

# Generalized Factorization in Hardy Spaces and the Commutant of Toeplitz Operators

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*Abstract.* Every classical inner function  $\varphi$  in the unit disk gives rise to a certain factorization of functions in Hardy spaces. This factorization, which we call the generalized Riesz factorization, coincides with the classical Riesz factorization when  $\varphi(z) = z$ . In this paper we prove several results about the generalized Riesz factorization, and we apply this factorization theory to obtain a new description of the commutant of analytic Toeplitz operators with inner symbols on a Hardy space. We also discuss several related issues in the context of the Bergman space.

## 1 Introduction

Let  $\mathbb{D}$  be the open unit disk in the complex plane  $\mathbb{C}$ . For  $0 < p < \infty$  the Hardy space  $H^p$  consists of analytic functions  $f$  in  $\mathbb{D}$  such that

$$\|f\|_p^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt < \infty.$$

It is well known that if

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in \mathbb{D},$$

then

$$\|f\|_2^2 = \sum_{k=0}^{\infty} |a_k|^2,$$

and  $H^2$  is a Hilbert space whose inner product,  $\langle \cdot, \cdot \rangle$ , is the polarization of the above norm. See [6] for the classical theory of  $H^p$  spaces.

Let  $H^\infty$  be the space of all bounded analytic functions in  $\mathbb{D}$ . For each  $\varphi \in H^\infty$  we define an operator  $T_\varphi$  on  $H^2$  as follows:

$$T_\varphi f = \varphi f, \quad f \in H^2.$$

For historical reasons  $T_\varphi$  will be called the *analytic Toeplitz operator* with symbol  $\varphi$ ;  $T_\varphi$  is simply the multiplication operator induced by  $\varphi$ .

The commutant of  $T_\varphi$ , denoted by  $(T_\varphi)'$ , is the algebra of all bounded linear operators  $S$  on  $H^2$  with  $ST_\varphi = T_\varphi S$ . A closed subspace  $M$  of  $H^2$  is called a *reducing subspace* of  $T_\varphi$  if it is invariant under both  $T_\varphi$  and  $T_\varphi^*$ . It is well known that a closed

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subspace of  $H^2$  is a reducing subspace of  $T_\varphi$  if and only if the orthogonal projection of  $H^2$  onto this subspace is in the commutant of  $T_\varphi$ .

In this paper we study  $T_\varphi$  when  $\varphi$  is an inner function. In this case the operator  $T_\varphi$  is an isometry, so that the Wold decomposition theorem (see [7]) determines a decomposition of  $H^2$  into the direct sum of singly generated subspaces invariant under  $T_\varphi$ . This decomposition leads to a certain factorization which generalizes the classical inner-outer factorization.

Throughout the paper we assume that  $\varphi$  is an inner function. For  $0 < p < \infty$  we say that a function  $f \in H^p$  is  $\varphi$ - $p$  inner if  $\|f\|_p = 1$  and

$$\int_0^{2\pi} |f(e^{it})|^p \varphi(e^{it})^n dt = 0$$

for all positive integers  $n$ . A function  $f \in H^p$  is called  $\varphi$ - $p$  outer if  $f = F \circ \varphi$ , where  $F \in H^p$  is outer.

It was proved in [8] that every  $f \in H^p$  admits a factorization  $f = hF \circ \varphi$ , where  $h$  is  $\varphi$ - $p$  inner and  $F \circ \varphi$  is  $\varphi$ - $p$  outer. Moreover, the  $\varphi$ - $p$  inner and outer factors are uniquely determined by  $f$  up to a unimodulus constant multiple.

We further investigate this factorization in Sections 2 and 3 and obtain several characterizations (and estimates) for  $\varphi$ - $p$  inner and outer functions. We then apply these results in Section 4 to obtain a description of the commutant of  $T_\varphi$ .

To state our main results, Theorems A, B and C below, we introduce a class of measures  $d\sigma_\zeta$  defined via Herglotz's theorem by

$$\operatorname{Re} \frac{\zeta + \varphi(z)}{\zeta - \varphi(z)} = \int_{\mathbb{T}} \operatorname{Re} \frac{w + z}{w - z} d\sigma_\zeta(w), \quad z \in \mathbb{D}, \zeta \in \mathbb{T},$$

where  $\mathbb{T}$  is the unit circle.

**Theorem A** For  $0 < p < \infty$  we have

(a) A function  $h \in H^p$  with  $\|h\|_p = 1$  is  $\varphi$ - $p$  inner if and only if

$$\int_{\mathbb{T}} |h(z)|^p d\sigma_w(z) = 1$$

for almost all  $w \in \mathbb{T}$ .

(b) If  $h$  is  $\varphi$ - $p$  inner, then

$$|h(z)|^p \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2}$$

for all  $z \in \mathbb{D}$ .

(c) If  $f \in H^p$  and  $f = hF \circ \varphi$  is the  $\varphi$ - $p$  factorization of  $f$ , then

$$F(z) = \exp \left\{ \frac{1}{p} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \left( \int_{\mathbb{T}} |f(\tau)|^p d\sigma_\zeta(\tau) \right) dm(\zeta) \right\}$$

for all  $z \in \mathbb{D}$ , where  $dm$  is the normalized Lebesgue measure on  $\mathbb{T}$ .

For any analytic function we will use  $M_f$  to denote the operator of multiplication by  $f$ .

**Theorem B** *There exists a sequence  $\{S_n\}$  of operators on  $H^2$  (depending on  $\varphi$ ) such that the following conditions are equivalent for a bounded linear operator  $S$  on  $H^2$ :*

- (a)  $S$  belongs to the commutant of  $T_\varphi$ .
- (b) There exists a sequence  $\{\varphi_n\}$  in  $H^2$  such that

$$S = M_{\varphi_1}S_1 + \cdots + M_{\varphi_n}S_n + \cdots,$$

where the series converges in the strong operator topology and the  $\varphi$ -2 outer part of each  $\varphi_n$  is bounded.

We will have explicit descriptions for the operators  $S_n$ . In general, a  $\varphi$ - $p$  inner function is not necessarily bounded. However, in the special case when  $\varphi$  is a Blaschke product of  $N$  zeros with  $N < \infty$ , we will see that the  $\varphi$ - $p$  inner part of each function in  $H^p$  is bounded, and the  $\varphi$ - $p$  outer part of  $f \in H^p$  is bounded if and only if  $f$  itself is bounded. In this particular case, our result realizes the commutant of  $T_\varphi$  as  $N$  copies of  $H^\infty$ .

In the final section of the paper we will discuss several related problems in the context of Bergman spaces. In particular, we will obtain several results about the reducing subspaces and commutants of analytic Toeplitz operators on the Bergman space induced by finite Blaschke products. One of the results proved in this section is the following.

