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NORM OF A LINEAR COMBINATION OF TWO OPERATORS ON A HILBERT SPACE

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Let $\alpha, \beta, \gamma, \delta$ be complex numbers such that $\gamma\delta \neq 0$. If A and B are bounded linear operators on the Hilbert space H such that $\gamma A + \delta B$ is right invertible then we study the operator norm of $(\alpha A + \beta B)(\gamma A + \delta B)^{-1}$ using the angle ϕ between two subspaces $\operatorname{ran} A$ and $\operatorname{ran} B$ or the angle $\psi = \psi(A, B)$ between two operators A and B where

$$\cos\psi(A, B) = \sup\Bigl\{\bigl|\langle Af, Bf\rangle\bigr|\Big/\bigl(\|Af\|\cdot\|Bf\|\bigr)\ ;\ f \in H,\ Af \neq 0,\ Bf \neq 0\Bigr\}.$$

1. INTRODUCTION

Let $B(H)$ be the set of all bounded linear operators on the Hilbert space H. Let $P \in B(H)$ satisfy $P^2 = P$ and let $Q = I - P$ where I denotes the identity operator on H. Denote by $\phi(H_1, H_2)$ the minimal angle between two subspaces H_1 and H_2 of H:

$$\cos\phi(H_1, H_2) = \sup_{0 \neq f \in H_1, 0 \neq g \in H_2} \frac{\bigl|\langle f, g\rangle\bigr|}{\|f\|\cdot\|g\|}.$$

Then $0 \leqslant \phi(H_1, H_2) \leqslant \pi/2$. Let $\operatorname{ran} P$ denote the range of P. If $\phi = \phi(\operatorname{ran} P, \operatorname{ran} Q) > 0$ then

$$\|P\| = \|Q\| = \csc\phi = \frac{1}{\sin\phi}$$

(see [**2**, p.339]). Let $J = P - Q$. Then

$$\|P\| = \|Q\| = \frac{1}{2}\Bigl(\|J\| + \frac{1}{\|J\|}\Bigr)$$

(see [**6**, Lemma 2], [**1**]). Hence

$$\|J\| = \|P\| + \sqrt{\|P\|^2 - 1} = (\csc + \cot)\phi = \cot\frac{\phi}{2}.$$

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Let α and β be complex numbers, and let t be a nonnegative number. Then we define a function $F(\alpha,\beta,t)$ which is the generalisation of $\max\bigl(|\alpha|,|\beta|\bigr) = F(\alpha,\beta,0)$.

DEFINITION 1. Let

$$F(\alpha,\beta,t) = \sqrt{\left|\frac{\alpha-\beta}{2}\right|^2 t + \left(\frac{|\alpha|+|\beta|}{2}\right)^2} + \sqrt{\left|\frac{\alpha-\beta}{2}\right|^2 t + \left(\frac{|\alpha|-|\beta|}{2}\right)^2}.$$

Then $F(\alpha,\beta,t)$ is a nondecreasing function of t, $t \geqslant 0$ and satisfies

$$\max\bigl(|\alpha|,|\beta|\bigr) = F(\alpha,\beta,0) \leqslant F(\alpha,\beta,t) < \infty, \quad (t \geqslant 0).$$

Feldman, Krupnik and Markus [**1**] established the following formula.

FELDMAN, KRUPNIK AND MARKUS FORMULA. Let $P \in B(H)$ satisfy $P \neq 0, I$ and $P^2 = P$. Let $Q = I - P$. Let $\alpha,\beta \in \mathbf{C}$. Then

$$\|\alpha P + \beta Q\| = F\bigl(\alpha,\beta,\|P\|^2 - 1\bigr).$$

Let $\phi = \phi(\operatorname{ran} P,\ \operatorname{ran} Q)$. Since $\|P\| = \csc\phi$, it follows that $\|P\|^2 - 1 = \cot^2\phi$. Hence $\|\alpha P + \beta Q\| = F\left(\alpha,\beta,\cot^2\phi\right)$.

DEFINITION 2. For two nonzero operators A, B on H, let $\psi(A,B)$ satisfy $0 \leqslant \psi(A,B) \leqslant \pi/2$ and

$$\cos\psi(A,B) = \sup_{Af\neq 0, Bf\neq 0} \frac{\bigl|\langle Af, Bf\rangle\bigr|}{\|Af\|\cdot\|Bf\|}.$$

Since $\cos\psi(A,B) \leqslant \cos\phi(\operatorname{ran} A, \operatorname{ran} B)$, it follows that $\psi(A,B) \geqslant \phi(\operatorname{ran} A, \operatorname{ran} B)$. We call $\psi(A,B)$ as the angle between two operators A and B. If $P^2 = P(\neq 0, I)$ and $Q = I - P$ then $\phi(\operatorname{ran} P, \operatorname{ran} Q) = \psi(P,Q)$, because if $h = Pf + Qg$ then

$$\frac{\bigl|\langle Pf, Qg\rangle\bigr|}{\|Pf\|\cdot\|Qg\|} = \frac{\bigl|\langle Ph, Qh\rangle\bigr|}{\|Ph\|\cdot\|Qh\|}.$$

In this paper, we shall study the operator norm of $(\alpha A + \beta B)(\gamma A + \delta B)^{-1}$. We use $\phi(\operatorname{ran} A, \operatorname{ran} B)$ in Section 2, and we use $\psi(A,B)$ in Section 4. Let $K = \overline{\operatorname{ran} A} \cap \overline{\operatorname{ran} B}$. In Section 2, we shall study in the case when $K \neq \overline{\operatorname{ran} A}$ and $K \neq \overline{\operatorname{ran} B}$. In Theorem 1, we shall use the Feldman, Krupnik and Markus formula [**1**] and Lemma 1 to establish the formula of the operator norm of $(\alpha A + \beta B)(\gamma A + \delta B)^{-1}$ using the angle $\phi(\operatorname{ran} A,\ \operatorname{ran} B)$ in the case when $K = \{0\}$ and $\gamma A + \delta B$ is right invertible. In Theorem 2, we shall use Theorem 1 to estimate the norm from below using the angle $\phi\bigl(\overline{\operatorname{ran} A} \ominus K, \overline{\operatorname{ran} B} \ominus K\bigr)$ in the case when K is a nonzero invariant subspace of $A(\gamma A + \delta B)^{-1}$. In Section 3, we shall study in the case when $K = \overline{\operatorname{ran} B}$. We shall consider the nilpotent operator B on H. The results in Sections 2 and 3 follow from the Feldman, Krupnik and Markus formula. In Section 4, if $\psi(A,B) > 0$ and $\gamma A + \delta B$ is right invertible or left invertible then we shall estimate $\left\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\right\|$ from above. In Theorem 3, we do not assume the boundedness of two operators A and B. As a corollary of Theorem 3, we shall show that if $\|A + B\| < \infty$ and $\psi(A,B) > 0$ then $\|A\| < \infty$ and $\|B\| < \infty$. The results in Section 4 do not follow from the Feldman, Krupnik and Markus formula.

