Can. J. Math. Vol. **48** (6), 1996 pp. 1196–1209

A CONSTRUCTIVE BRAUER-WITT THEOREM FOR CERTAIN SOLVABLE GROUPS

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ABSTRACT. Division algebras occurring in simple components of group algebras of finite groups over algebraic number fields are studied. First, well-known restrictions are presented for the structure of a group that arises once no further Clifford Theory reductions are possible. For groups with these properties, a character-theoretic condition is given that forces the p-part of the division algebra part of this simple component to be generated by a predetermined p-quasi-elementary subgroup of the group, for any prime integer p. This is effectively a constructive Brauer-Witt Theorem for groups satisfying this condition. It is then shown that it is possible to constructively compute the Schur index of a simple component of the group algebra of a finite nilpotent-by-abelian group using the above reduction and an algorithm for computing Schur indices of simple algebras generated by finite metabelian groups.

Let G be a finite group, $\mathbf{k}$ an algebraic number field, and let $\mathbf{k}G$ be the group algebra of the group G over $\mathbf{k}$. By the Artin-Wedderburn Theorem [CR, 3.28], we know that $\mathbf{k}G$ is semisimple. A simple component of $\mathbf{k}G$—one of its non-zero two-sided ideals—is always a principal ideal of the form $\mathbf{k}Ge$, where e is a centrally primitive idempotent of $\mathbf{k}G$. Being simple, $\mathbf{k}Ge$ is isomorphic to a matrix ring over a division algebra D finite-dimensional over $\mathbf{k}$. The square root of the dimension of D over its center is known as the *Schur index* of D over $\mathbf{k}$.

A standard result for computing the Schur index in the above situation in the case of a general finite group is the Brauer-Witt Theorem [W] (see also [Y, page 31]), stated in Section 2 of this paper. For each prime integer p, this theorem indicates the existence of a p-quasi-elementary section of G—see Definition 2 of this paper—that determines the p-part of the Schur index related to a given simple component of $\mathbb{Q}G$. (Recently, Schmid [Sch] has extended this theorem by identifying precise types of p-quasi-elementary groups that are the minimal groups one can reduce to using this theorem.) Complete descriptions of methods for the computation of Schur indices resulting from p-quasi-elementary groups have appeared in [Y], [L], and [Her], the last using an approach based on a paper of Janusz [J]. The methods used for these Schur index calculations rely heavily on number theoretic information because of the local norm calculations required for computing the index. However, the Brauer-Witt Theorem does not lend itself to an algorithmic method

Some of the results of this article were a part of the author's Ph. D. Thesis at the University of Alberta, Edmonton, Canada, Fall 1995.

The author would like to express his thanks to the referee, whose suggestions were incorporated to improve the final presentation.

Received by the editors July 5, 1995.

AMS subject classification: Primary: 20C15; Secondary: 16S34.

of determining sections of G suitable for computing the Schur index, so the methods available for computing Schur indices for general groups are not very practical.

There are several well-known theorems that indicate restrictions on the values of these Schur indices [I, Chap. 10]. Occasionally, group structure alone results in restrictions. The most notable of these restrictions is Roquette's Theorem that indicates Schur indices are at most 2 for nilpotent groups [Roq]. In addition to Roquette's Theorem, constructive computations of Schur indices that result from restrictions on the class of the finite group have appeared previously. Fontaine has demonstrated that Schur indices in the case of finite supersolvable groups can always be constructively computed using the Clifford theory reductions described in Section 1 of this paper [F, Prop. 3.4]. Janusz attained a constructive reduction for the Schur index question in the case where G has a normal subgroup N for which G/N is abelian using methods that involve projective representations [J2].

The present paper was motivated by the paper of Shirvani [Sh] which attained structural invariants for the metabelian case, and the author's work in [Her]. The main results of this paper are the following:

(1) For finite solvable groups G, there is a character-theoretic condition such that for any faithful irreducible character of G satisfying this condition, the Schur index of the simple component of $\mathbf{k}G$ corresponding to this character can be constructively computed using pre-determined subgroups of G (Theorem 4).

(2) The Schur indices of simple components of the group algebras of nilpotent-by-abelian groups over algebraic number fields can be constructively computed from a knowledge of the multiplication table of the group, using a reduction based on the above result (Section 3).

In order to deal constructively with simple components of $\mathbf{k}G$, we develop the following somewhat standard notation, based on based on [I] and [Y]. Let $\mathrm{Irr}(G)$ denote the set of irreducible complex characters of G. For any $\chi \in \mathrm{Irr}(G)$, let

$$e_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(1)\chi(g^{-1})g$$

denote the centrally primitive idempotent of $\mathbb{C}G$ determined by χ. It is well known that the simple component $\mathbb{C}Ge_\chi$ is a G-module affording the character $\chi(1)\chi$ [Y, Chapter 1]. For any subfield $\mathbf{k}$ of $\mathbb{C}$, let

$$a_\chi = \sum_{\sigma \in \mathfrak{G}} e_\chi^\sigma,$$

where $\mathfrak{G} = \mathrm{Gal}(\mathbf{k}(\chi)/\mathbf{k})$. Then a_χ is a centrally primitive idempotent of $\mathbf{k}G$, and the simple component $\mathbf{k}Ga_\chi$ of $\mathbf{k}G$ affords the character $\chi(1)\sum_{\sigma\in\mathfrak{G}}\chi^\sigma$ as a G-module.

