot \binom{\operatorname{codim}(F)}{q} \cdot \operatorname{vol}(F).$$

Proof. Fix a $\mathbb{Z}F$-orbit representative $b \in B^\beta$, chosen so that $\mathbb{P}_{F,b}^\beta \subseteq b + \widetilde{\mathbb{N}F}$. This implies that

$$0 \to P_{F,b}^\beta \to \widetilde{S}_F(b) \to \frac{\widetilde{S}_F(b)}{P_{F,b}^\beta} \to 0 \tag{6.2}$$

is a short exact sequence of toric modules. Since

$$\deg\left(\frac{\widetilde{S}_F(b)}{P_{F,b}^\beta}\right) = (b + \widetilde{\mathbb{N}F}) \backslash \mathbb{P}_{F,b}^\beta,$$

the definition of $\mathcal{C}_A(\beta)$ ensures that $\beta \notin \operatorname{qdeg}(\widetilde{S}_F(b)/P_{F,b}^\beta)$. Proposition 2.8 and (2.1) imply that (6.2) induces the isomorphism

$$\mathcal{H}_\bullet(P_{F,b}^\beta, \beta) \cong \mathcal{H}_\bullet^F(\widetilde{S}_F, \beta - b)(b).$$

As $\beta - b \in \mathbb{C}F$, Proposition 3.2 gives the decomposition (6.1), in light of Proposition 5.17. By [Wal07, Lemma 3.3],

$$\operatorname{rank} \mathcal{H}_0^F(\widetilde{S}_F, \beta - b)(b) = \operatorname{vol}(F).$$

Now the additivity of rank and (3.3) combine to complete the claim. □

Corollary 6.2. *If $\beta \in \mathcal{E}_A(M)$ is simple for M, then the rank jump of M at β is*

$$j_A(\beta) = |B^\beta| \cdot [\operatorname{codim}(F) - 1] \cdot \operatorname{vol}(F).$$

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Proof. Since β is simple for M, $P^\beta = P^\beta_F$ for some $F \preceq A$. Hence apply Theorem 6.1 to Corollary 5.20, noting that $B^\beta = B^\beta_F$. ☐

Example 6.3 (Continuation of Examples 4.5 and 5.7). With $b = \left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right]$, we have the set $\widehat{B}^\beta = \{\widehat{\beta}, \widehat{b}\}$. Both $P^\beta_{\mathcal{J}(\widehat{\beta})}$ and $P^\beta_{\mathcal{J}(\widehat{b})}$ are simple ranking toric modules. By Corollary 6.2,

$$j_{\widehat{\beta}}(\beta) = 1 \cdot [2-1] \cdot 1 = 1 \quad \text{and} \quad j_{\widehat{b}}(\beta) = 1 \cdot [1-1] \cdot 1 = 0.$$

It now follows from Proposition 5.17 that $j(\beta) = 1$. A similar calculation shows that $j(\beta') = 1$ for any $\beta' \in \mathcal{E}_A(S_A)$.

Example 6.4. Let

$$A = \begin{bmatrix} 2 & 3 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

with $\mathrm{vol}(A) = 24$, and consider the faces $F = [a_1\ a_2\ a_3]$ and $G = [a_3]$. Note that the semigroup $\mathbb{N}A$ of Example 4.5 embeds into the $\mathbb{N}A$ here. With $b = \left[\begin{smallmatrix}1\\1\\0\\0\end{smallmatrix}\right]$, the exceptional arrangement of S_A is $\mathcal{E}_A(S_A) = b + \mathbb{C}F$, where

$$-\mathrm{qdeg}(\mathrm{Ext}^i_R(S_A, R)(-\varepsilon_A)) = \begin{cases} b + \mathbb{C}F & \text{if } i = 8, \\ b + \mathbb{C}G & \text{if } i = 9, \\ \varnothing & \text{if } i > 9. \end{cases}$$

Thus the ranking slab stratification of $\mathcal{E}_A(S_A)$ is strictly finer than its arrangement stratification. Further, this finer stratification coincides with the rank stratification of $\mathcal{E}_A(S_A)$ inside the cone $\mathbb{R}_{\geqslant 0}A$. For $\beta \in b + \mathbb{C}G$, $|\widehat{B}^\beta| = 2$, while $|\widehat{B}^\beta| = 1$ for $\beta \in \mathcal{E}_A(S_A) \backslash [b + \mathbb{C}G]$. Calculations similar to those of Example 6.3 show that

$$j(\beta) = \begin{cases} 3 & \text{if } \beta \in b + \mathbb{C}G, \\ 1 & \text{if } \beta \in \mathcal{E}_A(S_A) \backslash [b + \mathbb{C}G]. \end{cases}$$

In particular, the rank of the A-hypergeometric system $\mathcal{H}_0(S_A, \beta) = M_A(\beta)$ is not constant on the slab $[b + \mathbb{C}F] \subseteq \mathcal{E}_A(S_A)$ (see Definition 4.8).

Example 6.5. Let $M = S_A$ for

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 4 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and consider the saturated faces $F_1 = [a_1]$ and $F_2 = [a_8]$. Here $\mathrm{vol}(A) = 20$, $\mathrm{vol}(F_i) = 1$, and $\mathrm{codim}(F_i) = 3$. Computations in Macaulay 2 [M2] with (4.2) reveal that

$$\mathcal{E}_A(S_A) = [\beta' + \mathbb{C}F_1] \cup [\beta' + \mathbb{C}F_2],$$

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where $\beta' = \begin{bmatrix} 1\\ \frac{1}{2}\\ 0 \end{bmatrix}$. With $b = \begin{bmatrix} 1\\ 0\\ 0\\ 0 \end{bmatrix}$ and $\beta \in \widetilde{\mathbb{N}A} \cap \mathcal{E}_A(S_A)$,

$$\mathcal{R}_A(S_A, \beta) = \begin{cases} \mathcal{E}_A(S_A) & \text{if } \beta = \beta', \\ \beta + \mathbb{C}F_1 & \text{if } \beta \in [\beta' + \mathbb{C}F_1] \backslash \beta', \\ \beta + \mathbb{C}F_2 & \text{if } \beta \in [\beta' + \mathbb{C}F_2] \backslash \beta', \end{cases}$$

$$\mathbb{P}^\beta = \begin{cases} [\beta + b + \mathbb{N}F_1] \cup [\beta + \mathbb{N}F_1] \cup [\beta + \mathbb{N}F_2] & \text{if } \beta = \beta', \\ [\beta + b + \mathbb{N}F_1] \cup \beta + \mathbb{N}F_1 & \text{if } \beta \in \widetilde{\mathbb{N}A} \cap [\beta' + \mathbb{C}F_1] \backslash \beta', \\ \beta + \mathbb{N}F_2 & \text{if } \beta \in \widetilde{\mathbb{N}A} \cap [\beta' + \mathbb{C}F_2] \backslash \beta', \end{cases}$$

and

$$\widehat{B}^\beta = \begin{cases} \{\widehat{\beta + b}, \widehat{\beta}\} & \text{if } \beta = \beta', \\ \{\widehat{\beta + b}, \widehat{\beta}\} & \text{if } \beta \in \widetilde{\mathbb{N}A} \cap [\beta' + \mathbb{C}F_1] \backslash \beta', \\ \{\widehat{\beta}\} & \text{if } \beta \in \widetilde{\mathbb{N}A} \cap [\beta' + \mathbb{C}F_2] \backslash \beta'. \end{cases}$$

By Corollary 6.2, for $\beta \in \widetilde{\mathbb{N}A} \cap [\beta' + \mathbb{C}F_2] \backslash \beta'$,

$$j(\beta) = [\operatorname{codim}(F_2) - 1] \cdot \operatorname{vol}(F_2) = [3 - 1] \cdot 1 = 2,$$

while for $\beta \in \widetilde{\mathbb{N}A} \cap [\beta' + \mathbb{C}F_1] \backslash \beta'$, $|B^\beta_{F_1}| = 2$, and

$$j(\beta) = 2 \cdot [\operatorname{codim}(F_1) - 1] \cdot \operatorname{vol}(F_1) = 2 \cdot [3 - 1] \cdot 1 = 4.$$

To compute the rank jump of S_A at β', we must move to the general case. We will see in Example 6.22 that $j(\beta') = 4$, which arises as the sum of the generic rank jumps along irreducible components of $\mathcal{R}_A(S_A, \beta')$ that is then corrected by error terms that arise from a spectral sequence calculation.