2. Norm formula using the angle ϕ between $\operatorname{ran} A$ and $\operatorname{ran} B$

Let $A, B \in B(H)$, $A \neq 0$, $B \neq 0$ and let $K = \overline{\operatorname{ran} A} \cap \overline{\operatorname{ran} B}$. In this section we shall study in the case when $K \neq \overline{\operatorname{ran} A}$ and $K \neq \overline{\operatorname{ran} B}$. In Theorem 1, if $K = \{0\}$ and $\operatorname{ran}(\gamma A + \delta B) = H$ then we shall use the Feldman, Krupnik and Markus formula [**1**], and establish the norm formula of $\left\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\right\|$ using the angle $\phi(\operatorname{ran} A,\ \operatorname{ran} B)$. In Theorem 2, if K is an invariant subspace of $A(\gamma A + \delta B)^{-1}$ then we shall estimate the norm from below using $\phi\left(\overline{\operatorname{ran} A} \ominus K, \overline{\operatorname{ran} B} \ominus K\right)$.

The operator $X \in B(H)$ is said to be right invertible if there exists an operator $Y \in B(H)$ such that $XY = I$. The operator Y is called the right inverse to X and is denoted by X^{-1}. Then $X^{-1} \in B(H)$ is not uniquely defined (see [**3**, Volume I, p.63]). If $\gamma A + \delta B$ is right invertible then

$$\left\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\right\| = \sup_{(\gamma A + \delta B) f \neq 0} \frac{\left\|(\alpha A + \beta B) f\right\|}{\left\|(\gamma A + \delta B) f\right\|},$$

where $(\gamma A + \delta B)^{-1}$ denotes one of the right inverses to $\gamma A + \delta B$.

Lemma 1. *Let $A, B \in B(H)$ satisfy $A \neq 0$, $B \neq 0$ and $\operatorname{ran}(A + B) = H$. The following assertions are equivalent:*

(1) $\operatorname{ran} A \cap \operatorname{ran} B = \{0\}$.

(2) $\overline{\operatorname{ran} A} \cap \overline{\operatorname{ran} B} = \{0\}$.

(3) $\phi(\operatorname{ran} A,\ \operatorname{ran} B) > 0$.

Suppose (1) to (3) hold. Let $(A + B)^{-1}$ denote one of the right inverses to $A + B$. Let $P = A(A + B)^{-1}$ and let $Q = B(A + B)^{-1}$. Then P and Q do not depend on the choice of $(A + B)^{-1}$. Then $P^2 = P \neq 0, I$, $P + Q = I$, $\operatorname{ran} P = \operatorname{ran} A$ and $\operatorname{ran} Q = \operatorname{ran} B$.

Proof: (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (3): Let $H_1 = \ker(A + B)$, and let $H_2 = H_1^{\perp}$. Then $H = H_1 \oplus H_2$. Since $(A + B)|_{H_1} = 0$, it follows that $A|_{H_1} = -B|_{H_1}$. By (1), $A|_{H_1} = B|_{H_1} = 0$. Let $T = (A + B)|_{H_2}$. Then $T \in B(H_2, H)$ and $\ker T = \{0\}$. Since $\operatorname{ran}(A + B) = H$, it follows that $\operatorname{ran} T = H$. By the open mapping theorem, there exists $S \in B(H, H_2)$ such that $ST = I_{H_2}$ and $TS = I_H$. Hence $(A + B)S = TS = I_H = I$. Hence S is a right inverse to $A + B$. Let C be one of the right inverses to $A + B$. Then $P + Q = (A + B)C = I$. Hence $A(C - S) = -B(C - S)$. By (1), $A(C - S) = -B(C - S) = 0$. Hence $P = AC = AS$ and $Q = BC = BS$. Hence P and Q do not depend on the choice of $(A + B)^{-1}$. By (1),

$$\operatorname{ran} P \cap \operatorname{ran}(I - P) = \operatorname{ran} P \cap \operatorname{ran} Q \subset \operatorname{ran} A \cap \operatorname{ran} B = \{0\}.$$

Since $P(I - P) = (I - P)P$, this implies that $P^2 = P$. Suppose $P = 0$. Then $AS = 0$ and hence $A|_{H_2} = AST = 0$. Since $A|_{H_1} = 0$, it follows that $A = 0$. This is a contradiction. Hence $P \neq 0$. Suppose $P = I$. Then $BS = Q = I - P = 0$ and hence

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$B|_{H_2} = BST = 0$. Since $B|_{H_1} = 0$, it follows that $B = 0$. This is a contradiction. Hence $P \neq I$. Since $P \neq 0, I$ and $Q = I - P$, it follows from Gohberg and Krein [**2**, p.339] that $\|P\| = \csc \phi(\operatorname{ran} P, \operatorname{ran} Q)$. Hence

$$\cos \phi(\operatorname{ran} P,\ \operatorname{ran} Q) = \frac{\sqrt{\|P\|^2 - 1}}{\|P\|} < 1.$$

Hence $\phi(\operatorname{ran} P,\ \operatorname{ran} Q) > 0$. Since $A|_{H_1} = B|_{H_1} = 0$, it follows that

$$\operatorname{ran} P = \operatorname{ran} AS = \operatorname{ran} A|_{H_2} = \operatorname{ran} A,$$
$$\operatorname{ran} Q = \operatorname{ran} BS = \operatorname{ran} B|_{H_2} = \operatorname{ran} B.$$

Hence

$$\phi(\operatorname{ran} A,\ \operatorname{ran} B) = \phi(\operatorname{ran} P,\ \operatorname{ran} Q) > 0.$$

