Glasgow Math. J. **53** (2011) 313–320. © Glasgow Mathematical Journal Trust 2010.
doi:10.1017/S001708951000073X.

MAXIMUM PRINCIPLES FOR SOME HIGHER-ORDER SEMILINEAR ELLIPTIC EQUATIONS

A. MARENO

Penn State University, Middletown, PA 17057, USA
e-mail: aum24@psu.edu

(Received 31 December 2009; accepted 28 June 2010; first published online 13 December 2010)

Abstract. We deduce maximum principles for fourth-, sixth- and eighth-order elliptic equations by modifying an auxiliary function introduced by Payne (*J. Analyse Math.* 30 (1976), 421–433). Integral bounds on various gradients of the solutions of these equations are obtained.

2010 *Mathematics Subject Classification.* 35B45, 35B50.

1. Introduction. It is well known that one can use classical maximum principles to develop maximum principles for auxiliary functions defined on the solution of certain elliptic equations of the order of higher than 2. In [**6**] Payne considers the semilinear equation

$$\Delta^2 u = f(u) \tag{1}$$

in a bounded domain Ω of $\mathbf{R}^n$. He shows that the maximum value, for auxiliary functions containing the terms $|\nabla^2 u|^2 - u_{,i}\Delta u_{,i}$, and for certain restrictions on $f'(u)$, is achieved on the boundary of the domain. An application of these techniques appears in [**4**] where the author modifies such functions to obtain results similar to those in [**6**] for a semilinear equation from thin plate theory. Other results for semilinear fourth-order equations can be found in [**5, 8, 9, 12**].

There are some works which primarily deal with fourth- and higher-order linear equations. For example, Dunninger in [**3**] developed maximum principles for equations of the form

$$\Delta^2 u - a\Delta u + bu = 0 \text{ in } \Omega, \tag{2}$$

where Ω is a bounded domain in $\mathbf{R}^n$, and other related linear fourth-order equations. In [**7**], the author obtains a maximum principle for a function defined on the solution of the sixth-order equation

$$\Delta^3 u - a\Delta^2 u + b\Delta u - cu = F(x) \tag{3}$$

in a bounded domain Ω of $\mathbf{R}^2$. Finally, in [**1**], the author extends the results in [**7**] to include the variable coefficient case and examines a related eighth-order linear equation of the form

$$\Delta^4 u - a\Delta^3 u + b\Delta^2 u - c\Delta u + du = f(x) \;\text{ in } \Omega \subset \mathbf{R}^2 \tag{4}$$

where Ω is bounded and b may be nonconstant.

One can find additional results for higher-order linear elliptic equations in [**2, 11**].

In this paper we show that one can obtain maximum principle results for the semilinear fourth-, sixth- and eighth-order versions of the constant-coefficient, linear elliptic equations (2), (3), (4), and for the sixth- and eighth-order generalizations of equation (1), by modifying the auxiliary functions used in [**6**]. From these maximum principles we deduce integral bounds on certain gradients of the solutions of several partial differential equations, subject to different boundary conditions.

2. Notation. Throughout this paper, the summation convention on repeated indices is used. Commas will denote partial differentiation. Additionally, we make the following identifications for the squares of the first, second, third, and fourth gradients of w:

$$\begin{aligned}|\nabla w|^2 &= w_{,i}w_{,i}\\ |\nabla^2 w|^2 &= w_{,ij}w_{,ij}\\ |\nabla^3 w|^2 &= w_{,ijk}w_{,ijk}\\ |\nabla^4 w|^2 &= w_{,ijkl}w_{,ijkl}\end{aligned}$$

for indices $i, j, k, l = 1, 2, \dots .n$.

Furthermore, we note that

$$\sum_{i,j,k,l} \text{ will denote } \sum_i \sum_j \sum_k \sum_l \text{ for indices } i, j, k, l = 1, 2, \dots .n.$$

Finally, we assume that Ω is a bounded domain in $\mathbf{R}^n$.

3. Fourth-order equations. Here, we consider the semilinear equation

$$\Delta^2 u - a\Delta u + bf(u) = 0 \text{ in } \Omega. \tag{5}$$

We now use a slight modification of an auxiliary function found in [**6**] to deduce the following result:

THEOREM 1. *Suppose that $u \in C^5(\Omega) \cap C^3(\bar{\Omega})$ is a solution of (5) and a, b are nonnegative constants. If $f \in C^1(\mathbf{R})$ satisfies $f'(u) \geq 0$, then the function*

$$V = |\nabla^2 u|^2 - u_{,i}(\Delta u)_{,i} + b\left(\frac{4-n}{n+2}\right)\int_0^u f(s)ds + \frac{n-4}{2(n+2)}(\Delta u)^2 + \frac{a}{2}|\nabla u|^2$$

takes its maximum value on $\partial\Omega$.