6.2 The general case

We are now prepared to compute the partial Euler–Koszul characteristics of ranking toric modules. The proof of our main theorem, Theorem 6.6, will be given at the end of this section, after a sequence of lemmas. The definitions of $\mathcal{J}(\beta)$, $\mathbb{E}^\beta_J$, and P^β_J can be found in Definitions 5.1 and 5.8, respectively.

Theorem 6.6. *For $J \subseteq \mathcal{J}(\beta)$, the partial Euler–Koszul characteristics of the ranking toric module P^β_J are determined by the combinatorics of $\mathbb{E}^\beta_J$.*

We compute the partial Euler–Koszul characteristics of the ranking toric module P^β_J will be achieved by homologically approximating P^β_J by simple ranking toric modules; note that the ranks of the Euler–Koszul homology modules of simple ranking toric modules have been computed in Theorem 6.1.

Definition 6.7. Let Δ be an oriented cell complex (e.g. CW, simplicial, polyhedral). Then Δ has the cochain complex

$$C^\bullet_\Delta : 0 \to \bigoplus_{\text{vertices } v \in \Delta} \mathbb{Z}_v \to \bigoplus_{\text{edges } e \in \Delta} \mathbb{Z}_e \to \cdots \to \bigoplus_{i\text{-faces } \sigma \in \Delta} \mathbb{Z}_\sigma \to \cdots \to 0,$$

where $\mathbb{Z}_\sigma \to \mathbb{Z}_\tau$ is multiplication by some integer $\operatorname{coeff}(\sigma, \tau)$. Let $\mathfrak{C}_\Delta$ be the category with the non-empty faces of Δ as objects and morphisms

$$\operatorname{Mor}_{\mathfrak{C}_\Delta}(\sigma, \tau) = \begin{cases} \{\sigma \subseteq \tau\} & \text{if } \sigma \subseteq \tau, \\ \varnothing & \text{otherwise.} \end{cases}$$

Fix an abelian category $\mathfrak{A}$ and suppose there is a covariant functor $\Phi : \mathfrak{C}_\Delta \to \mathfrak{A}$. Let $P_\sigma := \Phi(\sigma)$ for each $\sigma \in \Delta$. A sequence of morphisms in $\mathfrak{A}$

$$I^\bullet : 0 \to \bigoplus_{\text{vertices } v \in \Delta} P_v \to \bigoplus_{\text{edges } e \in \Delta} P_e \to \cdots \to \bigoplus_{i\text{-faces } \sigma \in \Delta} P_\sigma \to \cdots \to 0$$

is *cellular* and *supported on* Δ if the $P_\sigma \to P_\tau$ component of $I^\bullet$ is $\operatorname{coeff}(\sigma, \tau)\Phi(\sigma \subseteq \tau)$. Since $\mathfrak{A}$ is abelian, a cellular sequence is necessarily a complex.

In a manner analogous to Definition 6.7, a cellular complex supported on Δ can also be obtained from the chain complex $C_\bullet^\Delta$ of Δ and a contravariant functor $\Phi : \mathfrak{C}_\Delta \to \mathfrak{A}$. Further, we could replace $C_\bullet^\Delta$ in this construction with the *reduced* chain or cochain complexes of Δ. We say that a complex in $\mathfrak{A}$ is *cellular* if it can be constructed from the underlying topological data of a cell complex and a functor $\Phi : \mathfrak{C}_\Delta \to \mathfrak{A}$.

When Δ is a simplicial or polyhedral cell complex, $\operatorname{coeff}(\sigma, \tau)$ of Definition 6.7 is simply 1 if the orientation of τ induces the orientation of σ and -1 if it does not.

The generality in which we define cellular complexes is alluded to in the introduction of [JM08] and appears as [Mil09, Definition 3.2]. An introduction to these complexes, in the polyhedral case, can be found in [MS05, ch. 4].

Recall from Definition 5.15 that $\max(J)$ was defined so that it yields the smallest set of faces of A that determines a direct sum of simple ranking toric modules into which P_J^β embeds.

Notation 6.8. We wish to take intersections of faces in the set $\max(J)$. In order to keep track of which faces were involved in each intersection, set

$$\Delta_J^0 = \{F \in \mathcal{F}(\beta) \mid \exists (F, b) \in \max(J)\},$$
$$\Delta_J^p = \{s \subseteq \Delta_J^0 \mid |s| = p + 1\} \quad \text{and} \quad F_s = \bigcap_{G \in s} G \text{ for } s \in \Delta_J^p.$$

With $r + 1 = |\Delta_J^0|$, let $\Delta = \Delta_J^\beta$ be the standard r-simplex with vertices corresponding to the elements of Δ_J^0. To the p-face of Δ spanned by the vertices corresponding to the elements in $s \in \Delta_J^p$, assign the ranking toric module $P_{F_s,J}^\beta$. Choosing the natural maps $P_{F_s,J}^\beta \to P_{F_t,J}^\beta$ for $s \subseteq t$ induces a cellular complex supported on Δ,

$$I_J^\bullet : I_J^0 \to I_J^1 \to \cdots \to I_J^r \to 0 \tag{6.3}$$

with

$$I_J^p = \bigoplus_{s \in \Delta_J^p} P_{F_s,J}^\beta.$$

Lemma 6.9. *The cohomology of the cellular complex $I_J^\bullet$ of (6.3) is concentrated in cohomological degree zero and is isomorphic to P_J^β.*

Proof. Given $\alpha \in \mathbb{P}_J^\beta = \deg(P_J^\beta)$, let $F_{i_1}, \ldots, F_{i_k}$ be the faces $F \in \Delta_J^0$ such that $\alpha \in \mathbb{P}_{F,J}^\beta$. The degree-$\alpha$ part of $I_J^\bullet$ computes the cohomology of the $(k-1)$-subsimplex of Δ given by the

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vertices with labels corresponding to $F_{i_1}, \dots, F_{i_k}$; in particular, it is acyclic with 0-cohomology $\mathbb{C} \cong (P_J^\beta)_\alpha$. □

By construction, P_J^β is a $\mathbb{Z}^d$-graded monomial module over the saturated semigroup ring $\widetilde{S}_A$, and it can be translated by some $\alpha \in \mathbb{Z}^d$ so that $\deg(P_J^\beta(\alpha)) = \alpha + \mathbb{P}_J^\beta \subseteq \widetilde{\mathbb{N}A} = \deg(\widetilde{S}_A)$. After translation by α, (6.3) is similar to an *irreducible resolution*, as defined in [Mil02, Definition 2.1]. We continue to view P_J^β as an S_A-module, so we use maximal Cohen–Macaulay toric face modules instead of irreducible quotients of $\widetilde{S}_A$.