(3) $\Rightarrow$ (2): Suppose $\overline{\operatorname{ran} A} \cap \overline{\operatorname{ran} B} \neq \{0\}$. Then there exists an $h \in H$ and sequences $\{f_n\}, \{g_n\} \subset H$ such that $h \neq 0$, $\|Af_n - h\| \to 0$, $\|Bg_n - h\| \to 0$ as $n \to \infty$. Hence

$$\cos \phi(\operatorname{ran} A,\ \operatorname{ran} B) = \sup_{f,g \in H} \frac{|\langle Af, Bg\rangle|}{\|Af\| \cdot \|Bg\|} \geqslant \lim_{n\to\infty} \frac{|\langle Af_n, Bg_n\rangle|}{\|Af_n\| \cdot \|Bg_n\|} = \frac{|\langle h, h\rangle|}{\|h\| \cdot \|h\|} = 1.$$

Hence $\phi(\operatorname{ran} A,\ \operatorname{ran} B) = 0$. Lemma 1 is proved. □

The assertions in Lemma 1 are equivalent to the formula:

$$\left\|A(A+B)^{-1}\right\| = \csc \phi(\operatorname{ran} A, \operatorname{ran} B).$$

If $\alpha = \gamma = \delta = 1$ and $\beta = 0$ then the following Theorem 1 implies this formula. Let P_{H_1} (respectively P_{H_2}) denote the orthogonal projection from H onto H_1 (respectively H_2). By Lemma 1, if $H_1 \cap H_2 = \{0\}$ and $\operatorname{ran}(P_{H_1} + P_{H_2}) = H$ then $\phi(H_1, H_2) > 0$.

Theorem 1. *Let $\alpha, \beta, \gamma, \delta$ be complex numbers such that $\gamma\delta \neq 0$. Let $A, B \in B(H)$ satisfy $A \neq 0$, $B \neq 0$, $\operatorname{ran}(\gamma A + \delta B) = H$ and $\operatorname{ran} A \cap \operatorname{ran} B = \{0\}$. Then*

$$\left\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\right\| = F\left(\frac{\alpha}{\gamma},\ \frac{\beta}{\delta},\ \cot^2 \phi\right).$$

where $\phi = \phi(\operatorname{ran} A, \operatorname{ran} B) > 0$ and $(\gamma A + \delta B)^{-1}$ denotes one of the right inverses to $\gamma A + \delta B$.

Proof: It is sufficient to prove when $\gamma = \delta = 1$. Let $P = A(A+B)^{-1}$ and let $Q = B(A+B)^{-1}$, where $(A+B)^{-1}$ denote one of the right inverses to $A + B$. Then $P + Q = I$. By Lemma 1, if $A \neq 0$ and $B \neq 0$ then $P^2 = P \neq 0, I$, $\operatorname{ran} P = \operatorname{ran} A$ and $\operatorname{ran} Q = \operatorname{ran} B$. Since $\|P\| = \csc \phi(\operatorname{ran} P, \operatorname{ran} Q)$, it follows from the Feldman, Krupnik and Markus formula [**1**] that

$$\begin{aligned}\left\|(\alpha A + \beta B)(A+B)^{-1}\right\| &= \|\alpha P + \beta Q\| \\ &= F\left(\alpha,\ \beta,\ \|P\|^2 - 1\right) \\ &= F\left(\alpha,\ \beta,\ \cot^2 \phi(\operatorname{ran} P, \operatorname{ran} Q)\right) \\ &= F\left(\alpha,\ \beta,\ \cot^2 \phi(\operatorname{ran} A, \operatorname{ran} B)\right).\end{aligned}$$

Theorem 1 is proved. $\square$

In Theorem 1, if $AB = BA$ then

$$\left\|(\alpha A+\beta B)(\gamma A+\delta B)^{-1}\right\| = \left\|(\gamma A+\delta B)(\alpha A+\beta B)^{-1}\right\| = F\Big(\frac{\alpha}{\gamma},\ \frac{\beta}{\delta},\ \cot^2\phi\Big).$$

We have assumed that $\operatorname{ran} A \cap \operatorname{ran} B = \{0\}$. This is equivalent to the assertions in Lemma 1. If $(\alpha\delta - \beta\gamma)\gamma\delta \neq 0$ then $\alpha/\gamma - \beta/\delta \neq 0$ and hence the norm formula in Theorem 1 is equivalent to the assertions in Lemma 1. By Theorem 1, if $P \in B(H)$ satisfy $P^2 = P \neq I, 0$ and $Q = I - P$ then

$$\|P\| = \|Q\| = \csc\phi(\operatorname{ran} P, \operatorname{ran} Q),$$

which is in the book of Gohberg and Krein [**2**, p.339]. If $A = P$ and $B = Q$ then Theorem 1 becomes the Feldman, Krupnik and Markus formula ([**1**]).

In Theorem 1, if $A = P$ and $B = Q^*$ then $A + B$ becomes invertible and we can compute the norm $\left\|(\alpha A+\beta B)(A+B)^{-1}\right\|$ as the following. For all $f \in H$,

$$\begin{aligned}\big\langle (A^*A+B^*B)f, f\big\rangle &= \|Af\|^2 + \|Bf\|^2\\ &= \|Af\|^2 + \|f - A^*f\|^2\\ &= \|f\|^2 + \|Af - A^*f\|^2\\ &\geqslant \|f\|^2.\end{aligned}$$

Hence $A^*A + B^*B$ is invertible. Similarly, $AA^* + BB^*$ is invertible. Since

$$(A+B)(A^*+B^*) = AA^* + BB^*$$

and

$$(A^*+B^*)(A+B) = A^*A + B^*B,$$

it follows that $A+B$ is invertible. Since $\operatorname{ran} A \perp \operatorname{ran} B$, it follows that $\phi(\operatorname{ran} A, \operatorname{ran} B) = \pi/2$. By Theorem 1,

$$\left\|(\alpha A+\beta B)(A+B)^{-1}\right\| = F(\alpha,\beta,0) = \max\big(|\alpha|, |\beta|\big).$$

Let $P_0 = A(A+B)^{-1}$ and $Q_0 = B(A+B)^{-1}$. Since $(A+B)^*A = A^*(A+B)$, it follows that P_0 and Q_0 are selfadjoint. By Lemma 1, P_0 and Q_0 are selfadjoint idempotent.