**Theorem C** *Let  $B$  be a finite Blaschke product which vanishes at the origin,*

$$B(z) = z \prod_{k=1}^n \frac{z - a_k}{1 - \bar{a}_k z}, \quad |a_k| < 1, \quad k = 1, 2, \dots, n.$$

Then the subspace

$$M = \text{Span}\{B^m, m = 0, 1, \dots\}$$

is reducing for the operator  $T_B$  on  $A^2$ , and its orthogonal complement is given by

$$M^\perp = \text{Span}\left\{\frac{B^m}{1 - \bar{a}_k z} : 1 \leq k \leq n, m \geq 0\right\}.$$

Here  $\text{Span}$  denotes the closed linear span of a collection of vectors in  $A^2$ .

**Remark** The condition of vanishing at the origin is imposed just for convenience. It is not necessary for the first statement and without it the second one needs a small adjustment.

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## 2 Generalized Riesz Factorizations

In this section we present some facts about the Wold decomposition of the Toeplitz isometry  $T_\varphi$ , show how this decomposition leads to a natural generalization of the classical inner-outer factorization in Hardy spaces, and obtain several characterizations (and estimates) for the corresponding inner and outer functions.

For each  $0 < p < \infty$  we set

$$H^p[\varphi] = \{f \circ \varphi : f \in H^p\}.$$

Since  $\varphi$  is inner, the composition operator  $C_\varphi$  is bounded from above and below for all  $p$ ; see [4]. In particular, the space  $H^p[\varphi]$  is closed in  $H^p$ . The image of  $T_\varphi$  on  $H^p$ , denoted by  $\varphi H^p$ , is also a closed subspace of  $H^p$ . When  $p = 2$ , we will need the defect space

$$D[\varphi] = H^2 \ominus (\varphi H^2),$$

which is just the kernel of  $T_\varphi^*$ . Defect spaces are also called *model spaces* in the literature and are sometimes denoted by  $K_\varphi$ . They have been extensively studied by those who work on Hardy spaces and operator theory; see [9].

Recall that the Wold decomposition theorem [7] states that every isometry  $T: X \rightarrow X$  of a Hilbert space  $X$  determines the following decomposition of  $X$ :

$$X = X_0 \bigoplus_{n=0}^{\infty} T^n X_1,$$

where

$$X_1 = X \ominus TX$$

is the wandering subspace, and

$$X_0 = \bigcap_{n=0}^{\infty} T^n X$$

is the stable subspace.

In the special case when  $M = X$  is a closed subspace of  $H^2$  invariant under  $T = T_\varphi$ , the stable subspace is obviously trivial, so that

$$M = \bigoplus_{n=0}^{\infty} T_\varphi^n (M \ominus \varphi M).$$

In particular, if  $M = H^2$  and  $\{e_1, \dots, e_n, \dots\}$  is an orthonormal basis of the defect space of  $T_\varphi$ , the above decomposition implies that any function  $f \in H^2$  is uniquely represented in the form

$$f = \sum_{k=1}^{\infty} (f_k \circ \varphi) e_k,$$

where each  $f_k \circ \varphi \in H^2[\varphi]$  and

$$\|f\|^2 = \sum_{k=1}^{\infty} \|f_k\|^2.$$

Conversely, every sequence  $\{f_1, \dots, f_n, \dots\}$  of functions in  $H^2$  with

$$\sum_{n=1}^{\infty} \|f_n\|_2^2 < \infty$$

determines a function in  $H^2$  as follows:

$$f = \sum_{n=1}^{\infty} (f_n \circ \varphi) e_n.$$

This follows from the simple fact (see [8], for example) that if

$$f = \sum_{n=1}^{\infty} (f_n \circ \varphi) e_n, \quad g = \sum_{n=1}^{\infty} (g_n \circ \varphi) e_n,$$

then

$$(1) \quad \langle f, g \rangle = \sum_{n=1}^{\infty} \langle f_n, g_n \rangle.$$

As a consequence of equation (1) we obtain the following.

**Lemma 1** Suppose  $\{e_k\}$  is an orthonormal basis for  $D[\varphi]$ . For each  $k$  let  $H_k[\varphi]$  be the  $H^2$ -closure of the set of vectors of the form  $(p \circ \varphi)e_k$ , where  $p$  is a polynomial. Then  $H_k[\varphi] \perp H_m[\varphi]$  whenever  $k \neq m$ , and

$$H^2 = \bigoplus_k H_k[\varphi].$$

We will call the decomposition of  $H^2$  in the above lemma the Wold decomposition of  $H^2$  with respect to  $\varphi$ .

As another consequence of equation (1) we obtain the following well-known fact representing the Toeplitz operator with symbol  $\varphi$ :

$$(2) \quad (T_\varphi)^* f = T_{\bar{\varphi}} f = \sum_{n=0}^{\infty} [(Sf_n) \circ \varphi] e_n,$$

where the operator  $S$  defined by

$$Sf(z) = \frac{f(z) - f(0)}{z}$$

is the backward shift.

As a consequence of the representation (2) we obtain the following result proved in [7] and [10].

**Lemma 2** Suppose  $X$  is any subset of the defect space  $D[\varphi]$ . Let  $H_X$  be the closed linear span in  $H^2$  of the set of vectors of the form  $(p \circ \varphi)f$ , where  $f \in X$  and  $p$  is a polynomial. Then  $H_X$  is a reducing subspace of  $T_\varphi$ .

In particular, it follows from the above lemma that the commutant of  $T_\varphi$  contains a lot of orthogonal projections. We will need the following result from [8].

**Proposition 3** Suppose  $\{e_n\}$  is an orthonormal basis of the defect space  $D[\varphi]$  and  $0 < p < \infty$ . Then

(i) A function  $f$  in  $H^p$  is  $\varphi$ - $p$  inner if and only if

$$\|(h \circ \varphi)f\|_p = \|h\|_p$$

for every  $h \in H^p$ .

(ii) A function

$$f = \sum_{n=0}^{\infty} (f_n \circ \varphi)e_n$$

is  $\varphi$ -2 inner if and only if

$$\sum_{n=0}^{\infty} |f_n(e^{i\theta})|^2 = 1$$

for almost all  $\theta$ .

(iii) Every function  $f \in H^p$  admits a unique (up to a unimodular scalar factor) factorization  $f = hF \circ \varphi$ , where  $h$  is  $\varphi$ - $p$  inner and  $F \circ \varphi$  is  $\varphi$ - $p$  outer.

It follows from the representation (2) that each  $e_k$  is a  $\varphi$ -2 inner function. Combining this observation with part (i) of Proposition 3 above, we conclude that  $H_k[\varphi] = e_k H^2[\varphi]$  in Lemma 1, so that the Wold decomposition of  $H^2$  associated with  $\varphi$  can be written as

$$H^2 = \bigoplus_k (e_k H^2[\varphi]).$$

If  $\varphi(z) = z$ , the factorization mentioned in part (iii) of Proposition 3 coincides with the classical inner-outer factorization.

If  $\varphi$  is a finite Blaschke product of order  $N$ , then every  $\varphi$ - $p$  inner function is in  $H^\infty$ . In fact, the defect space consists of rational functions in this case, so any element of the defect space is bounded in the unit disk. If

$$h = \sum_{k=1}^N e_k h_k \circ \varphi$$

is  $\varphi$ -2 inner, then by part (ii) of Proposition 3, each  $h_k$  is bounded, so  $h$  is bounded as well. If  $0 < p < \infty$  and  $h = gF$  is  $\varphi$ - $p$  inner, where  $g$  is a classical inner function and  $F$  is outer, then  $F^{p/2}$  is  $\varphi$ -2 inner and hence bounded, which implies that  $h \in H^\infty$ .