Consider the $\mathbb{Z}^d$-graded double complex $E_0^{\bullet,\bullet}$ with $E_0^{p,-q} := \mathcal{K}_q(I_J^p, \beta)$. Let ${}_h\psi_0$ and ${}_v\psi_0$ denote the horizontal and vertical differentials of $E_0^{\bullet,\bullet}$, respectively. By the exactness of (6.3), taking homology of $E_0^{\bullet,\bullet}$ with respect to ${}_h\psi_0$ yields

$$ {}_hE_1^{p,-q} = \begin{cases} \mathcal{K}_q(P_J^\beta, \beta) & \text{if } p = 0 \text{ and } 0 \leqslant q \leqslant d, \\ 0 & \text{otherwise.} \end{cases} $$

Hence

$$ {}_hE_\infty^{p,-q} = {}_hE_2^{p,-q} = \begin{cases} \mathcal{H}_q(P_J^\beta, \beta) & \text{if } p = 0 \text{ and } 0 \leqslant q \leqslant d, \\ 0 & \text{otherwise.} \end{cases} \tag{6.4} $$

On the other hand, the first page of the vertical spectral sequence given by $E_0^{\bullet,\bullet}$ consists of Euler–Koszul homologies of simple ranking toric modules:

$$ {}_vE_1^{p,-q} = \mathcal{H}_q(I_J^p, \beta) = \bigoplus_{s \in \Delta_J^p} \mathcal{H}_q(P_{F_s,J}^\beta, \beta). \tag{6.5} $$

We now apply the decomposition of these homologies given in Theorem 6.1 to obtain a new description of the Euler–Koszul homology of the ranking toric module P_J^β.

Lemma 6.10. *The vertical spectral sequence obtained from the double complex*

$$ {}'E_0^{p,-q} = \bigoplus_{s \in \Delta_J^p} \bigoplus_{i+j=q} \mathbb{C}[x_{F_s^c}] \otimes_{\mathbb{C}} \mathcal{K}_i^{F_s}(P_{F_s,J}^\beta, \beta) \otimes_{\mathbb{Z}} \left(\textstyle\bigwedge^j \mathbb{Z}F_s^\perp\right) \tag{6.6} $$

(with differentials as in Lemma 3.5) has abutment

$$ {}'E_\infty^{p-q} \cong \begin{cases} \mathcal{H}_q(P_J^\beta, \beta) & \textit{if } p = 0, \\ 0 & \textit{otherwise.} \end{cases} $$

Proof. By Theorem 6.1 and Lemma 3.5, the vertical differentials ${}_v\psi_0$ of $E_0^{p,-q}$ are compatible with the quasi-isomorphism

$$ E_0^{p,\bullet} \simeq_{\text{qis}} \bigoplus_{s \in \Delta_J^p} \mathbb{C}[x_{F_s^c}] \otimes_{\mathbb{C}} \mathcal{K}_\bullet^{F_s}(P_{F_s,J}^\beta, \beta) \otimes_{\mathbb{Z}} \left(\textstyle\bigwedge^\bullet \mathbb{Z}F_s^\perp\right). \tag{6.7} $$

Since ${}_hE_\infty^{\bullet,\bullet}$ and ${}_vE_\infty^{\bullet,\bullet}$ converge to the same abutment, the result follows from (6.4). □

Note that the first page of the spectral sequence in Lemma 6.10 is

$$ {}'E_1^{p,-q} = \bigoplus_{s \in \Delta_J^p} \mathbb{C}[x_{F_s^c}] \otimes_{\mathbb{C}} \mathcal{H}_0^{F_s}(P_{F_s,J}^\beta, \beta) \otimes_{\mathbb{Z}} \left(\textstyle\bigwedge^q \mathbb{Z}F_s^\perp\right). \tag{6.8} $$

For $s \in \Delta_J^p$, let κ_s denote the differential of $\mathcal{K}_\bullet^{F_s}(P_{F_s,J}^\beta, \beta)$, and let ${}_v\delta$ and ${}_h\delta$ respectively denote the vertical and horizontal differentials of ${}'E_0^{\bullet,\bullet}$. If $i + j = q$ with $i, j \geqslant 0$, then, by Remark 3.3, the element

$$f \otimes a \otimes b \in \mathbb{C}[x_{F_s^c}] \otimes_{\mathbb{C}} \mathcal{K}_i^{F_s}(P_{F_s,J}^{\beta}, \beta) \otimes_{\mathbb{C}} (\textstyle\bigwedge^q \mathbb{Z}F_s^{\perp}) \subseteq {}'E_0^{p,-q}$$

has vertical differential

$$_v\delta(f \otimes a \otimes b) = f \otimes \kappa_s(a) \otimes b. \tag{6.9}$$

We will use the fact that (6.9) is an element of

$$\mathbb{C}[x_{F_s^c}] \otimes \mathcal{K}_{i-1}^{F_s}(P_{F_s,J}^{\beta}, \beta) \otimes_{\mathbb{C}} (\textstyle\bigwedge^j \mathbb{Z}F_s^{\perp}) \subseteq {}'E_0^{p,-q+1}$$

to show that ${}'E_{\bullet}^{\bullet,\bullet}$ degenerates quickly. This is the main technical result of this article.

Lemma 6.11. *The spectral sequence ${}'E_{\bullet}^{\bullet,\bullet}$ of Lemma 6.10 degenerates at the second page.*

Proof. For $\xi \in {}'E_0^{i,j}$, let $\overline{\xi}$ denote the image of ξ in ${}'E_2^{i,j}$, if it exists. Let δ_r denote the differential of ${}'E_r^{\bullet,\bullet}$, so $\delta_0 = {}_v\delta$.

To see that $\delta_2 = 0$, consider an element $\alpha \in {}'E_0^{p,-q}$ with $\overline{\alpha} \in {}'E_2^{p,-q}$. Then there is an element $\eta \in {}'E_0^{p+1,-q-1}$ such that ${}_v\delta(\eta) = {}_h\delta(\alpha)$, which is used to define

$$\delta_2(\overline{\alpha}) = \overline{{}_h\delta(\eta)}. \tag{6.10}$$

(Recall that (6.10) is independent of the choice of η.) We write $\alpha = \sum \alpha_{ij}^s$ as an element of (6.6). Note that

$$_v\delta(\alpha_{ij}^s) \in \mathbb{C}[x_{F_s^c}] \otimes_{\mathbb{C}} \mathcal{K}_{i-1}^{F_s}(P_{F_s,J}^{\beta}, \beta) \otimes_{\mathbb{Z}} (\textstyle\bigwedge^j \mathbb{Z}F_s^{\perp}),$$

so α_{ij}^s is in the kernel of ${}_v\delta$ for all s, i, j. By (6.8), α_{ij}^s is in the image of ${}_v\delta$ whenever $i > 0$. Hence without changing $\overline{\alpha}$, we may assume that, for all $s \in \Delta_J^p$, $\alpha_{ij}^s = 0$ when $i > 0$, so that

$$\alpha \in \bigoplus_{s \in \Delta_J^p} \mathbb{C}[x_{F_s^c}] \otimes_{\mathbb{C}} \mathcal{K}_0^{F_s}(P_{F_s,J}^{\beta}, \beta) \otimes_{\mathbb{Z}} (\textstyle\bigwedge^q \mathbb{Z}F_s^{\perp}).$$