Corollary 1. *Let $A, B \in B(H)$ satisfy $\operatorname{ran}(A+B) = H$. Let α and β be complex numbers. Let p, q, r, s be complex numbers satisfying $p+r = q+s = 1$, $ps - qr \neq 0$, $\operatorname{ran}(pA+qB) \neq H$, $\operatorname{ran}(rA+sB) \neq H$ and $\operatorname{ran}(pA+qB) \cap \operatorname{ran}(rA+sB) = \{0\}$. Then*

$$\left\|(\alpha A+\beta B)(A+B)^{-1}\right\| = F\Big(\frac{\alpha s-\beta r}{ps-qr}, \frac{p\beta-q\alpha}{ps-qr}, \cot^2\phi\Big),$$

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where $\phi = \phi\big(\operatorname{ran}(pA+qB),\ \operatorname{ran}(rA+sB)\big)$, *and* $(A+B)^{-1}$ *denote one of the right inverses to* $A+B$.

PROOF: Let $A' = pA + qB$ and let $B' = rA + sB$. Then Then $A + B = A' + B'$. Define α' and β' by

$$\alpha' = \frac{\alpha s - \beta r}{ps - qr} \quad \text{and} \quad \beta' = \frac{p\beta - q\alpha}{ps - qr}.$$

Then $\alpha A + \beta B = \alpha' A' + \beta' B'$. Since $\operatorname{ran} A' \neq H$, $\operatorname{ran} B' \neq H$ and $\operatorname{ran} A' + \operatorname{ran} B' = \operatorname{ran}(A+B) = H$, it follows from Theorem 1 that

$$\left\|(\alpha A + \beta B)(A+B)^{-1}\right\| = \left\|(\alpha' A' + \beta' B')(A'+B')^{-1}\right\| = F\big(\alpha', \beta', \cot^2 \phi(A', B')\big).$$

Corollary 1 is proved. ∎

There are many operators $A, B \in B(H)$ such that $\rho(A,B) = 1$ and $\rho(pA+qB, rA+sB) < 1$. If $p = s = 1$ and $q = r = 0$ then Corollary 1 implies Theorem 1. Let $K = \overline{\operatorname{ran} A} \cap \overline{\operatorname{ran} B}$. In Theorem 1, we have established the norm formula in the case when $K = \{0\}$. Next we shall consider the case when $K \neq \{0\}$. Then $\phi(\operatorname{ran} A, \operatorname{ran} B) = 0$. In the following Theorem 2, we shall estimate the norm $\left\|(\alpha A + \beta B)(A+B)^{-1}\right\|$ from below in the case when $K \neq \overline{\operatorname{ran} A}$ and $K \neq \overline{\operatorname{ran} B}$.

THEOREM 2. *Let* $\alpha, \beta, \gamma, \delta$ *be complex numbers such that* $\gamma\delta \neq 0$. *Let* $A, B \in B(H)$. *Let* $\gamma A + \delta B$ *be invertible. Let* $K = \overline{\operatorname{ran} A} \cap \overline{\operatorname{ran} B}$ *be an invariant subspace of* $A(\gamma A + \delta B)^{-1}$ *such that* $K \neq \overline{\operatorname{ran} A}$ *and* $K \neq \overline{\operatorname{ran} B}$. *Then*

$$\left\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\right\| \geqslant F\Big(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \cot^2 \phi\Big),$$

where $\phi = \phi\big(\overline{\operatorname{ran} A} \ominus K, \overline{\operatorname{ran} B} \ominus K\big) > 0$.

PROOF: It is sufficient to prove that when $\gamma = \delta = 1$

$$\left\|(\alpha A + \beta B)(A+B)^{-1}\right\| \geqslant F\left(\alpha, \beta, \cot^2 \phi\right).$$

Let $X = A(A+B)^{-1}$ and let $Y = B(A+B)^{-1}$. Then $X + Y = I$. Let P_K denote the orthogonal projection from H onto K and let $P_{K^\perp}$ denote the orthogonal projection from H onto $K^\perp$. Let $X_1 = P_{K^\perp} X|_{K^\perp}$ and let $Y_1 = P_{K^\perp} Y|_{K^\perp}$. We shall show that $\operatorname{ran} X_1 \cap \operatorname{ran} Y_1 = \{0\}$. Let $h \in \operatorname{ran} X_1 \cap \operatorname{ran} Y_1$. Then $h \in K^\perp$. There exist $f, g \in K^\perp$ such that $h = P_{K^\perp} X f = P_{K^\perp} Y g$. Then $Xf = h + P_K X f$ and $Yg = h + P_K Y g$. Hence $Xf - Yg \in K$. Since $Yg \in \operatorname{ran} B$, it follows that $Xf \in \operatorname{ran} A \ \cap\ \overline{\operatorname{ran} B} \subset K$. Hence $h \in K \cap K^\perp = \{0\}$. Therefore $\operatorname{ran} X_1 \cap \operatorname{ran} Y_1 = \{0\}$ and $X_1 + Y_1 = I|_{K^\perp}$. By Lemma 1, this implies that X_1 and Y_1 are idempotent operators on $K^\perp$, and $\phi(\operatorname{ran} X_1,\ \operatorname{ran} Y_1) > 0$. Then $\operatorname{ran} X_1$ and $\operatorname{ran} Y_1$ are closed subspaces of H. By Theorem 1,

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$$\begin{aligned}\left\|(\alpha A+\beta B)(A+B)^{-1}\right\| &= \|\alpha X+\beta Y\| \\ &\geqslant \sup_{f\in K^\perp}\frac{\left\|P_{K^\perp}(\alpha X+\beta Y)f\right\|}{\|f\|} = \sup_{f\in K^\perp}\frac{\left\|(\alpha X_1+\beta Y_1)f\right\|}{\|f\|} \\ &= \|\alpha X_1+\beta Y_1\|_{K^\perp} \\ &= F\left(\alpha,\beta,\cot^2\phi(\operatorname{ran}X_1,\operatorname{ran}Y_1)\right).\end{aligned}$$

If $f \in \overline{\operatorname{ran} X} \ominus K$ then there exists $g_n + h_n \in K \oplus K^\perp$ such that $\left\|f - X(g_n+h_n)\right\| \to 0$ as $n \to \infty$. Since K is an invariant subspaces of X, it follows that $Xg_n \in K$. Since $f \in K^\perp$, $X_1 h_n = P_{K^\perp} X h_n$, $P_{K^\perp} X g_n = 0$ and $\|P_{K^\perp}\| = 1$, it follows that

$$\|f - X_1 h_n\| = \left\|P_{K^\perp}\left(f - X(g_n+h_n)\right)\right\| \leqslant \left\|f - X(g_n+h_n)\right\|.$$

