THE FREE EXACT CATEGORY ON A LEFT EXACT ONE

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Abstract

We give an explicit one step description of the free (Barr) exact category on a left exact one. As an application we give a new two step construction of the free abelian category on an additive one.

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Introduction

Following a suggestion of A. Joyal, we construct the free exact category $\mathbf{A}_{ex}$ on a left exact one $\mathbf{A}$ by adding as new objects the equivalence relations in $\mathbf{A}$ and as new arrows between the new objects suitable classes of "compatible maps"; in this way equivalence relations gain quotients in the new category, but the problem is that this process gives rise to new equivalence relations which do not necessarily have coequalizers. We will show that to avoid this obstruction it suffices to consider *pseudo* equivalence relations (see Section 1 below). The appropriate universal property of our construction is that, for each exact category $\mathbb{E}$, we have an equivalence of categories between the category of left exact functors from $\mathbf{A}$ to $\mathbb{E}$ and the category of exact functors from $\mathbf{A}_{ex}$ to $\mathbb{E}$. This amounts to showing that the construction of $\mathbf{A}_{ex}$ defines a left biadjoint to the inclusion of the 2-category $\mathcal{E}x$ of exact categories and exact functors into the 2-category $\mathcal{L}ex$ of left exact categories and left exact functors, which, moreover, make the doctrine $\mathbf{M}$ it induces on $\mathcal{L}ex$ into a "Kock-Zöberlein doctrine" in the terminology of [4], such that the $\mathbf{M}$-algebras are precisely the exact categories. As an application, by using

Tierney's theorem [2] that an additive exact category is abelian, we use the construction $\mathbf{A}_{\mathrm{ex}}$ to produce a new two step description of the free abelian category on an additive one, perhaps a bit less mysterious than that contained in [1].

1. Pseudo equivalence relations

An *exact category* in the sense of Barr [2] is a left exact one (that is, with finite limits) such that:
a) every morphism $f: A \to B$ has a regular epi-mono factorization;
b) regular epis are pullback stable;
c) every equivalence relation is effective (that is, is a kernel pair).
We recall that an *equivalence relation* on a category $\mathbf{A}$ is a jointly monic pair $\mathbf{r} = \langle r_0, r_1 \rangle: R \to X$ such that, for all objects U, it induces an equivalence relation on the set $\mathbf{A}(U, X)$. If $\mathbf{A}$ is left exact, this is equivalent to requiring that $\mathbf{r}: R \to X$ should satisfy the following conditions:
R) there exists a map $r: X \to R$ such that $rr_0 = 1 = rr_1$;
S) there exists a map $s: R \to R$ such that $sr_0 = r_1$ & $sr_1 = r_0$;
T) if $d_0, d_1: R * R \to R$ is the pullback of r_0 by r_1 (in particular $d_1r_0 = d_0r_1$), then there exists a map $t: R * R \to R$ such that $tr_0 = d_0r_0$ & $tr_1 = d_1r_1$.
A *pseudo* equivalence relation is a pair $\mathbf{r} = \langle r_0, r_1 \rangle: R \to X$ such that for each U the image of the map $\mathbf{A}(U, R) \to \mathbf{A}(U, X) \times \mathbf{A}(U, X)$ induced by composition is an equivalence relation on the set $\mathbf{A}(U, X)$. As for the equivalence relations, if $\mathbf{A}$ is a left exact category, the notion of pseudo equivalence relation (p.e.r) is equivalent to the requirements **R)**, **S)**, **T)**, with the only difference that the maps r, s, t are not necessarily unique anymore. To simplify the understanding let us introduce the following (abuse of) notations for general pseudo relations $\mathbf{r}: R \to X$: if $\langle x, x' \rangle: U \to X$ is a pair of U-elements of X, we write xRx' if there exists $k: U \to R$ such that $x = kr_0$ and $x' = kr_1$. By the previous definition, $\mathbf{r}: R \to X$ is a p.e.r. if and only if for each U and for all U-elements x, x', x'', we have: xRx, $xRx' \Rightarrow x'Rx$, xRx' and $x'Rx'' \Rightarrow xRx''$. Finally, let us remark that a category is exact if and only if it is pseudo exact, that is, condition c) in the definition is replaced by: c') the regular image of every p.e.r. is effective.
If $\mathbf{A}$ is any category, let us consider the new category $\mathbf{A}_{\mathrm{ex}}$ whose *objects* are the p.e.r. $\mathbf{r} = \langle r_0, r_1 \rangle: R \to X$, which we write as $(X; R, \mathbf{r})$, and whose *arrows* $_{\llcorner}f_{\lrcorner}: (X; R, \mathbf{r}) \to (Y; S, \mathbf{s})$ are equivalence classes of arrows $f: X \to Y$ "compatible with the relations on X and Y", that is, such that for all U-elements $x, x': U \to X$ we have $xfSx'f$ or, equivalently, such that there exists $f': R \to S$ with $r_0 f = f's_0$ and