As the differential ${}_h\delta$ is induced by (6.3),

$$_h\delta(\alpha) \in \bigoplus_{s \in \Delta_J^{p+1}} \mathbb{C}[x_{F_s^c}] \otimes_{\mathbb{C}} \mathcal{K}_0^{F_s}(P_{F_s,J}^{\beta}, \beta) \otimes_{\mathbb{Z}} (\textstyle\bigwedge^q \mathbb{Z}F_s^{\perp}).$$

By hypothesis on α and (6.9), there is an element

$$\eta \in \bigoplus_{s \in \Delta_J^{p+1}} \mathbb{C}[x_{F_s^c}] \otimes_{\mathbb{C}} \mathcal{K}_1^{F_s}(P_{F_s,J}^{\beta}, \beta) \otimes_{\mathbb{Z}} (\textstyle\bigwedge^q \mathbb{Z}F_s^{\perp}) \tag{6.11}$$

such that ${}_v\delta(\eta) = {}_h\delta(\alpha)$. Set $\zeta = {}_h\delta(\eta)$ and note that $\delta_2(\overline{\alpha}) = \overline{\zeta}$. Using again the fact that the differential ${}_h\delta$ is induced by (6.3), applied now to (6.11), we see that

$$\zeta \in \bigoplus_{s \in \Delta_J^{p+2}} \mathbb{C}[x_{F_s^c}] \otimes_{\mathbb{C}} \mathcal{K}_1^{F_s}(P_{F_s,J}^{\beta}, \beta) \otimes_{\mathbb{Z}} (\textstyle\bigwedge^q \mathbb{Z}F_s^{\perp}).$$

Since ${}_v\delta(\zeta) = {}_h\delta^2(\alpha) = 0$, ζ is in the kernel of ${}_v\delta = \delta_0$. Hence (6.8) implies that $\delta_2(\overline{\alpha}) = \overline{\zeta}$ vanishes. $\square$

Lemma 6.12. *For $J \subseteq \mathcal{J}(\beta)$, the t^{th} partial Euler–Koszul characteristic of the ranking toric module P_J^{β} is given by*

$$\chi_t(P_J^{\beta}, \beta) = \sum_{p-q>-t} (-1)^{p-q+t+1} \operatorname{rank} \mathcal{H}_q(I_J^p, \beta) - \sum_{p-q=-t} \operatorname{rank}(\operatorname{image} \delta_1^{p,-q}). \tag{6.12}$$

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Proof. For $i \in \mathbb{N}$, let

$$r_i^k := \sum_{p-q=k} \operatorname{rank}({}'E_i^{p,-q}).$$

By Lemma 6.11, $r_2^k = r_\infty^k$. From the abutment (6.4), we see that $r_2^k = 0$ for $k > 0$. Since $\sum_{k\in\mathbb{Z}}(-1)^k r_1^k = 0$ by Lemma 5.12, also $\sum_{k\in\mathbb{Z}}(-1)^k r_2^k = 0$. Thus the tth partial Euler–Koszul characteristic of P_J^β can be expressed as

$$\begin{aligned}
\chi_t(P_J^\beta, \beta) &= \sum_{k=-d}^{-t} (-1)^{k+t} r_2^k \\
&= \sum_{k=-d}^{\infty} (-1)^{k+t+1} r_1^k - \sum_{k=-d}^{-t} (-1)^{k+t+1} r_2^k \\
&= \sum_{k=-t+1}^{\infty} (-1)^{k+t+1} r_1^k - \sum_{p-q=-t} \operatorname{rank}(\operatorname{image} \delta_1^{p,-q}).
\end{aligned}$$

Now (6.12) follows from the definition of r_1^k and the quasi-isomorphism (6.7), as the isomorphic first pages of the spectral sequences there are (6.5) and (6.8). □

We will compute the ranks of $\mathcal{H}_q(I_J^p, \beta)$ and the image of $\delta_1^{p,-q}$ from (6.12) in subsequent lemmas. The first is an immediate consequence of Theorem 6.1.

Lemma 6.13. *If $q \geqslant 0$, then*

$$\operatorname{rank} \mathcal{H}_q(I_J^p, \beta) = \sum_{s\in\Delta_J^p} |B_{F_s}^\beta| \cdot \binom{\operatorname{codim}(F_s)}{q} \cdot \operatorname{vol}(F_s).$$

Proof. By definition of I_J^p and additivity of rank,

$$\operatorname{rank} \mathcal{H}_q(I_J^p, \beta) = \sum_{s\in\Delta_J^p} \operatorname{rank} \mathcal{H}_q(P_{F_s,J}^\beta, \beta).$$

Now apply Theorem 6.1. □

The rank of the image of $\delta_1^{p,-q}$ is determined combinatorially because the spectral sequence rows ${}'E_1^{\bullet,-q}$ are cellular complexes.

Lemma 6.14. *The complexes ${}'E_1^{\bullet,-q}$ are cellular with support $\Delta = \Delta_J^\beta$ of Notation 6.8.*

Proof. In Notation 6.8, we constructed the cellular complex $I_J^\bullet$ from a labeling of the simplex $\Delta = \Delta_J^\beta$. If we assign in this construction

$$\mathbb{C}[x_{F^c}] \otimes_{\mathbb{C}} \mathcal{H}_q^F(P_F^\beta, \beta) \otimes_{\mathbb{Z}} \left(\textstyle\bigwedge^q \mathbb{Z}F^\perp\right)$$

in place of $P_{F,J}^\beta$ and use the induced maps, we obtain the cellular complex ${}'E_1^{\bullet,-q}$ with differential $\delta_1^{\bullet,-q}$, see (3.3). The existence and compatibility of the differentials follows from Lemma 3.5. □

Lemma 6.15. *The rank of the image of $\delta_1^{p,-q}$ is determined by the combinatorics of $\mathbb{E}_J^\beta$.*

Proof. By Lemma 6.14, the image of $\delta_1^{p,-q}$ is determined by the p-coboundaries of Δ and the corresponding labels of Δ, which come from $\mathbb{E}_J^\beta$. □

Proof of Theorem 6.6. If J is simple for M, the result follows from Theorem 6.1, so suppose that P_J^β is not simple for M. By Lemma 6.9, P_J^β is the 0-cohomology of the acyclic cellular complex $I_J^\bullet$ (6.3). Thus the abutment of the spectral sequences arising from the double complex $\mathcal{K}_\bullet(I_J^\bullet, \beta)$ is $\mathcal{H}_\bullet(P_J, \beta)$. By Lemma 6.10, the vertical spectral sequence obtained from the double complex ${}'E_0^{p,-q}$ of (6.6) has the same abutment. Since this spectral sequence degenerates on the second page by Lemma 6.11, Lemma 6.12 yields the formula (6.12), and by Lemmas 6.13 and 6.15, the summands of (6.12) are dependent only on the combinatorics of $\mathbb{E}_J^\beta$. □

6.3 Computing partial Euler–Koszul characteristics

Recall formula (6.12):

$$\chi_t(P_J^\beta, \beta) = \sum_{p-q>-t} (-1)^{p-q+t+1} \operatorname{rank} \mathcal{H}_q(I_J^p, \beta) - \sum_{p-q=-t} \operatorname{rank}(\operatorname{image} \delta_1^{p,-q}).$$

Lemma 6.13 computes the first summand, but Lemma 6.15 does not explicitly state the rank of the image of $\delta_1^{p,-q}$. A method to do this is provided by Proposition 6.18.