Hence $\overline{\operatorname{ran} X} \ominus K \subset \overline{\operatorname{ran} X_1} = \operatorname{ran} X_1$. Since K is an invariant subspace of X and $X + Y = I$, it follows that K is an invariant subspace of Y. By the similar proof, $\overline{\operatorname{ran} Y} \ominus K \subset \operatorname{ran} Y_1$. Since $A + B$ is invertible, it follows that $\operatorname{ran} X = \operatorname{ran} A$ and $\operatorname{ran} Y = \operatorname{ran} B$. Hence

$$\begin{aligned}1 &> \cos\phi(\operatorname{ran}X_1,\operatorname{ran}Y_1) \\ &\geqslant \cos\phi\left(\overline{\operatorname{ran}X}\ominus K,\overline{\operatorname{ran}Y}\ominus K\right) \\ &= \cos\phi\left(\overline{\operatorname{ran}A}\ominus K,\overline{\operatorname{ran}B}\ominus K\right).\end{aligned}$$

Hence

$$0 < \phi(\operatorname{ran}X_1,\operatorname{ran}Y_1) \leqslant \phi\left(\overline{\operatorname{ran}A}\ominus K,\overline{\operatorname{ran}B}\ominus K\right),$$

and thus

$$\infty > \cot\phi(\operatorname{ran}X_1,\operatorname{ran}Y_1) \geqslant \cot\phi\left(\overline{\operatorname{ran}A}\ominus K,\overline{\operatorname{ran}B}\ominus K\right).$$

Since $F(\alpha,\beta,t)$ is a nondecreasing function of t, $(t \geqslant 0)$, it follows that

$$\begin{aligned}\left\|(\alpha A+\beta B)(A+B)^{-1}\right\| &\geqslant F\left(\alpha,\beta,\cot^2\phi(\operatorname{ran}X_1,\operatorname{ran}Y_1)\right) \\ &\geqslant F\left(\alpha,\beta,\cot^2\phi\left(\overline{\operatorname{ran}A}\ominus K,\overline{\operatorname{ran}B}\ominus K\right)\right).\end{aligned}$$

Theorem 2 is proved. ∎

By Theorem 1, if $K = \{0\}$ then the equality holds in the inequality in Theorem 2. There are many operators $A, B \in B(H)$ such that $\phi(\operatorname{ran} A, \operatorname{ran} B) = 0$ and

$$\phi\left(\overline{\operatorname{ran}A}\ominus K,\overline{\operatorname{ran}B}\ominus K\right) > 0.$$

If $AB = BA$ then $A(\gamma A + \delta B)^{-1} B = BA(\gamma A + \delta B)^{-1}$ and hence $K = \overline{\operatorname{ran} A} \cap \overline{\operatorname{ran} B}$ is an invariant subspace of $A(\gamma A + \delta B)^{-1}$. Even when the conditions in Theorem 2 do not

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hold, the similar result holds as the followings. If $A, B \in B(H)$ satisfy $A \neq 0$, $B \neq 0$, $A + B$ is right invertible, and M is the closed subspace of H satisfying

$$\operatorname{ran} A \cap \overline{\operatorname{ran} B} \subset M \subset \overline{\operatorname{ran} B},$$

then

$$\left\|(\alpha A + \beta B)(A + B)^{-1}\right\| \geqslant F\left(\alpha, \beta, \cot^2 \phi\right),$$

where $\phi = \phi\left(P_{M^\perp} A(A + B)^{-1}|_{M^\perp}, P_{M^\perp} B(A + B)^{-1}|_{M^\perp}\right) > 0$.

3. Norm formula when $A = I$ and $B^n = 0$

Let $K = \overline{\operatorname{ran} A} \cap \overline{\operatorname{ran} B}$. In Section 2, we have considered two operators $A, B \in B(H)$ satisfying $K \neq \overline{\operatorname{ran} A}$ and $K \neq \overline{\operatorname{ran} B}$. In this section we shall consider two operators A and B satisfying $K = \overline{\operatorname{ran} B}$. In general, suppose $A_0, B_0 \in B(H)$, $\operatorname{ran} B_0 \subset \operatorname{ran} A_0$ and $\ker B_0 \neq \{0\}$. Then there are many $C_0 \in B(H)$ such that $C_0(\operatorname{ran} B_0) \subset \ker B_0$. Let $A = C_0 A_0$ and $B = C_0 B_0$. Then

$$C_0(\alpha A_0 + \beta B_0) = \alpha A + \beta B.$$

Since $\operatorname{ran} B_0 \subset \operatorname{ran} A_0$, it follows that $\operatorname{ran} B \subset \operatorname{ran} A$ and hence $K = \overline{\operatorname{ran} B}$. Since $C_0(\operatorname{ran} B_0) \subset \ker B_0$, it follows that $B^2 = 0$. Hence, in many cases, the linear combination $\alpha A + \beta B$ for $A, B \in B(H)$ satisfying $B^2 = 0$ appears. We shall prove the following Proposition 1 using the Feldman, Krupnik and Markus formula [**1**]. The first author ([**5**]) proved it in the different way.

Proposition 1. *Let $B \in B(H)$ satisfy $B^2 = 0$. Let α and β be complex numbers. Then*

$$\|\alpha I + \beta B\| = \sqrt{\left|\frac{\beta}{2}\right|^2 \|B\|^2 + |\alpha|^2} + \left|\frac{\beta}{2}\right| \|B\|.$$

Proof: It is sufficient to prove that when $\alpha = -1$ and $\beta = 2$:

$$\|2B - I\| = \sqrt{\|B\|^2 + 1} + \|B\|.$$

Let P_B denote the orthogonal projection from H onto $\overline{\operatorname{ran} B}$. For $\varepsilon > 0$ let

$$P_\varepsilon = P_B - \frac{B}{\varepsilon}.$$

Then

$$P_\varepsilon^2 = P_B - \frac{P_B B}{\varepsilon} - \frac{B P_B}{\varepsilon} = P_B - \frac{B}{\varepsilon} = P_\varepsilon.$$

Let $Q_\varepsilon = I - P_\varepsilon$. Then

$$2B - I = 2\varepsilon(P_B - P_\varepsilon) - (P_\varepsilon + Q_\varepsilon) = 2\varepsilon P_B - (1 + 2\varepsilon)P_\varepsilon - Q_\varepsilon.$$