$r_1 f = f's_1$; we consider equivalent two such arrows $f, g: X \to Y$ when, for all x: $U \to X$ we have $xfSxg$ or, equivalently, when there exists $\Sigma: X \to S$ such that $\Sigma s_0 = f$ and $\Sigma s_1 = g$.

Easily $\mathbf{A}_{ex}$ is a category, because the relation on maps described is a congruence, and there exists an obvious functor $\Gamma: \mathbf{A} \to \mathbf{A}_{ex}$ sending each object to the trivial equivalence relation on it.

In the next section we will show that *if $\mathbf{A}$ is a left exact category, then $\mathbf{A}_{ex}$ is its free exact completion*, that is, it is exact and it has the universal property stated in the introduction.

2. Exactness and universality

From now on let us suppose $\mathbf{A}$ is a left exact category. In this case can be useful to observe that if $(X; R, \mathbf{r})$ and $(Y; S, \mathbf{s})$ are two objects of $\mathbf{A}_{ex}$, then the product $\mathbf{r} \times \mathbf{s} = \langle r_0 \times s_0, r_1 \times s_1 \rangle: R \times S \to (X \times Y) \times (X \times Y)$ still is a p.e.r., and if $f: Z \to X$ is any morphism of $\mathbf{A}$, then the pullback of $\langle r_0, r_1 \rangle: R \to X \times X$ along $f \times f$ is also a p.e.r. on Z, which we denote $(Z; f^*R, \mathbf{r}^*)$; this can be easily proved by using the previous characterization of p.e.r.'s by U-elements. To show that $\mathbf{A}_{ex}$ is left exact, the naive idea is to handle a new object $(X; R, \mathbf{r})$ as if it were an old one X of $\mathbf{A}$, but with a new "equality" given by the p.e.r. So, for example, the pullback of two arrows $\llcorner f \lrcorner: (X; R, \mathbf{r}) \to (Z; T, \mathbf{t})$ and $\llcorner g \lrcorner: (Y; S, \mathbf{s}) \to (Z; T, \mathbf{t})$ will have underlying object $X \times_T Y$ defined as the pullback of $\langle t_0, t_1 \rangle: T \to Z \times Z$ along $f \times g$, and with the p.e.r. given by the pullback of $\langle r_0 \times s_0, r_1 \times s_1 \rangle: R \times S \to (X \times Y) \times (X \times Y)$ along $t^* \times t^*$, where $t^*: X \times_T Y \to X \times Y$ is the map defined by the first pullback. One can easily check that these definitions do not depend on the choice of f and g in their class and that, together with the definition of the terminal object in $\mathbf{A}_{ex}$ (as Γ of the terminal in $\mathbf{A}$), they yield the left exactness of $\mathbf{A}_{ex}$ as well that of Γ.