If  $\varphi$  is not a finite Blaschke product, a  $\varphi$ - $p$  inner function need not be bounded (see [8] and the example following Corollary 8 in the present paper). Still, we are able to obtain some growth estimates.

**Theorem 4** Let  $0 < p < \infty$  and  $h$  be a  $\varphi$ - $p$  inner function. Then

$$|h(z)|^p \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2}.$$

for all  $z \in \mathbb{D}$ .

**Proof** By Proposition 3, for every function  $g \in H^p$  we have

$$\|hg \circ \varphi\|_p = \|g\|_p.$$

In particular, this implies that

$$|h(z)| |g(\varphi(z))| \leq \frac{\|g\|_p}{(1 - |z|^2)^{1/p}}$$

for every  $z \in \mathbb{D}$ . Choosing

$$g(w) = \frac{1}{[1 - w\varphi(z)]^{2/p}}$$

leads to the desired estimate. ■

**Corollary 5** Let  $\varphi = Bg$ , where  $B$  is a Blaschke product and  $g$  is a singular inner function generated by a singular measure  $\mu$ . If  $w \in \mathbb{T}$  is neither a limit point of the zeros of  $B$  nor a point in the closure of the support of  $\mu$ , then any  $\varphi$ - $p$  inner function is bounded near  $w$ .

**Proof** It is well known that  $\varphi$  has an analytic extension to a neighborhood of  $w$ . In particular,  $\varphi$  has a finite angular derivative at  $w$ . The result then follows from Theorem 4. ■

We proceed to describe  $\varphi$ - $p$  inner and outer functions in terms of singular measures introduced by Clark in [2]. More specifically, if  $|w| = 1$ , then

$$\operatorname{Re} \frac{w + \varphi(z)}{w - \varphi(z)} = \frac{1 - |\varphi(z)|^2}{|w - \varphi(z)|^2} > 0$$

for all  $z$  in the unit disk. By Herglotz's theorem there is a nonnegative measure  $\sigma_w$  on the unit circle  $\mathbb{T}$  such that

$$\operatorname{Re} \frac{w + \varphi(z)}{w - \varphi(z)} = \int_{\mathbb{T}} \operatorname{Re} \frac{\zeta + z}{\zeta - z} d\sigma_w(\zeta)$$

for all  $z \in \mathbb{D}$ . Since  $\varphi$  is inner,  $\sigma_w$  is singular with respect to the normalized Lebesgue measure  $dm$  on  $\mathbb{T}$ . If  $E_w$  is the set of points in  $\mathbb{T}$  where  $\varphi$  has  $w$  as a nontangential limit, then (see [1])

$$(3) \quad \sigma_w(\mathbb{T} \setminus E_w) = 0.$$

Furthermore, if  $\varphi(0) = 0$ , then  $\sigma_w$  is a probability measure for all  $w \in \mathbb{T}$ .

These measures  $\sigma_w$  are related to the conditional expectation operator  $E(\cdot | \varphi)$  associated with the  $\sigma$ -algebra generated by  $\varphi$ :

$$E(f|\varphi)(w) = \int_{\mathbb{T}} f(\zeta) d\sigma_w(\zeta),$$

where  $f \in L^1(\mathbb{T})$  and  $w \in \mathbb{T}$ . This was proved by Alexandrov in [1], where he also proved that every  $f \in L^1(\mathbb{T})$  belongs to  $L^1(\mathbb{T}, \sigma_w)$  for almost all  $w \in \mathbb{T}$  and

$$(4) \quad \int_{\mathbb{T}} f(\zeta) dm(\zeta) = \int_{\mathbb{T}} \left( \int_{\mathbb{T}} f(\zeta) d\sigma_w(\zeta) \right) dm(w).$$

**Theorem 6** *Suppose  $0 < p < \infty$  and  $h$  is a unit vector in  $H^p$ . Then  $h$  is  $\varphi$ - $p$  inner if and only if*

$$\int_{\mathbb{T}} |h(z)|^p d\sigma_w(z) = 1$$

for almost all  $w \in \mathbb{T}$ .

**Proof** It follows from equations (3) and (4) that

$$\int_{\mathbb{T}} |h(z)|^p \varphi(z)^k dm(z) = \int_{\mathbb{T}} w^k \left( \int_{\mathbb{T}} |h(z)|^p d\sigma_w(z) \right) dm(w)$$

for all positive integers  $k$ . Thus, the uniqueness theorem implies that  $h$  is  $\varphi$ - $p$  inner if and only if

$$\int_{\mathbb{T}} |h(z)|^p d\sigma_w(z)$$

is a constant function of  $w$ . Since  $\|h\|_p = 1$ , we conclude that  $h$  is  $\varphi$ - $p$  inner if and only if

$$\int_{\mathbb{T}} |h(z)|^p d\sigma_w(z) = 1$$

for almost all  $w \in \mathbb{T}$ . ■

Note that statement (i) of Proposition 3 is a straightforward corollary of Theorem 6 and equation (4). This offers a proof of part (i) of Proposition 3 that is different from the one in [8].

**Theorem 7** *Suppose  $0 < p < \infty$ ,  $f \in H^p$ , and  $f = hF_p \circ \varphi$  is the  $\varphi$ - $p$  factorization of  $f$ . Then*

$$(5) \quad F_p(z) = \exp \left\{ \frac{1}{p} \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \left( \int_{\mathbb{T}} |f(\tau)|^p d\sigma_{\zeta}(\tau) \right) dm(\zeta) \right\}$$

for all  $z \in \mathbb{D}$ .



**Proof** It follows directly from the classical Riesz formula and (4) that the function given by the integral on the right-hand side of (5) is outer and in  $H^p$  and the absolute value of its nontangential boundary limit at almost all  $\zeta \in \mathbb{T}$  is

$$\left( \int_{\mathbb{T}} |f(\tau)|^p d\sigma_{\zeta}(\tau) \right)^{1/p}.$$

Indeed, if we denote the last function by  $\hat{F}_p(\zeta)$ , then by (4) it is in  $L^1(dm)$ , and, hence,  $\log^+ \hat{F}_p \in L^1(dm)$ . Since  $f \in H^p$ ,  $\log^- |f| \in L^1(dm)$ . Now, the concavity of the function  $\log$  and (4) show that  $\log^- \hat{F}_p \in L^1(dm)$ .