Since $\|\varepsilon P_\varepsilon\| = \|\varepsilon P_B - B\| \to \|B\|$ as $\varepsilon \to 0$, it follows from the Feldman, Krupnik and Markus formula [1] that

$$\begin{aligned}\|2B - I\| &= \lim_{\varepsilon\to 0}\left\|(1+2\varepsilon)P_\varepsilon + Q_\varepsilon\right\| \\ &= \lim_{\varepsilon\to 0} F\left(1+2\varepsilon, 1, \|P_\varepsilon\|^2 - 1\right) \\ &= \lim_{\varepsilon\to 0}\left(\sqrt{\varepsilon^2\left(\|P_\varepsilon\|^2 - 1\right) + (1+\varepsilon)^2} + \sqrt{\varepsilon^2\left(\|P_\varepsilon\|^2 - 1\right) + \varepsilon^2}\right) \\ &= \sqrt{\|B\|^2 + 1} + \|B\|.\end{aligned}$$

Proposition 1 is proved. ∎

Suppose A is invertible. Let $C = A^{-1}B$. If $A^{-1}(\operatorname{ran} B) \subset \ker B$ then $C^2 = 0$. In Proposition 1, we have considered $\|\alpha I + \beta C\|$. Then

$$\frac{\|\alpha I + \beta C\|}{\|A^{-1}\|} \leqslant \|\alpha A + \beta B\| \leqslant \|A\| \cdot \|\alpha I + \beta C\|.$$

For example, if A is invertible, $B^2 = 0$ and $AB = BA$ then $A^{-1}(\operatorname{ran} B) \subset \ker B$. If $\|A\| = \|A^{-1}\|$ then inequalities become equalities. Then A is a unitary operator.

PROPOSITION 2. *Let n be an integer satisfying $n \geqslant 2$. Let $B \in B(H)$ satisfy $B^n = 0$. Let α and β be complex numbers. Then*

$$\|\alpha I + \beta B\| \geqslant \sqrt{\left|\frac{\beta}{2}\right|^2 \left\|B|_{\overline{\operatorname{ran} B^{n-2}}}\right\| + |\alpha|^2} + \left|\frac{\beta}{2}\right| \left\|B|_{\overline{\operatorname{ran} B^{n-2}}}\right\|.$$

4. Norm formula using the angle ψ between A and B

In this section, we shall estimate the operator norm of $(\alpha A + \beta B)(\gamma A + \delta B)^{-1}$ from above in the case when $\psi(A, B) > 0$ where ψ is defined in Introduction. We do not assume the boundedness of two operators A and B on H. In the proof of Theorem 3 (1), we do not use the linearity of A and B. We do not use the Feldman, Krupnik and Markus formula [1] in this section.

DEFINITION 3. For $f, g \in H$, let

$$\rho(f,g) = \begin{cases} \dfrac{|\langle f, g\rangle|}{\|f\| \cdot \|g\|} & \text{if } f \neq 0 \text{ and } g \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\cos\phi(\operatorname{ran} A, \operatorname{ran} B) = \sup_{f,g\in H} \rho(Af, Bg) \geqslant \sup_{f\in H} \rho(Af, Bf) = \cos\psi(A, B).$$

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Lemma 2. *Let α and β be distinct complex numbers. Let x and ρ be real numbers satisfying $x \geqslant \max\bigl(|\alpha|, |\beta|\bigr)$ and $0 \leqslant \rho < 1$. The following assertions are equivalent:*

$$x \geqslant F\Bigl(\alpha, \beta, \frac{\rho^2}{1-\rho^2}\Bigr). \tag{1}$$

$$x^4 - \Bigl(\frac{\rho^2}{1-\rho^2}|\alpha-\beta|^2 + |\alpha|^2 + |\beta|^2\Bigr)x^2 + |\alpha\beta|^2 \geqslant 0. \tag{2}$$

$$\frac{|x^2 - \alpha\overline{\beta}|}{|\alpha-\beta|x} \geqslant \frac{1}{\sqrt{1-\rho^2}}. \tag{3}$$

$$\bigl(x^2 - |\alpha|^2\bigr)\bigl(x^2 - |\beta|^2\bigr) \geqslant \rho^2 \bigl|x^2 - \alpha\overline{\beta}\bigr|^2. \tag{4}$$

The equivalence holds for not only inequalities but also equalities.

Proof: (1) $\Leftrightarrow$ (2): Since $x \geqslant 0$, (1) is equivalent to

$$x^2 \geqslant \frac{|\alpha-\beta|^2}{2}\frac{\rho^2}{1-\rho^2} + \frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{\Bigl(\frac{|\alpha-\beta|^2}{2}\frac{\rho^2}{1-\rho^2} + \frac{|\alpha|^2+|\beta|^2}{2}\Bigr)^2 - |\alpha\beta|^2}.$$

Since $x \geqslant \max\bigl(|\alpha|, |\beta|\bigr)$, this is equivalent to (2).

(2) $\Rightarrow$ (3): By (2),

$$\bigl(x^2 - |\alpha|^2\bigr)\bigl(x^2 - |\beta|^2\bigr) \geqslant \frac{\rho^2}{1-\rho^2}|\alpha-\beta|^2 x^2.$$

Hence

$$\bigl|x^2 - \alpha\overline{\beta}\bigr|^2 - |\alpha-\beta|^2 x^2 \geqslant \frac{\rho^2}{1-\rho^2}|\alpha-\beta|^2 x^2.$$

This implies (3).

(3) $\Rightarrow$ (4): By (3),

$$\bigl|x^2 - \alpha\overline{\beta}\bigr|^2 - |\alpha-\beta|^2 x^2 \geqslant \rho^2 \bigl|x^2 - \alpha\overline{\beta}\bigr|^2.$$

This implies (4).