Definition 6.16. For $1 < j \leqslant |\Delta_J^p|$, a subset $\lambda \subseteq \{1, 2, \ldots, j\}$ corresponds to a subcomplex $\Delta(\lambda)$ of the simplex $\Delta = \Delta_J^\beta$, as described in Notation 6.8. If $j \in \lambda$ and there is a minimal generator of $H^p(\Delta(\lambda), \Delta(\lambda)\backslash\{j\}; \mathbb{C})$ of the form $\sum_{i\in\lambda} v_i \cdot [s_i]$, where all coefficients v_i are nonzero, then we say that λ is a *circuit* for j.

Notation 6.17. For $1 < j \leqslant |\Delta_J^p|$, let

$$\Upsilon_J^p(j) = \{\lambda \subseteq \{1, 2, \ldots, j\} \mid \lambda \text{ is a circuit for } j\}$$

denote the set of circuits for j, and set

$$\Upsilon_J^p(j,k) = \{\Lambda \subseteq \Upsilon_J^p(j) \mid |\Lambda| = k\}.$$

For $s \in \Delta_J^p$, $\lambda \in \Upsilon_J^p(j)$, and $\Lambda \in \Upsilon_J^p(j,k)$, set

$$\begin{aligned} s_\lambda &= \{s_i \mid i \in \lambda\}, \\ F(\Lambda) &= \bigcup_{\lambda\in\Lambda} \bigcup_{i\in\lambda} F_{s_i}, \\ N_J^p(s) &= \{(F,b) \in J \mid \exists t \in \Delta_J^p\backslash\{s\} \text{ with } (b + \mathbb{Z}F) \subseteq \mathbb{E}_{F_s,J}^\beta \cap \mathbb{E}_{F_t,J}^\beta \neq \varnothing\}, \\ N_J^p(\lambda) &= \{(F,b) \in N_J^p(s_j) \mid \exists i \in \lambda\backslash\{j\} \text{ with } (b + \mathbb{Z}F) \subseteq \mathbb{E}_{F_{s_i},J}^\beta\}, \\ N_J^p(\Lambda) &= \bigcap_{\lambda\in\Lambda} N_J^p(\lambda), \\ \nu^{p,-q}(\Lambda) &= \binom{\operatorname{codim}(F(\Lambda))}{q} \cdot \operatorname{rank} \mathcal{H}_0(P^\beta_{N_J^p(\Lambda)}, \beta) \quad \text{and} \\ \nu^{p,-q}(j) &= \sum_{k=1}^{|\Upsilon_J^p(j)|} \sum_{\Lambda\in\Upsilon_J^p(j,k)} (-1)^{|\Lambda|+1} \cdot \nu^{p,-q}(\Lambda). \end{aligned} \tag{6.13}$$

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Proposition 6.18. *Let* $P^{\beta}_{N^p_J(s)}$ *be the ranking toric module in (6.13). The rank of the image of* $\delta_1^{p,-q}$ *from (6.12) is equal to*

$$\operatorname{rank}(\text{image } \delta_1^{p,-q}) = \sum_{s\in\Delta_J^p} \binom{\operatorname{codim}(F_s)}{q} \cdot \operatorname{rank} \mathcal{H}_0(P^{\beta}_{N^p_J(s)}, \beta) - \sum_{j=2}^{|\Delta_J^p|} \nu^{p,-q}(j). \tag{6.14}$$

Further, (6.14) can be computed by combining Theorem 6.1 and induction on the dimension of P_J^{β}.

Before providing the proof of Proposition 6.18, we state two lemmas.

Lemma 6.19. *For* $s \in \Delta_J^p$,

$$\operatorname{rank}(\text{image } \delta_{1,s}^{p,-q}) = \binom{\operatorname{codim}(F_s)}{q} \cdot \operatorname{rank} \mathcal{H}_0(P^{\beta}_{N^p_J(s)}, \beta), \tag{6.15}$$

where $P^{\beta}_{N^p_J(s)}$ *is the ranking toric module given by (6.13).*

Proof. The rank of the image of $\delta_{1,s}^{p,-q}$ is $\binom{\operatorname{codim}(F_s)}{q} \cdot \operatorname{rank}(\text{image } \delta_{1,s}^{p,0})$ by Proposition 3.6. View the image of $\delta_{1,s}^{p,0}$ as a quotient of $P^{\beta}_{F_s,J}$. If α is one of its non-zero multigraded components, then it also appears in the degree set of another summand of ${}'E_1^{p,0}$. The collection of such degrees is exactly $\mathbb{P}^{\beta}_{N^p_J(s)}$. □

Lemma 6.20. *For* $1 < j \leqslant |\Delta_J^p|$,

$$\nu^{p,-q}(j) = \operatorname{rank}[(\text{image } \delta^{p,-q}_{1,\{s_1,\ldots,s_{j-1}\}}) \cap (\text{image } \delta^{p,-q}_{1,\{s_j\}})]. \tag{6.16}$$

Proof. To see this, notice first that, for $1 < j \leqslant |\Delta_J^p|$,

$$(\text{image } \delta^{p,-q}_{1,\{s_1,\ldots,s_{j-1}\}}) \cap (\text{image } \delta^{p,-q}_{1,\{s_j\}}) = \sum_{\lambda\in\Upsilon_J^p(j)} (\text{image } \delta^{p,-q}_{1,s_{\lambda\setminus\{j\}}}) \cap (\text{image } \delta^{p,-q}_{1,\{s_j\}})$$

is generated by the images coming from circuits for j. By Proposition 3.6, given a fixed circuit λ for j, the rank of

$$(\text{image } \delta^{p,-q}_{1,s_{\lambda\setminus\{j\}}}) \cap (\text{image } \delta^{p,-q}_{1,\{s_j\}}) \tag{6.17}$$

can be computed as the rank of

$$(\text{image } \delta^{p,0}_{1,s_{\lambda\setminus\{j+1\}}}) \cap (\text{image } \delta^{p,0}_{1,\{s_{j+1}\}}) \tag{6.18}$$

times the $\mathbb{Z}$-rank of

$$\bigcap_{i\in\lambda} [\textstyle\bigwedge^q \mathbb{Z}F_{s_i}^{\perp}] = \textstyle\bigwedge^q \mathbb{Z}F(\lambda)^{\perp}. \tag{6.19}$$

By the same reasoning used to obtain (6.15), the rank of $\mathcal{H}_0(P^{\beta}_{N^p_J(\lambda)}, \beta)$ equals the rank of (6.18). The $\mathbb{Z}$-rank of (6.19) is a binomial coefficient in the codimension in $\mathbb{C}^d$ of the span of the vectors in $F(\lambda)$, so the rank of (6.17) is

$$\binom{\operatorname{codim}(F(\lambda))}{q} \cdot \operatorname{rank} \mathcal{H}_0(P^{\beta}_{N^p_J(\lambda)}, \beta).$$

Further, for a collection of circuits $\Lambda \in \Upsilon_J^p(j,k)$, $\nu^{p,-q}(\Lambda)$ gives the rank of the intersection over Λ of the images of type (6.17), so the inclusion–exclusion principle yields (6.16). □