Further, it follows from (4) and (5) that

$$\begin{aligned} \int_{\mathbb{T}} |f(\tau)|^p |\hat{F}_p(\varphi(\tau))|^{-p} \varphi(\tau)^k dm(\tau) &= \int_{\mathbb{T}} |\hat{F}_p(\zeta)|^{-p} \zeta^k \left( \int_{\mathbb{T}} |f(\tau)|^p d\sigma_{\zeta}(\tau) \right) dm(\zeta) \\ &= \int_{\mathbb{T}} |\hat{F}_p(\zeta)|^{-p} |\hat{F}_p(\zeta)|^p \zeta^k dm(\zeta) \\ &= \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & \text{if } k = 0. \end{cases} \end{aligned}$$

This implies that  $f(\hat{F}_p \circ \varphi)^{-1}$  is  $\varphi$ - $p$  inner, so  $\hat{F}_p$  is the  $\varphi$ - $p$  outer part of  $f$ . ■

Note that if  $\varphi(z) = z$ , then  $\sigma_w$  is a point mass at  $w$  of mass 1 and (5) converts to the classical Riesz formula.

**Corollary 8** Suppose  $f \in H^p$ ,  $0 < r \leq p \leq +\infty$ , and  $f = h_r F_r \circ \varphi$  is the  $\varphi$ - $r$  factorization of  $f$ . Then  $F_r \in H^p$ .

**Proof** This follows directly from equation (4), the proof of Theorem 7, and Hölder’s inequality. ■

In view of the last corollary it is natural to ask if for  $0 < r \leq p$  the  $\varphi$ - $r$  inner part of an  $H^p$ -function  $f$  must be in  $H^p$ . For example, if  $f$  is bounded, Corollary 8 implies that its  $\varphi$ - $p$  outer part is bounded (in fact, its  $H^\infty$  norm does not exceed the  $H^\infty$  norm of  $f$ ) for all  $p$ . Must the  $\varphi$ - $p$  inner part of  $f$  be bounded? The following example shows that in general the answer is negative.

**Example** Assume that  $\varphi$  has the property that the measures  $\sigma_w$  are continuous (that is, they have no point masses) for almost all  $w \in \mathbb{T}$ ; see [1] for examples of such functions. Choose a set of circular arcs  $I_n$ ,  $n = 1, 2, \dots$ , satisfying the following conditions:

- (i)  $I_n \cap I_m = \emptyset$  for  $n \neq m$ ,
- (ii)  $|I_n| = 3^{-n}$  for all  $n$ .

Write  $V_n = \varphi^{-1}(I_n)$  and note that  $V_n$  are disjoint subsets of  $\mathbb{T}$  of positive Lebesgue measure. It follows from the definition of  $\sigma_w$  that

$$\sigma_w(\mathbb{T}) = \frac{1 - |\varphi(0)|^2}{|1 - \bar{w}\varphi(0)|^2} > 0$$

for all  $w \in \mathbb{T}$ . For each  $n$  we consider the points  $w \in I_n$  for which the measure  $\sigma_w$  is continuous; for each such point  $w$  there exists an arc  $J_w = (1, e^{i\theta(w)})$  with  $0 < \theta(w) < \pi/2$  and

$$\sigma_w(J_w) = \frac{1}{n}\sigma_w(\mathbb{T}).$$

We then define

$$\tilde{U}_n = \bigcup \{J_w : w \in I_n\},$$

and

$$U_n = \tilde{U}_n \cap V_n.$$

Since  $\sigma_w(\mathbb{T} \setminus V_n) = 0$  for all  $w \in I_n$ , we have  $\sigma_w(U_n) = \frac{1}{n}\sigma_w(\mathbb{T})$ . Thus equation (5) implies that  $U_n$  has positive Lebesgue measure for all  $n$ . Now define a function  $\rho$  on  $\mathbb{T}$  as follows:

$$\rho(w) = \begin{cases} 1 & \text{if } w \in (\mathbb{T} \setminus \bigcup_n V_n) \cup U_n \\ \frac{1}{n} & \text{if } w \in V_n \setminus U_n. \end{cases}$$

It is easily seen that  $\rho$  is a bounded positive function on  $\mathbb{T}$  and

$$\int_{\mathbb{T}} |\log \rho(z)| dm(z) \leq C \sum_{n=1}^{\infty} \frac{\log n}{3^n},$$

where  $C$  is a positive constant. So  $\log \rho$  is in  $L^1(\mathbb{T})$ , which implies that there is a bounded analytic function  $f$  in  $\mathbb{D}$  such that  $|f|$  coincides with  $\rho$  almost everywhere on  $\mathbb{T}$ ; here we denote the radial limits of  $f$  by the same letter  $f$ . Let  $0 < p < \infty$  and  $f = h_p F_p \circ \varphi$  be the  $\varphi$ - $p$  factorization of  $f$ . For each  $w \in I_n$ , an application of (4) and Theorem 7 gives

$$\begin{aligned} |F_p(w)| &= \left( \int_{E_w} |f(z)|^p d\sigma_w(z) \right)^{1/p} \\ &= \left( \frac{1}{n^p} \frac{n-1}{n} \frac{1 - |\varphi(0)|^2}{|1 - \bar{w}\varphi(0)|^2} + \frac{1}{n} \frac{1 - |\varphi(0)|^2}{|1 - \bar{w}\varphi(0)|^2} \right)^{1/p} \\ &\sim \begin{cases} n^{-1/p} & \text{if } p \geq 1 \\ n^{-1} & \text{if } 0 < p < 1. \end{cases} \end{aligned}$$

Since  $|U_n| > 0$ , the set of points  $\zeta_n \in U_n$  where the functions  $\varphi, f, F_p \circ \varphi$ , and  $h_p$  all have radial limits has positive measure. Since for all such  $\zeta_n$  we have  $w_n = \varphi(\zeta_n) \in I_n$ , the last estimate shows that

$$|h_p(w_n)| = \frac{\rho(w_n)}{|F_p(w_n)|} \sim \begin{cases} n^{1/p} & \text{if } p \geq 1 \\ n & \text{if } 0 < p < 1, \end{cases}$$

so that  $h_p$  is unbounded.

### 3 $\varphi$ -Multipliers

For  $0 < p \leq s \leq \infty$  we will denote by  $H_{\varphi,p,s}$  the collection of all functions in  $H^p$  whose  $\varphi$ - $p$  outer part is in  $H^s$ . For  $0 < r \leq p \leq \infty$  we say that an analytic function  $f$  in the unit disk is a  $\varphi$ -multiplier of type  $(p, r)$  if for every  $h \in H^p[\varphi]$  the product  $fh$  is in  $H^r$ .

**Theorem 9** *An analytic function  $f$  is a  $\varphi$ -multiplier of type  $(p, r)$  if and only if  $f \in H_{\varphi,r,t}$ , where  $t = pr/(p-r)$ .*

**Proof** Since constants are in  $H^p[\varphi]$  for all  $p$ , any  $\varphi$ -multiplier of type  $(p, r)$  must be in  $H^r$ . Let  $f \in H^r$  and  $f = h_r F_r \circ \varphi$  be the  $\varphi$ - $r$  factorization of  $f$ . For every  $g \in H^p$ , part (i) of Proposition 3 gives

$$\|fg \circ \varphi\|_r = \|F_r g\|_r \leq \|F_r\|_{\frac{pr}{p-r}} \|g\|_p \leq C \|F_r\|_{\frac{pr}{p-r}} \|g \circ \varphi\|_p.$$

Since  $g$  is arbitrary and the last inequality is sharp, the result follows. ■

**Corollary 10** *If  $0 < p \leq s \leq \infty$ , then  $H_{\varphi,p,s}$  is a subspace of  $H^p$ .*

**Proof** This is because the set of multipliers is obviously a linear space. ■

As a special case of the theorem above we see that  $\varphi$ -multipliers of type  $(p, p)$  (or simply  $\varphi$ - $p$ -multipliers) are those  $H^p$  functions whose  $\varphi$ - $p$  outer part is bounded in the unit disk.