$\mathbf{k}Ga_\chi$ is isomorphic to a ring of $n \times n$ matrices over a division algebra D, for some positive integer n. This division algebra D is finite dimensional over its center $\mathbf{k}(\chi)$, the field of character values of χ over $\mathbf{k}$ [I, Exercise 9.15]. Furthermore, the integer $m_\mathbf{k}(\chi)$ for which

$$|D: \mathbf{k}(\chi)| = m_\mathbf{k}(\chi)^2$$

is the Schur index of χ (or D) over $\mathbf{k}$ [CR, page 585]. In what follows, the goal is to find a constructive means of computing the Schur index $m_{\mathbf{k}}(\chi)$ that works for as large a class of groups as possible.

1. **Clifford Reductions.** Assume now that χ is a faithful character of the finite group G. Clifford's Theorem [I,6.2] implies that whenever $N \lhd G$ and the subalgebra $\mathbf{k}Ne_\chi$ of $\mathbf{k}Ge_\chi$ is not simple, then

$$\mathbf{k}Ge_\chi \cong (\mathbf{k}Ca_\lambda e_\chi)^{k\times k},$$

where C is the centralizer in the group G of a centrally primitive idempotent a_λ of $\mathbf{k}Ne_\chi$, and $k = |G : C|$ [P, 6.1.7]. The idempotent a_λ is the central idempotent of $\mathbf{k}N$ determined by some fixed irreducible constituent λ of χ_N. Furthermore, it follows from [I, 6.11] and the fact that C always contains the stabilizer of the character λ that $\chi = \psi^G$, for some irreducible character ψ of C lying over λ, and so we even know that $e_\chi = a_\psi$. Thus the division algebra part of $\mathbf{k}Ce_\chi$ is isomorphic to that of $\mathbf{k}Ge_\chi$, and so we might as well replace G by C and χ by ψ. In order to maintain our original assumption that χ is faithful, we can further replace G by $C/(\ker\psi)$.

This reduction of the Schur index problem to one concerning a smaller group is constructive, because the subgroup $C = C_G(a_\lambda)$ can be obtained (although with possibly quite a bit of work) from the multiplication table of G and from some familiarity with $\mathbf{k}$. One first computes the character tables of G and N and then uses the formula to write down the idempotents e_χ and all of the e_λ, $\lambda \in \mathrm{Irr}(N)$. Once one has found a λ for which $e_\chi e_\lambda$ is non-zero, one can use the character values of λ and the field $\mathbf{k}$ to determine a_λ. Then one can determine C by finding the centralizer in G of a_λ. Finally, ψ is obtained by using the character table of C to find an idempotent e_ψ such that $e_\chi e_\psi \neq 0$, $e_\lambda e_\psi \neq 0$, and $\chi(1) = |G : C|\psi(1)$.

Only so many non-trivial Clifford reductions of the above type are possible, of course, and so eventually one is reduced to the case where $\mathbf{k}Ne_\chi$ is a simple algebra, for every normal subgroup N of G. We will refer to a group that has a faithful character for which $\mathbf{k}Ga_\chi$ has this property as a *Clifford reduced* group over $\mathbf{k}$. The structure of Clifford reduced groups can be partly inferred from the structure of finite groups with the property that every normal abelian subgroup is cyclic, because Clifford reduced groups automatically have this property. Very precise structural information is available for these groups.

PROPOSITION. *[Hup, III.13.10] Suppose that G is a finite group such that every normal abelian subgroup of G is cyclic. Then $F(G)$ has a characteristic subgroup F of index at most 2. This subgroup F is the central product of a cyclic group $U = Z(F)$ with a group E that is the direct product of extraspecial p-groups of exponent p or 4, at most one extraspecial p-group for each prime integer p dividing the order of F. In particular, F/U is elementary abelian. Always, we have $F = C_{F(G)}\Big(\Phi\big(F(G)\big)\Big)$.*

If we further suppose that G is solvable, then further restrictions can be obtained.

LEMMA. *Assume that the finite group G is solvable, and that every normal abelian subgroup of G is cyclic. Let $F = C_{F(G)}\Big(\Phi\big(F(G)\big)\Big)$ be the characteristic subgroup of G given in the above. Then $Z(F) = C_G(F)$.*

PROOF. Let $U = Z(F)$. If $F = F(G)$, then since $C_G\big(F(G)\big) \subseteq F(G)$ when G is solvable [Sc, 7.4.7], we must have

$$U = C_G(F) \cap F = C_G(F),$$

and we are done.

If F has index 2 in $F(G)$, then note that the Fitting subgroup of $C_G(F)$ must be $F \cap C_G(F) = U$, because the Fitting subgroup of $C_G(F)$ is a normal nilpotent subgroup of G and hence lies inside F.