To show $\mathbf{A}_{ex}$ has the stated properties, we give the main definitions; details can be found in [3].

a) The regular epi-mono factorization of a morphism $\llcorner f \lrcorner: (X; R, \mathbf{r}) \to (Y; S, \mathbf{s})$ is
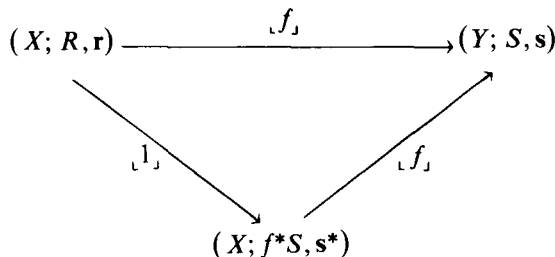
b) By the previous description of the regular epi-mono factorization, it follows that a morphism $\llcorner f \lrcorner: (X; R, \mathbf{r}) \to (Y; s, \mathbf{s})$ is a regular epi iff $\llcorner f \lrcorner: (X; fS, \mathbf{s}) \to (Y; S, \mathbf{s})$ is an iso, that is, if and only if there exist $s: Y \to X$ and $\Sigma: Y \to S$ such that $\Sigma s_0 = sf$ and $\Sigma s_1 = 1$. So, from the definition of pullbacks in $\mathbf{A}_{ex}$, condition b) for an exact category follows. It should be observed that a regular epi in $\mathbf{A}_{ex}$ is always isomorphic in $\mathbf{A}_{ex}$ to one with the same codomain and whose underlying morphism in $\mathbf{A}$ has a section.

c) A p.e.r. in $\mathbf{A}_{ex}$ is a pair $\langle \llcorner h_0 \lrcorner, \llcorner h_1 \lrcorner \rangle: (X; R, \mathbf{r}) \to (Y; S, \mathbf{s})$ satisfying:

R) there exists $\llcorner h \lrcorner: (Y; S, \mathbf{s}) \to (X; R, \mathbf{r})$ such that $\llcorner hh_0 \lrcorner = \llcorner 1 \lrcorner \& \llcorner hh_1 \lrcorner = \llcorner 1 \lrcorner$;

S) there exists $\llcorner k \lrcorner: (X; R, \mathbf{r}) \to (X; R, \mathbf{r})$ such that $\llcorner kh_0 \lrcorner = \llcorner h_1 \lrcorner \& \llcorner kh_1 \lrcorner = \llcorner h_0 \lrcorner$;

T) if we write $\llcorner d_0 \lrcorner, \llcorner d_1 \lrcorner: (X \times_S X; R \times_S R, \mathbf{k}) \to (X; R, \mathbf{r})$ for the pullback in $\mathbf{A}_{ex}$ of $\llcorner h_0 \lrcorner$ by $\llcorner h_1 \lrcorner$ (in particular $\llcorner d_1 h_0 \lrcorner = \llcorner d_0 h_1 \lrcorner$), there exists $\llcorner t \lrcorner: (X \times_S X; R \times_S R, \mathbf{k}) \to (X; R, \mathbf{r})$ such that $\llcorner d_0 h_0 \lrcorner = \llcorner th_0 \lrcorner$ and $\llcorner d_1 h_1 \lrcorner = \llcorner th_1 \lrcorner$. To construct the coequalizer of such a p.e.r., consider the pullbacks:
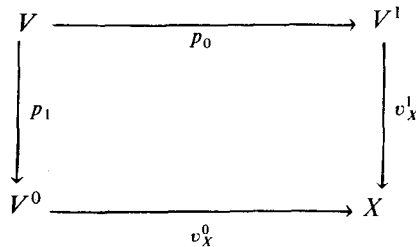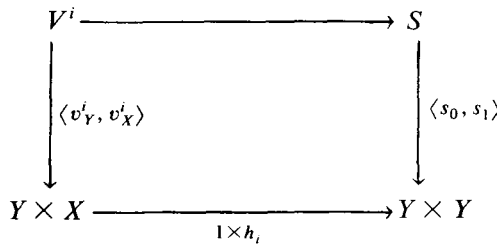
and define $v_0, v_1: V \to Y$ as $v_0 = p_1 v_Y^0$ and $v_1 = p_0 v_Y^1$. To show that $(Y; V, \mathbf{v})$ is an object of $\mathbf{A}_{ex}$ let us use elements; first, if $y, y': U \to Y$ are two U-elements of Y, then:

yVy' if and only if there exist $t_0: U \to V^0$ and $t_1: U \to V^1$ such that $t_0 v_X^0 = t_1 v_X^1$ and $t_0 v_Y^0 = y$, $t_1 v_Y^1 = y'$; if and only if there exist $x: U \to X$ and $k_0, k_1: U \to S$ such that $\langle y, xh_0 \rangle = k_0 \langle s_0, s_1 \rangle$ and $\langle y', xh_1 \rangle = k_1 \langle s_0, s_1 \rangle$; if and only if there exists $x: U \to X$ such that $ySxh_0$ and $y'Sxh_1$.

Now, to show conditions **R**), **S**) and **T**) for $(Y; V, v)$ is straightforward. For example, let us show **T**). Let yVy' and $y'Vy''$; then there exists $x: U \to X$ such that $ySxh_0$ and $y'Sxh_1$ and there exists $x': U \to X$ such that $y'Sx'h_0$ and $y''Sx'h_1$. Because of condition **T**) for $(Y; S, s)$, we have $x'h_0Sxh_1$; therefore there exists $\bar{x}$: $U \to : U \to X \times_S X$ such that $x = \bar{x}d_1$ and $x' = \bar{x}d_0$; because of condition **T**) for $\llcorner h_0 \lrcorner$, $\llcorner h_1 \lrcorner$, $x'' = \bar{x}t$ is such that $x''h_0Sxh_0$ and $x''h_1Sx'h_1$; so, $ySx''h_0$ and $y''Sx''h_1$, which means yVy''. In the same way, one can prove that $\llcorner 1 \lrcorner$: $(Y; S, s) \to (Y; V, v)$ is an arrow of $\mathbf{A}_{ex}$ and that $\ker(\llcorner 1 \lrcorner: (Y; S, s) \to (Y; V, v))$ is isomorphic to $\mathrm{Im}(\langle \llcorner h_0 \lrcorner, \llcorner h_1 \lrcorner \rangle: (X; R, r) \to (Y; S, s) \times (Y; S, s))$, that is, condition c').

To show the universal property of $\Gamma: \mathbf{A} \to \mathbf{A}_{ex}$ with respect to left exact functors F from $\mathbf{A}$ to any exact category $\mathbb{E}$, first observe that any object $(X; R, r)$ of $\mathbf{A}_{ex}$ appears as the coequalizer of Γr_0, $\Gamma r_1: \Gamma R \to \Gamma X$ and any morphism $\llcorner f \lrcorner$: $(X; R, r) \to (Y; S, s)$ appears in $\mathbf{A}_{ex}$ as the unique extension of $\llcorner f \lrcorner$ to the quotients of Γr_0, Γr_1 and Γs_0, Γs_1; the condition on the extension $\hat{F}: \mathbf{A}_{ex} \to \mathbb{E}$ to be exact forces us to define it on objects as

$$\hat{F}(X; R, r) = \mathrm{Coeq}(Fr_0, Fr_1: FR \to FX)$$

and similarly on morphisms.

Because F is left exact and because $\mathbb{E}$ is exact, $\hat{F}$ is well defined and by the previous observation it is unique up to isomorphisms, provided it is exact. It is easy to see that $\hat{F}$ preserves regular images. A bit more laborious is the verification of left exactness of $\hat{F}$. The following lemma is basically what we need to prove it.

LEMMA. $\hat{F}$ preserves mono.