In the case when  $\varphi$  is a finite Blaschke product any  $\varphi$ - $p$  inner function is bounded, so that the set of  $\varphi$ - $p$ -multipliers coincides with  $H^\infty$ .

**Corollary 11** *If  $f$  is a  $\varphi$ -multiplier of type  $(p, r)$ , then there is a constant  $C > 0$  such that*

$$|f(z)| \leq C \frac{(1 - |\varphi(z)|^2)^{1/p}}{(1 - |z|^2)^{1/r}}$$

for all  $z \in \mathbb{D}$ .

**Proof** Let  $f = h_r F_r \circ \varphi$  be the  $\varphi$ - $r$  factorization of  $f$ . By Theorem 9 the function  $F_r$  belongs to  $H^{\frac{pr}{p-r}}$ , so that

$$|F_r(\varphi(z))| \leq \frac{\|F_r\|_{\frac{pr}{p-r}}}{(1 - |\varphi(z)|^2)^{\frac{p-r}{pr}}}.$$

This together with Theorem 4 yields the desired estimate. ■

#### 4 The Commutant of $T_\varphi$

The Wold decomposition associated with  $\varphi$  in Section 2 depends on the choice of an orthonormal basis  $\{e_k\}$  for the defect space  $D[\varphi]$ . In the rest of this section we fix an orthonormal basis  $\{e_k\}$  of the defect space  $D[\varphi]$ . Let

$$H^2 = \bigoplus (e_k H^2[\varphi])$$

be the corresponding Wold decomposition of  $H^2$ . For each  $k$  we define an operator  $S_k$  on  $H^2$  by

$$S_k(f) = \frac{P_k(f)}{e_k}, \quad f \in H^2,$$

where  $P_k$  is the orthogonal projection from  $H^2$  onto  $H_k[\varphi] = e_k H^2[\varphi]$ .

**Lemma 12** *With the above notation we have*

- (a) *Each operator  $S_k$  is bounded on  $H^2$ .*
- (b) *Each  $S_k$  commutes with  $T_\varphi$ .*
- (c) *Each  $S_k$  maps  $H^\infty$  into  $H^\infty$ .*

**Proof** The first assertion follows from part (i) of Proposition 3 and the fact that each  $e_k$  is  $\varphi$ -2 inner, the second follows from Lemma 2, and the third can be found in [11]. ■

We now show how to combine the operators  $S_k$  with certain analytic multiplication operators to obtain all the operators in the commutant of  $T_\varphi$ . As a first step, fix any  $k$  and let  $h$  be a multiplier from  $H^2[\varphi]$  into  $H^2$ . By the closed graph theorem there exists a positive constant  $C$  such that  $\|hf \circ \varphi\| \leq C\|f \circ \varphi\|$  for all  $f \in H^2$ . This implies that the operator  $T = M_h S_k$  is bounded on  $H^2$ , where  $M_h$  is the operator of multiplication by  $h$ . In fact, if  $f \in H^2$ , we can write

$$f = \sum_k e_k f_k \circ \varphi.$$

The definition of  $S_k$  gives  $S_k(f) = f_k \circ \varphi$ , so by Proposition 3 we have

$$\|T(f)\| = \|hf_k \circ \varphi\| \leq C\|f_k \circ \varphi\| = C\|S_k(f)\| \leq C\|S_k\| \|f\|.$$

Since  $S_k$  commutes with  $T_\varphi$ , and  $M_h$  commutes with  $T_\varphi$  (at least on the dense subspace  $H^\infty$ ; we then apply part (c) of Lemma 12), we conclude that  $T$  belongs to the commutant of  $T_\varphi$ .

The next result shows that this procedure will produce all the operators in the commutant of  $T_\varphi$ .

**Theorem 13** A bounded linear operator  $S$  on  $H^2$  commutes with  $T_\varphi$  if and only if there exists a sequence  $\{\varphi_k\}$  of multipliers from  $H^2[\varphi]$  into  $H^2$  such that

$$S = M_{\varphi_1}S_1 + \cdots + M_{\varphi_k}S_k + \cdots ,$$

where the series converges in the strong operator topology.

**Proof** Assume that  $T$  is an operator of the form

$$T = \sum_k M_{\varphi_k}S_k,$$

where each  $\varphi_k$  is a multiplier from  $H^2[\varphi]$  into  $H^2$  and the series converges in the strong operator topology. We already knew that each term  $M_{\varphi_k}S_k$  is in the commutant of  $T_\varphi$ . Since the commutant is a linear space that is closed in the strong operator topology,  $T$  belongs to  $(T_\varphi)'$  as well.

Next assume that  $T$  is any operator in the commutant of  $T_\varphi$ . For each  $k$  let  $\varphi_k = T(e_k)$ . We first show that each  $\varphi_k$  is a multiplier from  $H^2[\varphi]$  into  $H^2$ . To this end we fix  $k$  and pick any  $f \in H^2$ . Write

$$f = \sum_{n=1}^{\infty} e_n f_n \circ \varphi,$$

where each  $f_n$  belongs to  $H^2$ . Since  $T$  commutes with  $T_\varphi$ , it follows easily that

$$T(e_n f_n \circ \varphi) = f_n \circ \varphi T(e_n).$$

This implies that

$$T(f) = \sum_{n=1}^{\infty} \varphi_n f_n \circ \varphi = \sum_{n=1}^{\infty} \varphi_n \frac{P_n(f)}{e_n}.$$

Fixing  $k$  and replacing  $f$  by  $e_k g \circ \varphi$ , where  $g \in H^2$ , we obtain

$$T(e_k g \circ \varphi) = \varphi_k g \circ \varphi.$$

Thus

$$\|\varphi_k g \circ \varphi\| \leq \|T\| \|e_k g \circ \varphi\| = \|T\| \|g \circ \varphi\|.$$

This shows that each  $\varphi_k$  is a multiplier from  $H^2[\varphi]$  into  $H^2$ . Our earlier calculation already gave

$$T(f) = \sum_k \varphi_k \frac{P_k(f)}{e_k} = \sum_k M_{\varphi_k}S_k(f)$$

for all  $f \in H^2$ . So we have

$$T = \sum_k M_{\varphi_k}S_k,$$

with the series converging in the strong operator topology. ■

**Corollary 14** *If  $B$  is a finite Blaschke product with  $n$  zeros, then there exist operators  $S_1, \dots, S_n$  such that the commutant of  $T_B$  on  $H^2$  consists of operators of the form*

$$S = M_{\varphi_1} S_1 + \dots + M_{\varphi_n} S_n,$$

where  $\varphi_1, \dots, \varphi_n$  are functions in  $H^\infty$ .