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Proof of Proposition 6.18. Recall from (6.8) that the domain of $\delta_1^{p,-q}$ is the direct sum

$$'E_1^{p,-q} = \bigoplus_{s\in\Delta_J^p} \mathbb{C}[x_{F_s^c}] \otimes_{\mathbb{C}} \mathcal{H}_0^{F_s}(P_{F_s,J}^{\beta}, \beta) \otimes_{\mathbb{Z}} (\textstyle\bigwedge^q \mathbb{Z}F_s^{\perp}).$$

For $S \subseteq \Delta_J^p$, let $\delta_{1,S}^{p,-q}$ denote the restriction of $\delta_1^{p,-q}$ to the summands in S. Order the elements of $\Delta_J^p = \{s_1, \ldots, s_{|\Delta_J^p|}\}$, so that

$$\begin{aligned}\operatorname{rank}(\operatorname{image} \delta_1^{p,-q}) = &\sum_{s\in\Delta_J^p} \operatorname{rank}(\operatorname{image} \delta_{1,s}^{p,-q}) \\ &- \sum_{j=2}^{|\Delta_J^p|} \operatorname{rank}[(\operatorname{image} \delta_{1,\{s_1,\ldots,s_{j-1}\}}^{p,-q}) \cap (\operatorname{image} \delta_{1,\{s_j\}}^{p,-q})]. \end{aligned} \tag{6.20}$$

Lemmas 6.19 and 6.20 respectively computed the summands of (6.20), resulting in (6.14). Thus it remains to show that (6.14) can be computed by combining Theorem 6.1 and induction on the dimension of P_J^{β}.

If a ranking toric module has dimension zero, then it is necessarily a simple ranking toric module, so Theorem 6.1 computes the rank of its Euler–Koszul homology modules. Thus, by induction on dimension, we can compute the summand in (6.15) corresponding to $s \in \Delta_J^p$ if the dimension of $P_{N_J^p(s)}$ is strictly less than the dimension of $P_{F_s,J}^{\beta}$.

If it is the case that the dimension of $P_{N_J^p(s)}$ equals the dimension of $P_{F_s,J}^{\beta}$, notice first that each pair $(F, b) \in N_J^p(s)$ has $F \preceq F_s$. This implies that $P_{N_J^p(s)}$ is a direct sum (as in Proposition 5.17) of the simple ranking toric module $P_{F_s,N_J^p(s)}$ and a lower-dimensional ranking toric module. Therefore induction together with Theorem 6.1 still completes the computation.

Finally, the same argument applies to computing the rank of $\mathcal{H}_0(P_{N_J^p(\Lambda)}^{\beta}, \beta)$ for $\Lambda \in \Upsilon_J^p(j,k)$, since $N_J^p(\Lambda) \subseteq N_J^p(s_j)$. $\square$

6.4 The combinatorics of rank jumps

By Lemma 5.12, our results on the partial Euler–Koszul characteristics of ranking toric modules reveal the combinatorial nature of rank jumps of the generalized A-hypergeometric system $\mathcal{H}_0(M, \beta)$.

Proof of Theorem 1.3. By (5.6) and Lemma 5.12, $j(\beta) = \chi_2(P^{\beta}, \beta)$, so the result is an immediate consequence of Theorem 6.6 and Proposition 6.18. $\square$

Example 6.21. If $\beta \in \mathbb{C}^d$ is such that $\Delta^0 = \{F_1, F_2\}$, the proof of Theorem 6.6 and § 6.3 show that the rank jump of M at β is

$$j(\beta) = \sum_{i=1}^{2} (|B_{F_i}^{\beta}| \cdot [\operatorname{codim}(F_i) - 1] \cdot \operatorname{vol}(F_i)) + |B_G^{\beta}| \cdot C^{\beta} \cdot \operatorname{vol}(G), \tag{6.21}$$

where $G = F_1 \cap F_2$ and the constant C^{β} is given by

$$C^{\beta} = \binom{\operatorname{codim}(G)}{2} - \operatorname{codim}(G) + 1 - \binom{\operatorname{codim}(F_1)}{2} - \binom{\operatorname{codim}(F_2)}{2} + \binom{\operatorname{codim}(\mathbb{C}F_1 + \mathbb{C}F_2)}{2}.$$

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Example 6.22 (Continuation of Example 6.5). With $b' = \beta' + b = \begin{bmatrix} 2\\1\\2\\0 \end{bmatrix}$, the set $\widehat{B}^{\beta'} = \{\widehat{\beta'}, \widehat{b'}\}$, and $P^{\beta'} = P^{\beta'}_{\mathcal{J}(\widehat{\beta'})} \oplus P^{\beta}_{\mathcal{J}(\widehat{b'})}$ by Proposition 5.17. By (6.21),

$$\begin{aligned} j_{\widehat{\beta'}}(\beta) &= \sum_{i=1}^{2} (|\widehat{\beta'} \cap B^{\beta}_{F_i}| \cdot [\operatorname{codim}(F_i) - 1] \cdot \operatorname{vol}(F_i)) + |\widehat{\beta'} \cap B^{\beta}_{G}| \cdot C^{\beta} \cdot \operatorname{vol}(G) \\ &= 2 + 2 + 1 \cdot (-2) \cdot 1 = 2, \end{aligned}$$

and $j_{\widehat{b'}}(\beta) = 2$ by Corollary 6.2. Thus Proposition 5.17 implies that the rank jump of the A-hypergeometric system $\mathcal{H}_0(S_A, \beta') = M_A(\beta')$ is $j(\beta') = 4$.

When $d = 3$ and $P^{\beta} = P^{\beta}_{\mathcal{J}(\widehat{b})}$ for some $\widehat{b} \in \widehat{B}^{\beta}$, [Oku06, Theorem 2.6] implies that the rank jump $j(\beta)$ of M at β corresponds to the reduced homology of the lattice $\mathcal{F}(\beta)$. The formula given by Okuyama involves this homology and the volumes of the one-dimensional faces of A in $\mathcal{F}(\beta)$. For higher-dimensional cases, the cellular structure of the complex $I^{\bullet}_{\mathcal{J}(\beta)}$ of Notation 6.8 shows that, in general, more information than the reduced homology of $\mathcal{F}(\beta)$ is needed to compute $j(\beta)$, or even a single $j_{\widehat{b}}(\beta)$.

Recall from Definition 4.7 that a ranking slab of M is a stratum in the coarsest stratification of $\mathcal{E}_A(M)$ that respects a specified collection of subspace arrangements. We are now prepared to prove Corollary 1.4, which states that the ranking slab stratification of $\mathcal{E}_A(M)$ refines its rank stratification. From this it follows that each $\mathcal{E}^i_A(M) = \{\beta \in \mathbb{C}^d \mid j(\beta) > i\}$ is a union of ranking slabs, making each a union of translated linear subspaces of $\mathbb{C}^d$.

Proof of Corollary 1.4. If $\beta, \beta' \in \mathbb{C}^d$ belong to the same ranking slab, then the ranking lattices $\mathbb{E}^{\beta} = \mathbb{E}^{\beta'}$ coincide by Proposition 5.4. By Theorem 1.3, the rank jumps $j(\beta)$ and $j(\beta')$ coincide as well. ☐

Corollary 6.23. *For all integers $i \geqslant 0$, $\mathcal{E}^i_A(M)$ is a union of translates of linear subspaces that are generated by faces of A.*

Proof. This is an immediate consequence of Theorem 4.3 and Corollary 1.4. ☐

The following is the second example promised at the end of § 4, showing that the rank of $\mathcal{H}_0(M, \beta)$ need not be constant on a slab (see Definition 4.8). Further, this example shows that neither the arrangement stratification of $\mathcal{E}_A(M)$ nor its refinement given by the Ext modules in (4.2) determine its rank stratification.