Suppose $C_G(F)$ properly contains U. Let X/U be a chief section of G with $X \subseteq C_G(F)$. Because G is solvable, X/U is a non-trivial p-group, for some prime p. Since U is the Fitting subgroup of $C_G(F)$, U is also the Fitting subgroup of X, and so X cannot centralize U. This contradicts the assumption that X centralizes F, and so we conclude that $U = C_G(F)$. ■

The above lemma allows one to come to certain conclusions about the structure of certain sections of the group G.

PROPOSITION. *Let G be a solvable group, and suppose that every normal abelian subgroup of G is cyclic. Let $F = C_{F(G)}\Big(\Phi\big(F(G)\big)\Big)$. Then if we let $U = Z(F)$ and $C = C_G(U)$, we have that G/C is abelian, and C/F is isomorphic to a solvable subgroup of the direct product of finite symplectic groups of the form* $\mathrm{Sp}(2n, p)$, *for any prime integer p and power n such that $|F : U|$ is divisible by p^{2n}.*

2. A Condition on Characters.

In this section, we assume that G is a finite solvable group that is Clifford reduced over an algebraic number field $\mathbf{k}$ with respect to a faithful irreducible character χ. Our goal is to find conditions which guarantee that a further constructive reduction of the Schur index problem is possible.

Fix a prime integer p throughout. The following definition will make it easier to state our results.

DEFINITION 1. Let χ be a faithful irreducible character of a finite group G, and let $\mathbf{k}$ be an algebraic number field for which $\mathbf{k} = \mathbf{k}(\chi)$. Let $e = \exp(G)$. The *p'-splitting field of $\mathbf{k}Ge_\chi$ over $\mathbf{k}$* is the unique subextension K of $\mathbf{k}(\zeta_e)$ containing $\mathbf{k}$ such that

$$[\mathbf{k}(\zeta_e) : K] \text{ is a power of } p, \text{ and}$$
$$[K : \mathbf{k}] \text{ is relatively prime to } p.$$

Because cyclotomic extensions are always abelian Galois extensions when the base field has characteristic zero, it follows that the K defined above is the field fixed by the unique

Sylow p-subgroup of $\mathrm{Gal}(\mathbf{k}(\zeta_e)/\mathbf{k})$, and so K is unique. The property of this K that we need is that the Schur index of the division algebra part of

$$KGe_\chi \cong K \otimes_{\mathbf{k}} \mathbf{k}Ge_\chi$$

will be precisely the p-part of $m_{\mathbf{k}}(\chi)$.

In order to state the Brauer-Witt Theorem, we need the following definition.

DEFINITION 2. A finite group is called *p-quasi-elementary* if it is isomorphic to the split extension of a cyclic group of order relatively prime to p by a finite p-group.

THEOREM (BRAUER-WITT). *Let χ be an irreducible character of a finite group G. Let $\mathbf{k}$ be an algebraic number field satisfying $\mathbf{k}(\chi) = \mathbf{k}$. Let K be the p'-splitting field for $\mathbf{k}Ge_\chi$ over $\mathbf{k}$.*

Then there exists a p-quasi-elementary section H of G and a character $\xi \in \mathrm{Irr}(H)$ such that

$$m_K(\xi) = \big(m_{\mathbf{k}}(\chi)\big)_p.$$

Proofs of the Brauer-Witt Theorem are based on Brauer's Induction Theorem [CR, 15.9], which is used to establish the existence of the section H of the correct type and the desired character ξ [Y, Chapter 3]. This approach does not lead by itself to an algorithmic method of finding such a section H from a knowledge of the subgroup structure of G, and thus has limited practical applications. In what follows, we will show that the Brauer-Witt Theorem can be made constructive when χ satisfies a certain character-theoretic condition.

Assuming that the solvable group G is Clifford reduced over $\mathbf{k}$ with respect to a faithful irreducible character χ, let K be the p'-splitting field for $\mathbf{k}Ge_\chi$ over $\mathbf{k}$. Since the p-part of the Schur index of $\mathbf{k}Ge_\chi$ is exactly the Schur index of KGe_χ, we may replace $\mathbf{k}$ by K.

The first step is to re-do the Clifford Theory reductions of the previous chapter with respect to the new field K. Assume we have completed this process, so that for every normal subgroup N of G, KNe_χ is a simple algebra. In particular, every normal abelian subgroup of G is cyclic, so G has normal subgroups F, U, and C satisfying the following conditions of the previous section:

(1) F is a characteristic nilpotent subgroup of G having index at most 2 in $F(G)$;
(2) $U = Z(F)$ is cyclic, and F/U is elementary abelian;
(3) F is the central product of a group E with U, where E is a direct product of extraspecial q-groups having exponent q (or 4 when q is 2), for some prime integers q; and
(4) $C = C_G(U)$.

We now establish various character identities for irreducible characters of the above subgroups of G.