The commutant of an analytic Toeplitz operator on the Hardy space induced by a finite Blaschke product has been studied extensively in the literature. We mention here the papers [3] [7] [5] [12]. In particular, Thomson [12] gives an explicit description of the commutant of  $T_B$  when  $B$  is a Blaschke product with two zeros; Cowen [3] describes the commutant of  $T_B$  for a finite Blaschke product in terms of the Riemann surface generated by  $\varphi$ .

Our result differs from the known ones in that it clearly relates the commutant of  $T_\varphi$  to multiplication operators. In particular, it realizes the commutant of  $T_\varphi$  as  $N$  copies of  $H^\infty$  when  $\varphi$  is finite Blaschke product with  $N$  zeros. More precisely, the commutant of  $T_\varphi$  in this case is an  $H^\infty$ -module generated by  $N$  elements.

We also mention that the commutant of a general isometry (such as  $T_\varphi$  in the paper) can be described in terms of block matrices; see [3] or [5].

## 5 Some Results for the Bergman space

It is very natural to ask how much of what we have done can be extended to the case of the Bergman space. Although most of our results do not have obvious generalizations, we show in this section that some ideas can still be pursued in the Bergman space setting. We will obtain several results about the reducing subspaces and commutants of analytic Toeplitz operators on the Bergman space whose symbols are finite Blaschke products.

Recall that for  $0 < p < \infty$  the Bergman space  $A^p$  consists of analytic functions  $f$  in  $\mathbb{D}$  such that

$$\|f\|_p = \left[ \int_{\mathbb{D}} |f(z)|^p dA(z) \right]^{1/p} < \infty,$$

where  $dA$  is the normalized area measure on  $\mathbb{D}$ . We will only consider the Hilbert space  $A^2$ , whose inner product is given by

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} dA(z),$$

and the associated norm will be denoted by  $\|\cdot\|$ .

Given any  $\varphi \in H^\infty$ , we still use  $M_\varphi$  or  $T_\varphi$  to denote the operator of multiplication by  $\varphi$  on  $A^2$ . All operators in this section act on the Bergman space  $A^2$  unless otherwise specified.

It was shown in [13] that the operator of multiplication by a Blaschke product  $B$  of order 2 acting on the Bergman space has exactly two reducing subspaces, and the two reducing subspaces were described in [13] in terms of the geodesic midpoint of the two zeros of  $B$ . Here we will construct a non-trivial reducing subspace

(and its orthogonal complement) of  $M_B$  for any finite Blaschke product  $B$  (with more than one zero, of course). When  $B$  has only two zeros, this construction gives an alternative description of the reducing subspaces obtained in [13], and it produces a new matricial representation of the commutant of  $M_B$ .

If  $B$  is a finite Blaschke product with  $n$  zeros, then so is the function

$$B_1 = \frac{B - B(0)}{1 - \overline{B(0)}B}.$$

Since  $T_B$  and  $T_{B_1}$  have the same commutant and the same lattice of reducing subspaces, we may assume, without loss of generality, that  $B(0) = 0$ . This condition will be assumed throughout the rest of this section.

**Theorem 15** *Let  $B$  be a finite Blaschke product which vanishes at the origin,*

$$(6) \quad B(z) = z \prod_{k=1}^n \frac{z - a_k}{1 - \overline{a_k}z}, \quad |a_k| < 1, \quad k = 1, 2, \dots, n.$$

Then the subspace

$$M = \text{Span}\{B^m, m = 0, 1, \dots\}$$

is reducing for the operator  $T_B$  on  $A^2$ , and its orthogonal complement is given by

$$M^\perp = \text{Span}\left\{ \frac{B^m(z)}{1 - \overline{a_k}z} : 1 \leq k \leq n, m \geq 0 \right\}.$$

Here Span denotes the closed linear span of a collection of vectors in  $A^2$ .

**Proof** Since  $M$  is clearly invariant under  $T_B$ , it suffices to show that  $M^\perp$  is also invariant under  $T_B$ .

We first prove that the functions in  $M^\perp$  that are analytic in the closed disk are dense in  $M^\perp$ . Let  $P$  and  $P_1$  be the orthogonal projections of  $A^2$  onto  $M$  and  $M^\perp$ , respectively. Then  $P_1(\mathcal{P}[z])$  is dense in  $M^\perp$ , because the space  $\mathcal{P}[z]$  of polynomials is dense in  $A^2$ . Since  $B(0) = 0$ , the functions

$$\frac{1}{\sqrt{n}} B^m, \quad m = 0, 1, 2, \dots,$$

form an orthonormal basis of  $M$ . For a polynomial  $q$  we have

$$Pq = \frac{1}{n} \sum_{k=0}^{\infty} \left( \int_{\mathbb{D}} q(z) \overline{B^k(z)} dA(z) \right) B^k.$$

If  $k > \text{deg}(q)$ , then the integral on the right-hand side above vanishes. Thus,

$$(7) \quad Pq = B' \sum_{k=0}^{\text{deg}(q)} \lambda_k B^k,$$

where  $\lambda_k$  are constants. This shows that  $Pq$  is analytic in the closed disk, so that  $P_1q = q - Pq$  is analytic in  $\mathbb{D}$ .

We next give an explicit description of  $M^\perp$ . Take  $f \in M^\perp$  and assume that  $f$  is analytic in  $\mathbb{D}$ . By the Green-Stokes formula,

$$\begin{aligned} 0 &= \int_{\mathbb{D}} f(z) \overline{B'(z)B(z)^k} dA(z) \\ &= \frac{1}{2\pi i(k+1)} \int_{\mathbb{T}} f(z) \overline{B(z)^{k+1}} dz \\ &= \frac{1}{k+1} \int_{\mathbb{T}} zf(z) \overline{B(z)^{k+1}} dm(z), \end{aligned}$$

where  $dm$  is the normalized Lebesgue measure on  $\mathbb{T}$ . Thus, the function  $zf(z)$  is orthogonal to  $B^m$  in  $H^2$  for every  $m > 0$ , and clearly, this also holds for  $m = 0$ . It is well known (see [8] for example) that the orthogonal complement in  $H^2$  of the set  $\{B^m : m \geq 0\}$  is spanned (in the  $H^2$  topology) by the set

$$\left\{ \frac{z}{1 - \bar{a}_k z} B^m(z) : k = 1, 2, \dots, n, m = 0, 1, \dots \right\}.$$

Since multiplication by  $z$  is an isometry in  $H^2$ , and since  $H^2$  convergence implies  $A^2$  convergence, we conclude that  $f$  can be approximated in the  $A^2$ -norm by functions of the form

$$\frac{B^m(z)}{1 - \bar{a}_k z}, \quad 1 \leq k \leq n, m \geq 0.$$

It follows that

$$(8) \quad M^\perp \subset \text{Span} \left\{ \frac{B^m(z)}{1 - \bar{a}_k z} : 1 \leq k \leq n, m \geq 0 \right\}.$$

Here the span is taken in  $A^2$ .