PROOF. The proof reduces to observing that $\llcorner f \lrcorner: (X; R, r) \to (Y; S, s)$ is mono if and only if $\llcorner 1 \lrcorner: (X; R, r) \to (X; f^*R, r^*)$ is an iso in $\mathbf{A}_{ex}$, and to prove that in an exact category if $p: U \to U'$ and $q: V \to V'$ are regular epis and $f: U \to V$ is a map such that the pullback of $\ker(q) \to V \times V$ along $f \times f$ is $\ker(p) \to U \times U$, then the map $f': U' \to V'$ induced by f is mono.

The following lemma, whose proof should now be easy, is basically what we need to prove that the endofunctor on $\mathcal{L}ex$ defined by the construction $\mathbf{A}_{ex}$ is in fact a Kock-Zöberlein doctrine (see [4]):

LEMMA. *Let* **A** *be a left exact category and let* Γ: **A** $\to$ **A**$_{ex}$ *be the left exact embedding previously defined. Then,* **A** *is an exact category if and only if* Γ *has a left exact left adjoint.*

3. The free abelian category

From the construction of the last section, it is clear that if **A** is a left exact and *additive* category, then the exact completion **A**$_{ex}$ still is additive; so, by Tierney's theorem quoted in the introduction, **A**$_{ex}$ is abelian and, in fact, **A**$_{ex}$ is the free abelian category on the additive and left exact (= with kernels) category **A**. The only thing to check is that the equivalence relation between the morphisms is a congruence with respect to the additive structure on **A**, so that the additive structure on **A**$_{ex}$ can be defined naturally as $_\llcorner f_\lrcorner + _\llcorner g_\lrcorner = _\llcorner f + g_\lrcorner$ and everything else works. Now, if we want to construct the free abelian category on an additive one, all we need to do is to freely add kernels to an additive category and then to apply the previous construction of the exact completion.

The free additive category with kernels **A**$_{ker}$ on an additive category **A** can be constructed as follows: *objects* are arrows α: $X \to X'$ of **A**, which we denote by (X, X', α): *arrows* $_\llcorner f_\lrcorner$: $(X, X', \alpha) \to (Y, Y', \beta)$ are equivalence classes of arrows f: $X \to Y$ of **A** such that there exists f': $X' \to Y'$ with $f\beta = \alpha f'$; we consider equivalent two such arrows f, g: $X \to Y$ when "they have the same restriction to the kernels", that is when there exists Σ: $X' \to Y$ such that $\alpha\Sigma = f - g$. the relation between morphisms being homotopy, **A**$_{ker}$ is an additive category. One can easily show that the kernel of a morphism $_\llcorner f_\lrcorner$: $(X, X', \alpha) \to (Y, Y', \beta)$ can be defined as 1: $(X, Y \times X', \langle f, \alpha \rangle) \to (X, X', \alpha)$. Thus **A**$_{ker}$ is an additive category with kernels.

There exists an obvious additive functor Γ: **A** $\to$ **A**$_{ker}$ sending each object X of **A** in 0: $X \to X$, which gives rise to an equivalence of categories between the category of additive functors from **A** to **B** (**B** being an additive category with kernels) and the category of additive kernel preserving functors from **A**$_{ker}$ to **B**. For the proof it suffices to observe that each object (X, X', α) of **A**$_{ker}$ is isomorphic to the kernel $(X, X', \langle \alpha, 0 \rangle)$ of $\Gamma\alpha$: $\Gamma X \to \Gamma X'$, and that each arrow $_\llcorner f_\lrcorner$: $(X, X', \alpha) \to (Y, Y', \beta)$ appears in **A**$_{ker}$ as the unique restriction to ker($\Gamma\alpha$) and ker($\Gamma\beta$). This observation forces us to define the extension $\hat{F}$: **A**$_{ker} \to$ **B** of an additive functor F from **A** to an additive category with kernels **B**, as $\hat{F}(X, X', \alpha)$ = ker($F\alpha$) and similarly on morphisms. With a little effort one can check that $\hat{F}$ is additive and preserves kernels. So, by starting with an additive category **A**, we can describe the free abelian on it as **A**$_{ab}$ = (**A**$_{ker}$)$_{ex}$, which therefore must be equivalent to that constructed in [1].

References

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