(4) $\Rightarrow$ (2): By (4),

$$\bigl(1-\rho^2\bigr)\bigl(x^2 - |\alpha|^2\bigr)\bigl(x^2 - |\beta|^2\bigr) \geqslant \rho^2|\alpha-\beta|^2 x^2.$$

This implies (2). Lemma 2 is proved. ∎

Lemma 3. *Let $\alpha, \beta, \gamma, \delta$ be complex numbers satisfying $\gamma\delta \neq 0$, and let f, g be nonzero elements in the Hilbert space H satisfying $\rho = \rho(f, g) < 1$. Then*

$$\frac{\|\alpha f + \beta g\|}{\|\gamma f + \delta g\|} \leqslant F\Bigl(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \frac{\rho^2}{1-\rho^2}\Bigr).$$

PROOF: It is sufficient to prove when $\alpha \neq \beta$ and $\gamma = \delta = 1$. Let $\rho = \rho(f, g)$, and let $x = F\left(\alpha, \beta, (\rho^2/1 - \rho^2)\right)$. By Lemma 2, $\left(x^2 - |\alpha|^2\right)\left(x^2 - |\beta|^2\right) = \rho^2|x^2 - \alpha\overline{\beta}|^2$. Hence

$$\left(x^2 - |\alpha|^2\right)\left(x^2 - |\beta|^2\right)\|f\|^2\|g\|^2 = |x^2 - \alpha\overline{\beta}|^2\left|\langle f, g\rangle\right|^2.$$

Hence

$$\begin{aligned}\left(x^2 - |\alpha|^2\right)\|f\|^2 + \left(x^2 - |\beta|^2\right)\|g\|^2 + 2\mathrm{Re}\left(\left(x^2 - \alpha\overline{\beta}\right)\langle f, g\rangle\right) \\ \geqslant 2\sqrt{x^2 - |\alpha|^2}\sqrt{x^2 - |\beta|^2}\|f\| \cdot \|g\| - 2|x^2 - \alpha\overline{\beta}| \cdot \left|\langle f, g\rangle\right| = 0.\end{aligned}$$

Therefore

$$\|\alpha f + \beta g\|^2 \leqslant x^2\|f + g\|^2.$$

Lemma 3 is proved. ∎

THEOREM 3. *Let A and B be nonzero linear operators on H satisfying $\psi(A, B) > 0$. Let $\alpha, \beta, \gamma, \delta$ be complex numbers such that $\gamma\delta \neq 0$. Then*

(1) *$\gamma A + \delta B \neq 0$ and*

$$\sup_{(\gamma A + \delta B)f \neq 0} \frac{\left\|(\alpha A + \beta B)f\right\|}{\left\|(\gamma A + \delta B)f\right\|} \leqslant F\left(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \cot^2\psi(A, B)\right).$$

(2) *If $\gamma A + \delta B$ is right invertible then*

$$\left\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\right\| \leqslant F\left(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \cot^2\psi(A, B)\right),$$

where $(\gamma A + \delta B)^{-1}$ denotes one of the right inverses to $\gamma A + \delta B$.

(3) *If $\gamma A + \delta B$ is left invertible then*

$$\left\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\right\| \leqslant \left\|(\gamma A + \delta B)(\gamma A + \delta B)^{-1}\right\| F\left(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \cot^2\psi(A, B)\right),$$

where $(\gamma A + \delta B)^{-1}$ denotes one of the left inverses to $\gamma A + \delta B$.

PROOF: It is sufficient to prove when $\gamma = \delta = 1$. We shall prove (1). Suppose $f \in H$ satisfies $Af \neq 0$, $Bf \neq 0$ and $\rho(Af, Bf) < 1$. By Lemma 3,

$$\frac{\left\|(\alpha A + \beta B)f\right\|}{\left\|(A + B)f\right\|} \leqslant F\left(\alpha, \beta, \frac{\rho(Af, Bf)^2}{1 - \rho(Af, Bf)^2}\right).$$

Since $y = F\left(\alpha, \beta, (x^2/1 - x^2)\right)$ is a nondecreasing function of x, $0 \leqslant x < 1$, and $y \geqslant \max\left(|\alpha|, |\beta|\right)$,

$$\begin{aligned}\sup_{(A+B)f \neq 0} \frac{\left\|(\alpha A + \beta B)f\right\|}{\left\|(A + B)f\right\|} &\leqslant \sup_{f \in H} F\left(\alpha, \beta, \frac{\rho(Af, Bf)^2}{1 - \rho(Af, Bf)^2}\right) \\ &= F\left(\alpha, \beta, \frac{\cos^2\psi(A, B)}{1 - \cos^2\psi(A, B)}\right) \\ &= F\left(\alpha, \beta, \cot^2\psi(A, B)\right).\end{aligned}$$

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Secondly we prove (2). For every $g \in H$, let $f = (A+B)^{-1}g$. Since $(A+B)^{-1}$ is the right inverse to $A+B$, it follows that

$$\sup_{g \neq 0} \frac{\left\|(\alpha A + \beta B)(A+B)^{-1}g\right\|}{\|g\|} \leqslant \sup_{(A+B)f \neq 0} \frac{\left\|(\alpha A + \beta B)f\right\|}{\left\|(A+B)f\right\|}.$$

By (1), this implies (2).

Finally we prove (3). Let $c = \left\|(A+B)(A+B)^{-1}\right\|$ and let $f = (A+B)^{-1}g$. Since $(A+B)^{-1}$ is the left inverse to $A+B$, it follows that

$$\begin{aligned}\sup_{g \neq 0} \frac{\left\|(\alpha A + \beta B)(A+B)^{-1}g\right\|}{\|g\|} &\leqslant c \sup_{(A+B)(A+B)^{-1}g \neq 0} \frac{\left\|(\alpha A + \beta B)(A+B)^{-1}g\right\|}{\left\|(A+B)(A+B)^{-1}g\right\|} \\ &\leqslant c \sup_{(A+B)f \neq 0} \frac{\left\|(\alpha A + \beta B)f\right\|}{\left\|(A+B)f\right\|}.\end{aligned}$$

By (1), this implies (3). Theorem 3 is proved. ∎

By Theorem 1, if $\psi(A,B) = \phi(\operatorname{ran} A, \operatorname{ran} B)$ then the equality holds in Theorem 3 (2). In many cases $\psi(A,B) = \phi(\operatorname{ran} A, \operatorname{ran} B)$. Let P be an analytic projection on the weighted L^2 space. Helson and Szegő [4] used the equivalence of $\psi(P, I-P) > 0$ and $\|P\| < \infty$.