Conversely, if

$$f(z) = \frac{B^m(z)}{1 - \bar{a}_k z}$$

for some  $1 \leq k \leq n$  and  $m \geq 0$ , then the arguments in the previous paragraph used in reverse order show that  $f \in M^\perp$ . This proves that the linear span in (8) is contained in  $M^\perp$ . Combining this with the conclusion of the previous paragraph, we obtain

$$M^\perp = \text{Span} \left\{ \frac{B^m(z)}{1 - \bar{a}_k z} : 1 \leq k \leq n, m \geq 0 \right\}.$$

This description of  $M^\perp$  clearly shows that  $M^\perp$  is invariant under  $T_B$ . ■

**Corollary 16** *Let  $B$  be a finite Blaschke product in the form (6). Then*

$$A^2 = \text{Span} \{B'B^m : m \geq 0\} \oplus \text{Span} \left\{ \frac{B^m(z)}{1 - \bar{a}_k z} : 1 \leq k \leq n, m \geq 0 \right\}.$$



In particular, if  $B$  is a Blaschke product of order 2,

$$B(z) = z \frac{z - a}{1 - \bar{a}z},$$

then the spaces

$$\text{Span}\{B'B^m : m \geq 0\}, \quad \text{Span}\left\{\frac{B^m(z)}{1 - \bar{a}z} : m \geq 0\right\}$$

are reducing for  $T_B$ . By [13] these must be the only two reducing subspaces for  $T_B$ ; and they must coincide with the descriptions given in [13].

We now proceed to give a matricial description of the commutant of  $T_B$  on  $A^2$  when  $B$  is a Blaschke product of order 2 whose zeros are 0 and  $a$ .

Denote by  $\mu$  the pull-back measure on  $\mathbb{D}$  induced by  $B$ . So for a Borel subset  $C \in \mathbb{D}$  we have

$$\mu(C) = \int_{B^{-1}(C)} dA(z).$$

The measure  $\mu$  is absolutely continuous with respect to area measure,

$$d\mu(z) = \rho(z) dA(z),$$

where

$$\rho(z) = \sum_{B(\tau)=z} \frac{1}{|B'(\tau)|^2},$$

with

$$B(\tau) = \tau \frac{a - \tau}{1 - \bar{a}\tau}.$$

A calculation shows that

$$\rho(z) = \sum_{B(\tau)=z} \frac{1}{|B'(\tau)|^2} = \frac{(1 - |a|^2)^2 + |1 - \bar{a}\tau|^4}{|\bar{a}\tau^2 - 2\tau + a|^2}.$$

From this we derive the following lower estimate for the weight function  $\rho$ ,

$$(9) \quad \rho(z) \geq \frac{(1 - |a|)^2(1 + |a|^2)}{(1 + |a|)^2}.$$

Consider the function

$$(10) \quad g(z) = 1 - \frac{\overline{B'(0)}}{2} B' = \frac{1}{1 - \bar{a}z} \left( \frac{2 - |a|^2}{2} + \frac{\bar{a}^2}{2} B \right).$$

By Theorem 15,  $g \in \text{Span}\{B^k(z)/(1 - \bar{a}z) : k \geq 0\}$ . Actually, it is easy to check that

$$\text{Span}\{gB^k : k \geq 0\} = \text{Span}\{B^k(z)/(1 - \bar{a}z) : k \geq 0\}.$$

It is also easy to see that

$$(11) \quad 1 - |a| \leq |g(z)| \leq \frac{1}{1 - |a|}$$

for all  $z \in \mathbb{D}$ .

**Lemma 17** *Let*

$$(12) \quad f = \frac{1}{\sqrt{2}}B'f_1 \circ B + g f_2 \circ B,$$

where  $f_1$  and  $f_2$  are functions in  $H^\infty$ . Then

$$(13) \quad \|f\|^2 = \|f_1\|^2 - \frac{|a|^2}{2}\|f_2\|^2 + \|f_2\|_\mu^2,$$

where

$$\|f_2\|_\mu^2 = \int_{\mathbb{D}} |f_2|^2 d\mu.$$

**Proof** By Theorem 15 and (10), we have

$$\begin{aligned} \|f\|^2 &= \frac{1}{2}\|B'f_1 \circ B\|^2 + \|g f_2 \circ B\|^2 \\ &= \|f_1\|^2 + \int_{\mathbb{D}} \left| 1 - \frac{\overline{B'(0)}}{2}B'(z) \right|^2 |f_2(B(z))|^2 dA(z) \\ &= \|f_1\|^2 + \|f_2 \circ B\|^2 + \frac{|a|^2}{2}\|f_2\|^2 \\ &\quad - 2 \operatorname{Re} \left( \frac{B'(0)}{2} \int_{\mathbb{D}} |f_2(B(z))|^2 \overline{B'(z)} dA(z) \right). \end{aligned}$$

Write

$$f_2(z) = \sum_{k=0}^{\infty} c_k z^k,$$

and let

$$h(z) = \sum_{k=0}^{\infty} \frac{c_k}{k+1} z^{k+1}$$

be the antiderivative of  $f_2$ . We have by Green-Stokes formula

$$\begin{aligned} \int_{\mathbb{D}} |f_2(B(z))|^2 \overline{B'(z)} dA(z) &= \frac{1}{2\pi i} \int_{\mathbb{T}} f_2(B(z)) \overline{h(B(z))} dz \\ &= \overline{B'(0)} \sum_{k=0}^{\infty} \frac{|c_k|^2}{k+1} = \overline{B'(0)} \|f_2\|^2. \end{aligned}$$

The last two strings of equalities then imply that

$$\begin{aligned} \|f\|^2 &= \|f_1\|^2 - \frac{|a|^2}{2}\|f_2\|^2 + \|f_2 \circ B\|^2 \\ &= \|f_1\|^2 - \frac{|a|^2}{2}\|f_2\|^2 + \|f_2\|_\mu^2. \quad \blacksquare \end{aligned}$$

As a direct consequence of Lemma 17 we obtain the following lower estimate for the composition operator with symbol  $B$ .

**Corollary 18** Let  $B$  be a Blaschke product of order 2 vanishing at 0 and  $a \in \mathbb{D}$ . Then

$$\|f\|_\mu = \|f \circ B\| \geq \gamma(a)\|f\|$$

for every  $f \in A^2$ , where

$$\gamma(a) = \sqrt{\frac{|a|^2}{2} + \frac{(1 - |a|)^4(1 + |a|^2)}{(1 + |a|)^2}}.$$

**Proof** It suffices to prove the result for polynomials  $f$ . By Lemma 17 and the estimates in (9) and (11),

$$\begin{aligned} \|f \circ B\|^2 - \frac{|a|^2}{2}\|f\|^2 &= \|gf \circ B\|^2 \\ &= \int_{\mathbb{D}} |g(z)|^2 |f(B(z))|^2 dA(z) \\ &\geq (1 - |a|)^2 \int_{\mathbb{D}} |f(B(z))|^2 dA(z) \\ &= (1 - |a|)^2 \int_{\mathbb{D}} |f(z)|^2 \rho(z) dA(z) \\ &\geq \frac{(1 - |a|)^4(1 + |a|^2)}{(1 + |a|)^2} \|f\|^2, \end{aligned}$$

and the desired result follows. ■

As another corollary we extend (13) to the whole space  $A^2$ .