PROPOSITION 3. *Let K be the p'-splitting field for the simple algebra $\mathbf{k}Ge_\chi$. Suppose that KGe_χ has been fully reduced using Clifford theory, and let F, U, and C be the subgroups of G defined above. Let S be a transversal of C in G. Then*

(i) $\chi_U = k\sum_{s\in S}\lambda^s$, *for some faithful irreducible character λ of U and some integer $k > 0$;*

(ii) $G/C \cong \mathrm{Gal}\big(K(\lambda)/K\big)$*;*

(iii) $\chi = \psi^G$, *for some $\psi \in \mathrm{Irr}(C)$;*

(iv) $\chi_F = d\sum_{s\in S}\varphi^s$, *for some faithful irreducible character φ of F, and some integer $d > 0$.*

(v) *φ and ψ may be chosen so that $\varphi_U = f\lambda$, with $f^2 = [F : U]$, and $\psi_F = d\varphi$;*

(vi) $K(\lambda) = K(\varphi) = K(\psi) = K(\zeta_u)$, *where $u = |U|$.*

PROOF. Since $U \lhd G$, the algebra KUe_χ is simple under our assumptions. Thus

$$\chi_U = k\sum_{\sigma\in\mathcal{G}}\lambda^\sigma,$$

where λ is some irreducible character of U, $\mathcal{G} = \mathrm{Gal}\big(K(\lambda)/K\big)$, and k is some positive integer. Because all of the characters λ^σ lying under χ are Galois conjugate, all of their kernels are the same. Thus

$$\ker\lambda = \bigcap_{\sigma\in\mathcal{G}}\ker\lambda^\sigma = \ker\chi_U = U\cap\ker\chi = U\cap 1 = 1,$$

so λ is a faithful irreducible character of U. Since λ is a faithful linear character, the stabilizer of λ in G must exactly be the centralizer in G of the cyclic group U; namely, C. The character-theoretic version of Clifford's Theorem gives the decomposition

$$\chi_U = k\sum_{s\in S}\lambda^s,$$

where S is a transversal of C in G. This proves (i). (iii) also follows because χ is always induced from the stabilizer in this situation [I, 6.11]. Mapping G onto permutations on the set $\{\lambda^\sigma : \sigma\in\mathcal{G}\}$ gives a natural isomorphism of G/C with $\mathcal{G}$, namely $gC \mapsto \sigma(g)$, where $\sigma(g)$ is defined by $\lambda^g = \lambda^{\sigma(g)}$. This proves (ii).

Now, $\chi = \psi^G$, for some $\psi \in \mathrm{Irr}(C)$ such that $\psi_U = k\lambda$. This forces $K(\lambda) \subseteq K(\psi)$. If $K(\lambda) \neq K(\psi)$, then there is a galois automorphism fixing λ but not ψ, and so there is more than one ψ lying over λ and inducing χ. This would contradict [I, 6.11(c)]. (This method originates in [C, Lemma 1.1].) Thus $K(\lambda) = K(\psi)$.

As in the case of U, $F \lhd G$ implies that

$$\chi_F = d\sum_{\tau\in\mathcal{H}}\varphi^\tau,$$

where φ is some irreducible character of F, $\mathcal{H} = \mathrm{Gal}\big(K(\varphi)/K\big)$, and d is some positive integer. As above, we can show that φ is faithful. Because F is nilpotent of class 2, it follows from [I, 2.31] that any faithful character of F vanishes off $Z(F) = U$. Thus we can choose $\varphi \in \mathrm{Irr}(F)$ so that $\varphi_U = f\lambda$, with $f^2 = [F:U]$. Since φ must vanish off U, we have that $K(\varphi) = K(\lambda)$, and the stabilizer of φ is exactly C. This proves (vi), and (iv) follows as in the above with λ replaced by φ.

As $\psi_F = d\varphi^s$, for some $s \in S$, the fact that $(\psi_F)_U$ is a multiple of λ implies that $\psi_F = d\varphi$. Statement (v) follows. ∎

One final remark is needed before the proof of the main result of this section. From our assumption concerning K, we see that by part (ii) of the above proposition, G/C must be a p-group. Thus for any Sylow p-subgroup of G, we have $G = CP$. Since non-identity elements of a transversal of C in G must lie outside C, we can choose the transversal S such that $S \subseteq P$, for any previously chosen Sylow p-subgroup P.

THEOREM 4. *Let K be the p'-splitting field for the simple algebra* $\mathbf{k}Ge_\chi$. *Suppose that KGe_χ has been fully reduced using Clifford theory, and let F, U, and C be the subgroups of G defined above. Let P be a Sylow p-subgroup of G for the fixed prime p. Suppose that*

$$H = UP = U_{p'} \rtimes P,$$

where $U_{p'}$ is the p-complement of the cyclic group U.

Then

(i) H is a p-quasi-elementary subgroup of G;

(ii) If $\psi_F = d\varphi$, with $(d,p) = 1$, then there exists a $\xi \in \mathrm{Irr}(H)$ such that $(\chi_H, \xi) \not\equiv 0 \bmod p$.

(iii) If $\psi_F = \varphi$, then there exists a $\xi \in \mathrm{Irr}(H)$ such that $(\chi_H, \xi) \not\equiv 0 \bmod p$ and $K(\xi) = K$.