Corollary 2. *Let A and B be linear operators on H. If $\|A+B\| < \infty$ and $\psi(A,B) > 0$ then $\|A\| < \infty$ and $\|B\| < \infty$.*

Proof: By Theorem 3 (1), if $\psi(A,B) > 0$ and $\|A+B\| < \infty$ then

$$\left\|(\alpha A + \beta B)f\right\| \leqslant F\Big(\alpha, \beta, \cot^2 \psi(A,B)\Big)\left\|(A+B)f\right\| \leqslant F\Big(\alpha, \beta, \cot^2 \psi(A,B)\Big)\|A+B\| \cdot \|f\|,$$

for all $f \in H$. Hence

$$\|\alpha A + \beta B\| \leqslant F\Big(\alpha, \beta, \cot^2 \psi(A,B)\Big)\|A+B\| < \infty,$$

for every complex numbers α and β. Corollary 2 is proved. ∎

Lemma 4. *Let A and B be nonzero operators on H satisfying $\psi(A,B) > 0$. Let $\alpha, \beta, \gamma, \delta$ be complex numbers such that $\gamma\delta \neq 0$. Then*

$$\begin{aligned}\sup_{t \in \mathbf{C}} \sup_{(\gamma t A + \delta B)f \neq 0} \frac{\left\|(\alpha t A + \beta B)f\right\|}{\left\|(\gamma t A + \delta B)f\right\|} &= \sup_{t \in \mathbf{C}} \sup_{(\gamma A + \delta t B)f \neq 0} \frac{\left\|(\alpha A + \beta t B)f\right\|}{\left\|(\gamma A + \delta t B)f\right\|} \\ &= F\Big(\frac{\alpha}{\gamma}, \frac{\beta}{\delta}, \cot^2 \psi(A,B)\Big).\end{aligned}$$

Proof: It is sufficient to prove that

$$\sup_{t \in \mathbf{C}} \sup_{(tA+B)f \neq 0} \frac{\left\|(\alpha t A + \beta B)f\right\|}{\left\|(tA+B)f\right\|} = F\left(\alpha, \beta, \cot^2 \psi(A,B)\right).$$

Let

$$c = \sup_{t\in\mathbf{C}} \sup_{(tA+B)f\neq 0} \frac{\left\|(\alpha t A + \beta B)f\right\|}{\left\|(tA+B)f\right\|}.$$

We shall prove that $c = F\big(\alpha,\beta,\cot^2\psi(A,B)\big)$. By Theorem 3 (1), $c \leqslant F\big(\alpha,\beta,\cot^2\psi(A,B)\big)$. Hence it is sufficient to prove that $c \geqslant F\big(\alpha,\beta,\cot^2\psi(A,B)\big)$. Since

$$c \geqslant \frac{\|\alpha t A f + \beta B f\|}{\|tAf + Bf\|}, \quad (t \in \mathbf{C}),$$

it follows that

$$|t|^2\big(c^2 - |\alpha|^2\big)\|Af\|^2 + \big(c^2 - |\beta|^2\big)\|Bf\|^2 + 2\mathrm{Re}\Big(t\big(c^2 - \alpha\overline{\beta}\big)\langle Af, Bf\rangle\Big) \geqslant 0,$$

for all $t \in \mathbf{C}$. Hence

$$\left|c^2 - \alpha\overline{\beta}\right|^2 \left|\langle Af, Bf\rangle\right|^2 \leqslant \big(c^2 - |\alpha|^2\big)\big(c^2 - |\beta|^2\big)\|Af\|^2\|Bf\|^2.$$

Hence

$$\rho(Af,Bf)^2 \left|c^2 - \alpha\overline{\beta}\right|^2 \leqslant \big(c^2 - |\alpha|^2\big)\big(c^2 - |\beta|^2\big).$$

By Lemma 2,

$$c \geqslant F\left(\alpha,\beta,\frac{\rho(Af,Bf)^2}{1-\rho(Af,Bf)^2}\right),$$

for all $f \in H$. Hence

$$c \geqslant \sup_{f\in H} F\left(\alpha,\beta,\frac{\rho(Af,Bf)^2}{1-\rho(Af,Bf)^2}\right).$$

Since $y = F\Big(\alpha,\beta,(x^2/1-x^2)\Big)$ is a nondecreasing function of x, $0 \leqslant x < 1$, it follows that

$$c \geqslant F\Big(\alpha,\beta,\frac{\cos^2\psi(A,B)}{1-\cos^2\psi(A,B)}\Big) = F\big(\alpha,\beta,\cot^2\psi(A,B)\big).$$

Lemma 4 is proved. ∎

Corollary 3. *Let A and B be nonzero linear operators on H satisfying $\psi(A,B) > 0$, $AB = BA = 0$ and $\gamma A + \delta B$ is right invertible. Let $\alpha,\beta,\gamma,\delta$ be complex numbers such that $\gamma\delta \neq 0$. Then*

$$\left\|(\alpha A + \beta B)(\gamma A + \delta B)^{-1}\right\| = F\Big(\frac{\alpha}{\gamma},\frac{\beta}{\delta},\cot^2\psi(A,B)\Big).$$

Proof: It is sufficient to prove when $\gamma = \delta = 1$. Suppose $A+B$ is right invertible. By Lemma 4, it is sufficient to prove the equality:

$$\sup_{(A+B)f\neq 0} \frac{\left\|(\alpha A + \beta B)f\right\|}{\left\|(A+B)f\right\|} = \sup_{t\in\mathbf{C}} \sup_{(tA+B)f\neq 0} \frac{\left\|(\alpha t A + \beta B)f\right\|}{\left\|(tA+B)f\right\|}.$$

Since $A+B$ is right invertible, it follows that $\operatorname{ran} A + \operatorname{ran} B = \operatorname{ran}(A+B) = H$. Since A and B are linear operators satisfying $AB = BA = 0$, it follows that

$$\begin{aligned}
\sup_{(A+B)f\neq 0} \frac{\left\|(\alpha A+\beta B)f\right\|}{\left\|(A+B)f\right\|} &\leqslant \sup_{t\in\mathbf{C}}\ \sup_{(tA+B)f\neq 0} \frac{\left\|(\alpha tA+\beta B)f\right\|}{\left\|(tA+B)f\right\|} \\
&\leqslant \sup_{Ag+Bh\neq 0} \frac{\|\alpha Ag+\beta Bh\|}{\|Ag+Bh\|} \\
&= \sup_{Ag+Bh\neq 0, g\in\operatorname{ran} A, h\in\operatorname{ran} B} \frac{\|\alpha Ag+\beta Bh\|}{\|Ag+Bh\|} \\
&= \sup_{(A+B)(g+h)\neq 0, g\in\operatorname{ran} A, h\in\operatorname{ran} B} \frac{\left\|(\alpha A+\beta B)(g+h)\right\|}{\left\|(A+B)(g+h)\right\|} \\
&= \sup_{(A+B)f\neq 0} \frac{\left\|(\alpha A+\beta B)f\right\|}{\left\|(A+B)f\right\|}.
\end{aligned}$$

Corollary 3 is proved. ∎

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