Example 6.24. Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 & 2 & 3 & 2 & 2 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 2 & 5 & 7 & 7 \end{bmatrix}$$

with $\operatorname{vol}(A) = 185$, and label the faces $F_1 = [a_1\ a_2\ a_3\ a_4]$, $F_2 = [a_5\ a_6\ a_7\ a_8]$, and $F_3 = [a_9]$. With

$$\beta' = \begin{bmatrix} 1\\1\\0 \end{bmatrix} \quad \text{and} \quad \mathcal{P}_A = \left\{ \begin{bmatrix} 2\\3\\3 \end{bmatrix}, \begin{bmatrix} 3\\2\\3 \end{bmatrix}, \begin{bmatrix} 3\\5\\3 \end{bmatrix}, \begin{bmatrix} 5\\3\\3 \end{bmatrix}, \begin{bmatrix} 3\\3\\5 \end{bmatrix}, \begin{bmatrix} 5\\5\\6 \end{bmatrix} \right\},$$

the exceptional arrangement of S_A is

$$\mathcal{E}_A(S_A) = (\beta' + \mathbb{C}F_3) \cup \mathcal{P}_A.$$

For $\beta \in \mathbb{R}_{\geqslant 0}A \cap \mathcal{E}_A(M)$,

$$\mathcal{R}_A(S_A, \beta) = \begin{cases} \beta & \text{if } \beta \in \mathcal{P}_A, \\ \beta' + \mathbb{C}F_3 & \text{if } \beta \in [\beta' + \mathbb{C}F_3]\backslash\beta', \\ \bigcup_{i=1}^{3}[\beta' + \mathbb{C}F_i] & \text{if } \beta = \beta', \end{cases}$$

so by the proof of Theorem 6.6, the rank jump of M at $\beta \in \mathcal{E}_A(M)$ is

$$j(\beta) = \begin{cases} 1 & \text{if } \beta \in [\beta' + \mathbb{C}F_3]\backslash\beta', \\ 2 & \text{otherwise.} \end{cases}$$

Here the arrangement stratification of $\mathcal{E}_A(S_A)$ agrees with the one given by the Ext modules that determine it, but $j(-)$ is not constant on the slab $[\beta' + \mathbb{C}F_3] \subseteq \mathcal{E}_A(S_A)$.

To show that all of the arrangements in the definition of ranking slabs (Definition 4.7) are necessary to obtain a refinement of the rank stratification of $\mathcal{E}_A(M)$, we include the following example. Here, $j(\beta)$ changes where components of $\mathcal{E}_A(S_A)$ that correspond to different Ext modules intersect.

Example 6.25. Let

$$A = \begin{bmatrix} 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 3 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$F = [a_1\ a_2]$, $G = [a_3\ a_4\ a_5\ a_6]$, and $\beta' = \left[\begin{smallmatrix} 1 \\ 1 \\ 0 \\ 0 \end{smallmatrix}\right]$. Here $\mathrm{vol}(A) = 21$, and the exceptional arrangement of S_A is

$$\mathcal{E}_A(S_A) = [\beta' + \mathbb{C}F] \cup [\beta' + \mathbb{C}G],$$

where

$$-\mathrm{qdeg}(\mathrm{Ext}^i_R(S_A, R)(-\varepsilon_A)) = \begin{cases} b + \mathbb{C}G & \text{if } i = 7, \\ b + \mathbb{C}F & \text{if } i = 8, \\ \varnothing & \text{if } i > 8. \end{cases}$$

By Corollary 6.2 and Example 6.21,

$$j(\beta) = \begin{cases} 9 & \text{if } \beta = \beta', \\ 6 & \text{if } \beta \in [\beta' + \mathbb{C}F]\backslash\beta', \\ 4 & \text{if } \beta \in [\beta' + \mathbb{C}G]\backslash\beta'. \end{cases}$$

We include a final example to show that $j(\beta)$ is not determined simply by $\widetilde{\mathbb{N}A}\backslash\mathbb{N}A$, the holes in the semigroup $\mathbb{N}A$.

Example 6.26. The matrix

$$A = \begin{bmatrix} 2 & 3 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

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has volume 16. The exceptional arrangement of $M = S_A$ is the union of four lines and a point:

$$\mathcal{E}_A(S_A) = \left(\begin{bmatrix}1\\1\\0\end{bmatrix} + \mathbb{C}F\right) \cup \left(\begin{bmatrix}1\\1\\0\end{bmatrix} + \mathbb{C}G\right) \cup \left(\begin{bmatrix}1\\0\\0\end{bmatrix} + \mathbb{C}[a_5]\right) \cup \left(\begin{bmatrix}0\\1\\0\end{bmatrix} + \mathbb{C}[a_6]\right) \cup \left\{\begin{bmatrix}2\\2\\1\end{bmatrix}\right\},$$

where $F = [a_1\ a_2]$ and $G = [a_3\ a_4]$. The generic rank jumps along each component are as follows:

$$\begin{bmatrix}1\\1\\0\end{bmatrix} + \mathbb{C}F \mapsto 3, \quad \begin{bmatrix}1\\0\\0\end{bmatrix} + \mathbb{C}[a_5] \mapsto 1, \quad \begin{bmatrix}1\\1\\0\end{bmatrix} + \mathbb{C}G \mapsto 3, \quad \begin{bmatrix}0\\1\\0\end{bmatrix} + \mathbb{C}[a_6] \mapsto 1 \quad \text{and} \quad \begin{bmatrix}2\\2\\1\end{bmatrix} \mapsto 2.$$

These generic rank jumps are achieved everywhere except at the points

$$\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0\\0\\-1\end{bmatrix}.$$

The point that may be unexpected in this collection is $b = \left[\begin{smallmatrix}0\\0\\-1\end{smallmatrix}\right]$. Both of the components of $\mathcal{E}_A(S_A)$ that contain b have generic rank jumps of 1; however, $j(b) = 2$. This is because the ranking arrangement of S_A has three components that contain b:

$$\mathcal{R}_A(S_A, b) = (b + \mathbb{C}[a_5]) \cup (b + \mathbb{C}[a_6]) \cup (b + \mathbb{C}[a_1\ a_2\ a_3\ a_4]). \tag{6.22}$$

If the plane $(b + \mathbb{C}[a_1\ a_2\ a_3\ a_4])$ were not in $\mathcal{R}_A(S_A, \beta)$, then the rank jump of S_A at b would only be 1 by (6.21). Thus the hyperplane in (6.22), although unrelated to the holes in $\mathbb{N}A$, accounts for the higher value of $j(b)$.

The rank jumps at the other parameters are

$$j\left(S_A, \begin{bmatrix}1\\0\\0\end{bmatrix}\right) = j\left(S_A, \begin{bmatrix}0\\1\\0\end{bmatrix}\right) = 3 \quad \text{and} \quad j\left(S_A, \begin{bmatrix}1\\1\\0\end{bmatrix}\right) = 5.$$

The algebraic upper semi-continuity of the rank of $\mathcal{H}_0(M, \beta)$ implies that most of the codimension-one components of the ranking arrangement $\mathcal{R}_A(M)$ do not increase the rank of $\mathcal{H}_0(M, \beta)$. It would interesting to know if the set of such hyperplanes can be identified.