PROOF. (i) is clear, since U is a cyclic normal subgroup of G.

To prove (ii), we let Q be the normal subgroup $H \cap F$ of G, and we compare expressions for the characters $(\chi_F)_Q$ and $(\chi_H)_Q$. Suppose that we have chosen a transversal S of C in G so that $S \subseteq P$. Then we have that

$$\begin{aligned}(\chi_F)_Q &= d\sum_{s\in S}(\varphi^s)_Q \\ &= \frac{df}{\sqrt{[Q:U]}}\sum_{s\in S}\theta^s,\end{aligned}$$

for some $\theta \in \mathrm{Irr}(Q)$ that is fully ramified with respect to U and lies over λ. On the other hand, suppose

$$\chi_H = \sum_{i=1}^{r} c_i \xi_i,$$

for some positive integers $c_1, \ldots, c_r$, and irreducible characters $\xi_1, \ldots, \xi_r$ of Q. Since the Galois conjugates of θ are the only irreducible characters of Q lying under χ, they are also the only irreducible characters of Q that lie under any of the characters $\xi_1, \ldots, \xi_r$. Since $\{\theta^s : s \in S\}$ is exactly the set of Galois conjugates of θ, and each $s \in H$, we conclude that for each $i = 1, \ldots, r$, we have

$$(\xi_i)_Q = p^{k_i}\sum_{s\in S}\theta^s,$$

for some integer $k_i \geq 0$. Therefore

$$\begin{aligned}(\chi_H)_Q &= \sum_{i=1}^{r} c_i(\xi_i)_Q, \\ &= \sum_{i=1}^{r} c_i(p^{k_i} \sum_{s\in S} \theta^s) \\ &= \sum_{i=1}^{r} c_i p^{k_i}(\sum_{s\in S} \theta^s).\end{aligned}$$

Comparing the two expressions for $\chi_Q = (\chi_F)_Q = (\chi_H)_Q$, we conclude that

$$\sum_{i=1}^{r} c_i p^{k_i} = \frac{df}{\sqrt{[Q:U]}}.$$

By our assumption that d is relatively prime to p, the second number has to be relatively prime to p. This implies that at least one of the numbers $c_i p^{k_i}$ in the sum on the left has to be relatively prime to p. Without loss of generality, assume $c_1 p^{k_1}$ is relatively prime to p. Then we must have $k_1 = 0$ and $c_1 = (\xi_1, \chi_H) \not\equiv 0 \bmod p$. This $\xi = \xi_1$ is the irreducible character of H required for (ii). (Note that we have also shown that for this character ξ,

$$\xi_Q = \sum_{s\in S} \theta^s.)$$

To prove (iii), note that the assumption that $\psi_F = \varphi$ is equivalent to

$$KFa_\varphi \cong KCa_\psi,$$

since $K(\psi) \cong K(\varphi)$ and both algebras have dimension $\psi(1)^2 = \varphi(1)^2$ over their centers. Our assumption on G then gives $KFe_\chi = KCe_\chi$, and so

$$KCe_\chi \cap KHe_\chi = KFe_\chi \cap KHe_\chi.$$

The subalgebra KHe_χ need not be simple because H need not be normal in G. However, we know from $\chi_H = \sum_i c_i\xi_i$ that

$$KHe_\chi = \oplus_i KHa_{\xi_i}e_\chi,$$

because KHe_χ is a homomorphic image of KH as an algebra. (The simple components of KH that occur in this sum will be exactly those whose associated irreducible characters are constituents of χ_H.) Let $\theta \in \mathrm{Irr}(Q)$ be as above. Since θ vanishes off U, it follows that $I_H(\theta) = H \cap C$. Let $R = H \cap C$, and note that FR/F is a Sylow p-subgroup of C/F.

Now let $\xi \in \mathrm{Irr}(H)$ be the character found in the proof of part (ii). From the remark that

$$\xi_H = \sum_{s\in S} \theta^s,$$

it follows from $R = I_H(\theta)$ that there is a character $\eta \in \mathrm{Irr}(R)$ such that $\xi = \eta^H$ and $\eta_Q = \theta$.

Consider the subalgebra KRe_χ. We have that $KRe_\chi \subseteq KCe_\chi$, so there are algebra inclusions

$$KQe_\chi \subseteq KRe_\chi \subseteq KFe_\chi.$$

Since $\varphi_Q = f\theta$ with $f = \sqrt{|F:Q|}$, we know that

$$[KFe_\chi : KQe_\chi] = |F:Q|$$

is relatively prime to p. Thus the index of KQe_χ in the subalgebra KRe_χ of KFe_χ has to be relatively prime to p. But R/Q is a p-group, and K is the p'-splitting field, so $[KRe_\chi : KQe_\chi]$ must be a power of p. This forces $KRe_\chi = KQe_\chi$.