**Corollary 19** Every function  $f \in A^2$  is uniquely represented in the form (12), where  $f_1$  and  $f_2$  are in  $A^2$  and the norms of  $f$ ,  $f_1$ , and  $f_2$  satisfy (13).

**Proof** If  $f$  is a polynomial, (7) shows that  $f_1$  is a polynomial. Since

$$\left\| \frac{1}{\sqrt{2}} B' f_1 \circ B \right\| = \|f_1\|,$$

we have  $\|f_1\| \leq \|f\|$ . Furthermore, if  $z \in \mathbb{D}$ , then  $|g(z)| \geq 1 - |a|$  by (11). Therefore,

$$\|f_2 \circ B\| \leq \frac{1}{1 - |a|} \|f\|.$$

Since

$$|B'(z)| \leq 2 \frac{1 + |a|}{(1 - |a|)^2}$$

for all  $z \in \mathbb{D}$ , we have

$$\|f_2\| \leq 2 \frac{1 + |a|}{(1 - |a|)^3} \|f\|.$$

The desired result now follows from Corollary 16, Lemma 17, and the density of polynomials in  $A^2$ . ■

It follows from Corollary 19 that  $A^2$  is isometric to  $A^2 \times A^2$  equipped with the norm given by (13). We call this space  $A^2_{2,B}$ . The standard norm on  $A^2 \times A^2$  is given by

$$\|(f, g)\|_{A^2 \times A^2} = (\|f\|_{A^2}^2 + \|g\|_{A^2}^2)^{1/2}.$$

The following simple estimate gives the relation between these two norms.

**Proposition 20** *Let  $\gamma$  be the constant defined in Corollary 18. Then*

$$(14) \quad \gamma(a)(1 - |a|)\|(f_1, f_2)\|_{A^2 \times A^2} \leq \|(f_1, f_2)\|_{A^2_{2,B}} \leq \frac{\|(f_1, f_2)\|_{A^2 \times A^2}}{1 - |a|}$$

for all functions  $f_1$  and  $f_2$  in  $A^2$ .

**Proof** Let

$$f = \frac{1}{\sqrt{2}}B'f_1 \circ B + gf_2 \circ B.$$

Since

$$\|(f_1, f_2)\|_{A^2_{2,B}}^2 = \|f\|^2 = \|f_1\|^2 + \int_{\mathbb{D}} |g(z)|^2 |f_2(B(z))|^2 dA(z),$$

the estimate in (11), together with the fact that the operator of composition by  $B$  on  $A^2$  has norm 1 (see [4]), gives us

$$\begin{aligned} \int_{\mathbb{D}} |g(z)|^2 |f_2(B(z))|^2 dA(z) &\leq \frac{1}{(1 - |a|)^2} \int_{\mathbb{D}} |f_2(B(z))|^2 dA(z) \\ &\leq \frac{1}{(1 - |a|)^2} \|f_2\|^2. \end{aligned}$$

This establishes the upper bound in (14). To prove the lower bound we use the other part of (11) and Corollary 18.

$$\begin{aligned} \int_{\mathbb{D}} |g(z)|^2 |f_2(B(z))|^2 dA(z) &\geq (1 - |a|)^2 \int_{\mathbb{D}} |f_2(B(z))|^2 dA(z) \\ &\geq (1 - |a|)^2 \gamma(a)^2 \|f\|^2. \end{aligned} \quad \blacksquare$$

Let  $\mathbb{A}_n$  stand for the space of  $n \times n$  matrices of bounded analytic functions in the unit disk. For

$$A = A(z) = [a_{i,j}(z)]_{i,j=1,2}$$

we define

$$\|A\|_{\mathbb{A}_n} = \sup\{\|A(z)\zeta\|_{\mathbb{C}^n} : z \in \mathbb{D}, \zeta \in \mathbb{C}^n, \|\zeta\|_{\mathbb{C}^n} = 1\},$$

where vectors in  $\mathbb{C}^n$  are written as columns and

$$\|\zeta\|_{\mathbb{C}^n} = \sqrt{|\zeta_1|^2 + \dots + |\zeta_n|^2}$$

for  $\zeta \in \mathbb{C}^n$ .

Since  $A^2$  is isometric to  $A^2_{2,B}$ , it follows immediately that every operator  $S$  on  $A^2$  which commutes with  $T_B$  corresponds to a matrix  $[a_{ij}(z)]_{i,j=1,2}$  of analytic functions in  $\mathbb{D}$ , where

$$(15) \quad S\left(\frac{B'}{\sqrt{2}}\right) = \frac{B'}{\sqrt{2}}a_{11} \circ B + ga_{12} \circ B, \quad S(g) = \frac{B'}{\sqrt{2}}a_{21} \circ B + ga_{22} \circ B.$$

We write this correspondence in the form

$$[a_{ij}] = \mathbb{F}(S).$$

By Lemma 17, the functions  $a_{ij}$  above are bounded in  $\mathbb{D}$ . Thus  $\mathbb{F}$  maps  $(T_B)'$  into  $\mathbb{A}_2$ . The following theorem states that  $\mathbb{F}$  is an isomorphism from  $(T_B)'$  onto  $\mathbb{A}_2$  and gives a norm estimate.

**Theorem 21** *Let  $B$  be a Blaschke product of order 2 which vanishes at 0 and  $a$ . Then the commutant of  $T_B$  is isomorphic to the space  $\mathbb{A}_2$ . The isomorphism  $\mathbb{F}$  is given by (15) and satisfies the estimate*

$$(1 - |a|)^2\gamma(a)\|\mathbb{F}(S)\|_{\mathbb{A}_2} \leq \|S\| \leq \frac{1}{(1 - |a|)^2\gamma(a)}\|\mathbb{F}(S)\|_{\mathbb{A}_2}$$

for all  $S \in (T_B)'$ .

**Proof** Given  $f \in A^2$ , we can write

$$f = \frac{1}{\sqrt{2}}B'f_1 \circ B + gf_2 \circ B.$$

By (14), we have

$$\begin{aligned} \frac{\|Sf\|}{\|f\|} &= \frac{\|\mathbb{F}(S)(f_1, f_2)\|_{A^2_{2,B}}}{\|(f_1, f_2)\|_{A^2_{2,B}}} \\ &\leq \frac{1}{(1 - |a|)^2\gamma(a)} \frac{\|\mathbb{F}(S)(f_1, f_2)\|_{A^2 \times A^2}}{\|(f_1, f_2)\|_{A^2 \times A^2}} \\ &\leq \frac{1}{(1 - |a|)^2\gamma(a)} \|\mathbb{F}(C)\|_{\mathbb{A}_2}. \end{aligned}$$

The lower estimate can be established in a similar way using the other part of (14). ■

Once again, we remark that other representations of the commutant  $(T_B)'$  are possible.

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