7. The isomorphism classes of A-hypergeometric systems

When $M = S_A$, the results of § 6 apply to the A-hypergeometric system $\mathcal{H}_0(S_A, \beta) = M_A(\beta)$. For a face τ of A,

$$E_\tau(\beta) = \{\lambda \in \mathbb{C}\tau \mid \beta - \lambda \in \mathbb{N}A + \mathbb{Z}\tau\}/\mathbb{Z}\tau$$

is a finite set. It is shown in [Sai01, ST01] that $M_A(\beta)$ and $M_A(\beta')$ are isomorphic as D-modules precisely when $E_\tau(\beta) = E_\tau(\beta')$ for all faces τ of A. We will now use Euler–Koszul homology to give a simple proof of one direction of this equivalence; first we exhibit a complementary relationship between $E_\tau(\beta)$ and $\mathbb{E}_\tau^\beta$ (see (5.2)).

Observation 7.1. It is shown in Theorem 1.3 that as τ runs through the faces of A, the sets

$$\mathbb{E}_\tau^\beta = \mathbb{Z}^d \cap (\beta + \mathbb{C}\tau) \backslash (\mathbb{N}A + \mathbb{Z}\tau) = \beta - \{\lambda \in \mathbb{C}\tau \mid \beta - \lambda \in \mathbb{Z}^d \backslash (\mathbb{N}A + \mathbb{Z}\tau)\}$$

determine the rank jump of $M_A(\beta)$ at β. Notice that $(\beta - \mathbb{E}_\tau^\beta)/\mathbb{Z}\tau$ is the complement of $E_\tau(\beta)$ in the group $(\mathbb{Z}^d \cap \mathbb{Q}\tau)/\mathbb{Z}\tau$.

Lemma 7.2. *If $\beta, \beta' \in \mathbb{C}^d$ are such that $\beta' - \beta = A\lambda$ for some $\lambda \in \mathbb{N}^n$, then the map defined by right multiplication $\partial_A^\lambda : M_A(\beta) \to M_A(\beta')$ is an isomorphism of D-modules if and only if $\beta' \notin \operatorname{qdeg}(S_A/\langle \partial_A^\lambda \rangle)$.*

Proof. See [SW09, Remark 3.6]. □

For a vector $v \in \mathbb{C}^n$, let the *support* of v be the subset $\operatorname{supp}(v) = \{a_i \mid v_i \neq 0\}$ of the columns of A.

Lemma 7.3. *If $\beta, \beta' \in \mathbb{C}^d$ are such that $\beta' - \beta = A\lambda$ for some $\lambda \in \mathbb{N}^n$ and $E_\tau(\beta) = E_\tau(\beta')$ for all faces τ of A, then the map defined by right multiplication $\partial_A^\lambda : M_A(\beta) \to M_A(\beta')$ is an isomorphism.*

Proof. If ∂_A^λ is not an isomorphism, then $\beta' \in \operatorname{qdeg}(S_A/\langle \partial_A^\lambda \rangle)$ by Lemma 7.2. By the definition of quasidegrees, there exist vectors $v \in \mathbb{N}^n$ and $\gamma \in \mathbb{C}\tau$ for some face τ such that $\beta' = Av + \gamma$, $\operatorname{supp}(\lambda) \not\subseteq \tau$, and $v_i \leqslant \lambda_i$ for all i. Hence $\beta' - \gamma \in \mathbb{N}A$, so $\gamma + \mathbb{Z}\tau \in E_\tau(\beta')$. Further, the condition $v_i \leqslant \lambda_i$ for all i implies that $\gamma + \mathbb{Z}\tau \notin E_\tau(\beta)$. □

Notation 7.4. If a vector $\lambda \in \mathbb{N}^n$ is such that the map given by right multiplication $\partial_A^\lambda : M_A(\beta) \to M_A(\beta + A\lambda)$ is an isomorphism, let $\partial_A^{-\lambda}$ denote its inverse.

Theorem 7.5. *The A-hypergeometric systems $M_A(\beta)$ and $M_A(\beta')$ are isomorphic if and only if $E_\tau(\beta) = E_\tau(\beta')$ for all faces τ of A.*

Proof. The proof of the 'only-if' direction holds as in [Sai01] without a homogeneity assumption on A because it involves only the construction of formal solutions of $M_A(\beta)$.

For the 'if' direction, suppose that $E_\tau(\beta) = E_\tau(\beta')$ for all faces τ of A. As stated in [Sai01, Proposition 2.2], $E_A(\beta) = E_A(\beta')$ implies by definition that $\beta' - \beta \in \mathbb{Z}A$, so $\beta' - \beta = A\lambda$ for some $\lambda \in \mathbb{Z}^n$. There are unique vectors $\lambda_+, \lambda_- \in \mathbb{N}^n$ with disjoint support such that $\lambda = \lambda_+ - \lambda_-$. In light of Lemma 7.3, we may assume that both λ_+ and λ_- are non-zero. We claim that at least one of $\partial_A^{-\lambda_-}\partial_A^{\lambda_+}$ or $\partial_A^{\lambda_+}\partial_A^{-\lambda_-}$ defines an isomorphism from $M_A(\beta)$ to $M_A(\beta')$.

If $\partial_A^{\lambda_+} : M_A(\beta) \to M_A(\beta + A\lambda_+)$ or $\partial_A^{\lambda_+} : M_A(\beta - A\lambda_-) \to M_A(\beta')$ defines an isomorphism, then the 'only-if' direction and Lemma 7.3 imply that $\partial_A^{-\lambda_-}\partial_A^{\lambda_+}$ or $\partial_A^{\lambda_+}\partial_A^{-\lambda_-}$, respectively, give the desired isomorphism. We are left to consider the case when $\partial_A^{\lambda_+}$ does not define an isomorphism from either domain. By Lemma 7.2, this is equivalent to

$$\beta + A\lambda_+ \in \operatorname{qdeg}(S_A/\langle \partial_A^{\lambda_+} \rangle) \quad \text{and} \quad \beta' \in \operatorname{qdeg}(S_A/\langle \partial_A^{\lambda_+} \rangle). \tag{7.1}$$

From the right side of (7.1), we see that the non-empty face $\eta := \operatorname{supp}(\lambda_-)$ is such that $\beta' + \mathbb{C}\eta \subseteq \operatorname{qdeg}(S_A/\langle \partial_A^{\lambda_+} \rangle)$, so $E_\eta(\beta') \neq \varnothing$. However, the shift $A\lambda_+$ in the left side of (7.1) implies that $(\beta + \mathbb{C}\eta) \cap (\mathbb{N}A + \mathbb{Z}\eta) = \varnothing$. Thus $E_\tau(\beta) = \varnothing$, which is a contradiction. □

It is not yet understood how the holomorphic solution space of $M_A(\beta)$ varies as a function of β; different functions of β suggest alternative behaviors. Walther showed in [Wal07] that the reducibility of the monodromy of $M_A(\beta)$ varies with β in a lattice-like fashion. When the convex hull of A and the origin is a simplex, Saito used the sets $E_\tau(\beta)$ to construct a basis of holomorphic solutions of $M_A(\beta)$ with a common domain of convergence [Sai02]. Thus Theorem 1.3 and the complementary relationship in Observation 7.1 between the $E_\tau(\beta)$ and the ranking lattices of S_A at β suggest that the ranking slabs give the coarsest stratification over which there could be a constructible sheaf of solutions for the hypergeometric system.

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