Note that each of the characters ξ_i occurring in the above sum is faithful on Q by the formula $(\xi_i)_Q = p^{k_i} \sum_S \theta^s$. Since KQe_χ is a simple subalgebra of KHe_χ, this means that each of the maps $KQe_\chi \longrightarrow KQa_{\xi_i}e_\chi$ has to be an isomorphism. (Hence we can interpret the inclusion

$$KQe_\chi \longrightarrow \oplus_i KHa_{\xi_i}e_\chi$$

as a diagonal embedding.) Since $KRe_\chi = KQe_\chi$, we have that $KRa_\xi e_\chi = KQa_\xi e_\chi$ is isomorphic to the simple component KRa_η of KR. The center of KRa_η is an isomorphic copy of $K(\eta)$, and so since $KQe_\chi \cong KQa_\xi e_\chi \cong KRa_\eta$, we conclude that $K(\theta) = K(\eta)$.

Finally, note that since ξ is induced from $I_H(\theta) = R$, ξ vanishes off R. Since H/R acts as galois automorphisms on the subfield $K(\theta) = KUa_\xi e_\chi$, H/R acts as galois automorphisms on $K(\eta) = Z(KRa_\eta e_\chi)$. Thus $K(\xi) \cong Z(KHa_\xi e_\chi)$ is the subfield of $K(\theta)$ fixed by H/R. Of course, this is exactly K, as required. ∎

COROLLARY 4.1. *Under the conditions of Theorem 4, if we have* $\psi_F = \varphi$, *then* $\big(m_{\mathbf{k}}(\chi)\big)_p = m_{\mathbf{k}}(\xi)$.

PROOF. The conclusion of Theorem 4, part (iii) is exactly what is needed to apply [Y, Corollary 3.8], from which we get $m_K(\chi) = m_K(\xi)$. The corollary follows because Schur indices of characters of p-quasi-elementary groups have to be powers of p and $[K:\mathbf{k}]$ is prime to p, which forces $m_K(\xi) = m_{\mathbf{k}}(\xi)$ by [Rit, Theorem 2(c)]. ∎

We are now motivated to find conditions for the group G that ensure that the character φ of F extends to its stabilizer C once G has been reduced using Clifford theory. Exact conditions for this are unknown to this author, although a considerable amount of literature is available on conditions for character extension. (See, for instance, [G] and [I2]). Of course, if C/F is cyclic, then it is well known that φ extends to C [I, 11.22].

3. **The Nilpotent-by-Abelian Group Case.** In this section, we assume that G is a finite nilpotent-by-abelian group that is Clifford reduced with respect to its faithful irreducible character χ. We conclude the paper by showing that for such a nilpotent-by-abelian group, the same p-quasi-elementary group H as in Theorem 4 above can be used to compute the Schur index of χ.

From the structure results in Section 1, we first note that the following holds for nilpotent-by-abelian groups.

PROPOSITION 5. *Suppose that G is a finite nilpotent-by-abelian group and $\chi \in \mathrm{Irr}(G)$. Then $\mathbf{k}Ge_\chi$ Clifford reduces to a simple component of the group algebra $\mathbf{k}\overline{G}$, where $\overline{G}$ is a nilpotent-by-abelian group of derived length at most 3.*

PROOF. Assume that $\mathbf{k}Ge_\chi$ has been fully reduced using Clifford theory, and let the subgroups F, U, and C be as above. Since the class of nilpotent-by-abelian groups is closed under taking sections, we still have that G is nilpotent-by-abelian. We know that $F = F(G) \cap C$ and G/C is abelian, so our assumption that $G/F(G)$ is abelian implies that

$$G' \subseteq F(G) \cap C = F,$$

and so G/F is abelian. The proposition follows because F is a nilpotent group of class at most 2. ∎

(We remark that if G is any solvable group, the Clifford theory reductions on $\mathbf{k}Ga_\chi$ end in a simple component of the group algebra of a group having derived length at most the derived length of $G/F(G)$ plus 3.)

Now, let K be the p'-splitting field for $\mathbf{k}Ge_\chi$, and suppose that KGe_χ has been fully reduced using Clifford Theory. Let F, U, and C be the characteristic subgroups of G determined in Section 1, and let $\varphi \in \mathrm{Irr}(F)$, $\lambda \in \mathrm{Irr}(U)$, and $\psi \in \mathrm{Irr}(C)$ satisfy the conclusions of Proposition 3. By the proof of Proposition 5 we know that G/F is abelian.

LEMMA 6. *Under the assumptions of the preceding paragraph, there exists a normal subgroup B and an irreducible character β of B such that*

(i) $F \subseteq B \subseteq C$;

(ii) $\chi = \beta^G$, $\psi = \beta^C$; and

(iii) $\beta_F = \varphi$.

PROOF. Because G/F is abelian, G is a relative M-group with respect to F [I, 6.22], which means that for every character $\chi' \in \mathrm{Irr}(G)$, there is a subgroup B' containing F and a $\beta \in \mathrm{Irr}(B')$ such that $\beta'^G = \chi'$ and $\beta'_F \in \mathrm{Irr}(F)$. Assuming B and β satisfy this for the character χ, we see that $\chi = \beta^G$. Since G/F is abelian, B has to be a normal subgroup of G. The assumption that KGe_χ is fully reduced using Clifford theory then implies that KBe_χ is simple, hence $KBe_\chi \cong KBa_\beta$. Since $\chi = \beta^G$, B is the stabilizer of β in G, and so

$$G/B \cong \mathrm{Gal}(K(\beta)/K).$$

Since the G-conjugates of λ are the only irreducible characters of F lying under χ, we must have $\beta_F = \varphi^g$, for some $g \in G$. This φ^g is thus invariant in B. Since C is the stabilizer of every φ^g in G, this implies that $B \subseteq C$. As $\beta^{g^{-1}}$ also induces χ, replacing β by $\beta^{g^{-1}}$ gives us the condition $\beta_F = \varphi$. Since β^C is irreducible and lies over φ, we get that $\beta^C = \psi$ because $\psi \in \mathrm{Irr}(C)$ is unique with this property. All three conclusions have now been shown. ∎

The next step is to reduce to a group having a normal p-complement.

LEMMA 7. *Suppose that G is a nilpotent-by-abelian group with faithful irreducible character χ. Assume KGe_χ has been fully reduced using Clifford theory, where K is the p'-splitting field of $\mathbb{Q}(\chi)Ge_\chi$. Let M be a subgroup of G so that M/F is the Sylow p-subgroup of G/F, F as above. Then $KMe_\chi = KGe_\chi$.*

PROOF. Let B be as in Lemma 6. Since $G/B \cong \mathrm{Gal}\big(K(\beta)/K\big)$ is a p-group, we can choose a transversal X of B in G consisting of elements of M. Since φ extends to $\beta \in \mathrm{Irr}(B)$, we see that

$$KBe_\chi \cong KFe_\chi \otimes_{K(\varphi)} K(\beta),$$

because $\beta(1) = \varphi(1)$. Let D be the subgroup of B for which D/F is a Sylow p-subgroup of B/F. Note that KDe_χ must be simple because $D \lhd G$. Since φ extends to D, the same reasoning as above shows that

$$KDe_\chi \cong KFe_\chi \otimes_{K(\varphi)} K(\beta_D).$$

Now we see that $[KBe_\chi : KDe_\chi] = [K(\beta) : K(\beta_D)]$, which is a power of p by our assumption on K. However, since $[KBe_\chi : KDe_\chi]$ divides $|B : D|$, it is relatively prime to p. We conclude that $KDe_\chi = KBe_\chi$. Since $\beta^G = \chi$, we have

$$\begin{aligned} KGe_\chi &= \oplus_{x\in X}(KBe_\chi)x \\ &= \oplus_{x\in X}(KDe_\chi)x \\ &= KMe_\chi, \end{aligned}$$

since X is also a transversal of D in M. ∎

Using Lemma 7, we now assume G/F is a p-group by replacing G by M if necessary. If P is a Sylow p-subgroup of G, then we have that $G = FP$, and so since the p'-part of F is normal in G, G can be written as

$$G = N \rtimes P,$$

with the order of N being relatively prime to P. Together with the assumption that KGe_χ is fully reduced using Clifford theory, this presentation forces the p-part of the subgroup F to be cyclic. (This observation follows from [MW, Corollary 1.10 (iii)] and the fact that every chief section of such a group G that lies over N must have prime order p.) Suppose that B is the normal subgroup of G identified in Lemma 6 that possesses an irreducible character β extending φ and inducing χ. The next step is to arrange that the field of character values $K(\beta)$ is a cyclotomic extension of the p'- splitting field K.

Let $e = \exp(B)$. Since B/F is a p-group, we must have $e = p^c \exp(F)$, for a non-negative integer c. From our knowledge of the structure of F we find that $\exp(F)$ is either $u = |U|$ or $2u$. Furthermore, $\exp(F)$ can be $2u$ only when p is odd, 4 divides $|N|$, and 4 does not divide $|U|$. The inclusions

$$K(\varphi) = K(\zeta_u) \subseteq K(\beta) \subseteq K(\zeta_e)$$

are obvious, with both $[K(\zeta_e) : K(\beta)]$ and $[K(\beta) : K(\zeta_u)]$ equaling powers of the prime p. Since e is either $p^c u$ or $2p^c u$, we have that whenever p is an odd prime, the subfields of $K(\zeta_{p^c u})$ (or $K(\zeta_{p^c u}, \zeta_4)$ when $e = 2p^c u$) that have index a power of p over $K(\zeta_u)$ are all of the form $K(\zeta_{p^\ell u})$, for some integer ℓ such that $0 \leq \ell \leq c$. (This is immediate because the p-part of the respective Galois groups are cyclic.) So in the case where p is odd, $K(\beta) = K(\zeta_{p^\ell u})$ for some integer ℓ, and ℓ can be determined easily from e and $|G : B|$.

If $p = 2$, then the subfields of $K(\zeta_{2^c u})$ lying over $K(\zeta_u)$ do not have to be of the form $K(\zeta_{2^\ell u})$, for $1 \leq \ell \leq c$, unless 4 divides u. If 4 divides u, then we determine that $K(\beta)$ is a cyclotomic extension of K exactly as above. Assuming that 4 does not divide u, we "inflate" the group G to a new group, $\hat{G}$, and arrange that 4 divides the order of the necessary cyclic normal subgroup of $\hat{G}$. If $|U|$ is even, define $\hat{G}$ to be the central product of G with the abstract group

$$D_8 := \langle x, y \mid x^4 = 1 = y^2, x^y = x^{-1} \rangle$$

—a copy of the dihedral group of order 8. If $|U|$ is odd, then define $\hat{G}$ to be the direct product of G with the same group D_8. $K\hat{G}$ has a simple component that is isomorphic to the tensor product

$$KGe_\chi \otimes_K K^{2\times 2}.$$

This simple component is associated with a faithful representation of $\hat{G}$. Not every normal abelian subgroup of $\hat{G}$ is cyclic, since D_8 does not have this property. However, if we set $\hat{F} = \langle F, x \rangle$, $\hat{U} = \langle U, x \rangle$, and $\hat{B} = \langle B, x \rangle$, then any faithful character associated with the above simple component is induced from a faithful irreducible character $\hat{\beta}$ of $\hat{B}$. Since $K(\hat{\beta})$ is a subfield of $K(\zeta_{2^c u})$ containing $K(\zeta_u, \zeta_4)$, $K(\hat{\beta})$ has to be a cyclotomic extension of K, as required. Since the Schur index of the new simple component is the same as that of KGe_χ, it suffices to do our computations based on the slightly larger nilpotent-by-abelian group $\hat{G}$, with the above definitions for the corresponding subgroups and their characters.

Once we have that the character field $K(\beta)$ is a cyclotomic field, we can "inflate" the group G in order to apply Theorem 4 directly. (In the special case, replace G by $\hat{G}$, B by $\hat{B}$, β by $\hat{\beta}$, and etc.) Let z be a root of unity in the center of KBe_χ so that

$$Z(KBe_\chi) = K(z).$$

Let $\tilde{G}$ be the finite subgroup of the group of units of KGe_χ generated by z and G. Then it is easy to see that $KGe_\chi = K\tilde{G}e_\chi$. Furthermore, the subgroups $\tilde{F} = \langle z, F \rangle$, $\tilde{U} = \langle z \rangle$, and $\tilde{B} = \langle z, B \rangle$ all generate simple subalgebras of $K\tilde{G}e_\chi$, as do all normal subgroups of $\tilde{G}$ lying between $\tilde{U}$ and $\tilde{F}$. Because we have arranged that $K(\beta) \cong Z(K\tilde{F}e_\chi)$, we now have $K\tilde{F}e_\chi = K\tilde{B}e_\chi$. This allows us to use the proof of Theorem 4 (without the assumption that every normal abelian subgroup of $\tilde{G}$ is cyclic in the special case), and conclude that there is an irreducible character of $\tilde{H} = \tilde{U}\tilde{P}$ that has the same Schur index as that of $K\tilde{G}e_\chi$, for some Sylow p-subgroup $\tilde{P}$ of $\tilde{G}$. Thus we have constructively reduced the problem of determining the Schur index of KGe_χ to determining the Schur index of a simple component of the group algebra of a p-quasi-elementary group. Calculations necessary for computing the index of such an algebra have been described in [Y], [L], and [Her].

This completes the outline of the algorithm for computing Schur indices of simple components of a group algebra of a nilpotent-by-abelian group over an algebraic number field. We conclude this section with the following example that indicates the limitations of this process for general solvable groups.

For example, let E be an extraspecial group of order 27 and exponent 3. The outer automorphisms of E that fix the center of E form a group isomorphic to $SL(2,3)$. Let G be the semi-direct product $E \rtimes SL(2,3)$ in which $SL(2,3)$ acts as these outer automorphisms. The irreducible characters of G which lie over a fixed faithful irreducible character φ of E are in one-to-one correspondence with characters of $SL(2,3)$. These characters are all faithful. Simple components of $\mathbb{Q}(\zeta_3)G$ corresponding to these characters do not reduce using Clifford theory, in fact, $E = F$ and $G = C$ in the above notation. There are three characters of G that extend φ, corresponding to the linear characters of $SL(2,3)$. For these characters χ_i, $i = 1,2,3$, a reduction using Theorem 4 is possible, and it shows that the 2- and 3-parts of $m_{\mathbb{Q}(\zeta_3)}(\chi_i)$ are Schur indices of simple components of rational group algebras over the groups $C_3 \rtimes Q_8$ and $E \rtimes C_3$, respectively. Since the quaternion component of $C_3 \rtimes Q_8$ is not the one in question, and $E \rtimes C_3$ is a 3-group, we must have $m_{\mathbb{Q}}(\chi_i) = 1$ for $i = 1,2,3$. On the other hand, for the three characters of G lying over φ that satisfy $(\chi_4)_F = 2\varphi$, $(\chi_5)_F = 2\varphi$, and $(\chi_6)_F = 3\varphi$, the reduction fails because the hypothesis is not satisfied. Of course the group H can be computed, but we cannot guarantee the existence of a $\xi \in \mathrm{Irr}(H)$ that will have the right Schur index.

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