Proof. A straightforward calculation shows that

$$\Delta V = 2|\nabla^3 u|^2 - \frac{6}{n+2}(\Delta u)_{,i}(\Delta u)_{,i} + \frac{6b}{n+2}f'|\nabla u|^2 + a|\nabla^2 u|^2 + a\left(\frac{n-4}{n+2}\right)(\Delta u)^2.$$

Using the well-known inequalities,

$$|\nabla^3 u|^2 \geq \frac{3}{n+2}(\Delta u)_{,i}(\Delta u)_{,i}, \tag{6}$$

$$|\nabla^2 u|^2 \geq \frac{1}{n}(\Delta u)^2 \text{ (see [\textbf{10}])} \tag{7}$$

https://doi.org/10.1017/S001708951000073X Published online by Cambridge University Press

and our assumptions on a, b and $f'(u)$, we see that ΔV is subharmonic in Ω and so the conclusion follows. $\square$

4. Sixth-order equations. In this section of the paper, we deduce maximum principles for the solutions of two classes of semilinear sixth-order partial differential equations. First we introduce the equation

$$\Delta^3 u + f(u) = 0 \text{ in } \Omega. \tag{8}$$

Now we establish the following theorem:

THEOREM 2. *Let u in $C^7(\Omega) \cap C^5(\bar{\Omega})$ be a solution of (8). Suppose that $f \in C^1(\mathbf{R})$ satisfies $f'(u) \leq 0$ and $n \leq 3$. Then the function*

$$P = |\nabla^3 u|^2 + \frac{1}{2} u_{,i}(\Delta^2 u)_{,i} - (\Delta u)_{,ij} u_{,ij} - \frac{1}{4}(\Delta u)_{,i}(\Delta u)_{,i}$$

takes its maximum value on $\partial\Omega$.

The proof of Theorem 2 requires the lemma below which extends inequality (6).

LEMMA 1. *Let w be an arbitrary function in $C^4(\Omega)$. Then*

$$|\nabla^4 w|^2 \geq \frac{6}{n+5}(\Delta w)_{,ij}(\Delta w)_{,ij}. \tag{9}$$

Proof. Let $\epsilon > 0$ be arbitrary. Then

$$\sum_{i,j,k,l} [w_{,ijkl} - \epsilon\{(\Delta w)_{,il}\delta_{jk} + (\Delta w)_{,ij}\delta_{lk} + (\Delta w)_{,ik}\delta_{jl} + (\Delta w)_{,jk}\delta_{il} + (\Delta w)_{,jl}\delta_{ik} + (\Delta w)_{,kl}\delta_{ij}\}]^2 \geq 0.$$

This inequality reduces to the following:

$$w_{,ijkl}w_{,ijkl} - 12\epsilon(\Delta w)_{,ij}(\Delta w)_{,ij} + 6\epsilon^2(n+5)(\Delta w)_{,ij}(\Delta w)_{,ij} \geq 0.$$

Viewing the previous expression as a quadratic expression in ϵ, we see that the discriminant of this expression leads to the inequality

$$|\nabla^4 w|^2 = w_{,ijkl}w_{,ijkl} \geq \frac{6}{n+5}(\Delta w)_{,ij}(\Delta w)_{,ij},$$

as was to be shown.

Now we prove Theorem 2 by showing that P is subharmonic in Ω. First we calculate

$$P_{,l} = 2u_{,ijk}u_{,ijkl} + \frac{1}{2}u_{,il}(\Delta^2 u)_{,i} + \frac{1}{2}u_{,i}(\Delta^2 u)_{,il} - (\Delta u)_{,ij}u_{,ijl} - (\Delta u)_{,ijl}u_{,ij} - \frac{1}{2}(\Delta u)_{,il}(\Delta u)_{,i}.$$

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Therefore,

$$\begin{aligned}\Delta P &= 2u_{,ijkl}u_{,ijkl} + 2u_{,ijk}(\Delta u)_{,ijk} + \frac{1}{2}(\Delta u)_{,i}(\Delta^2 u)_{,i} + u_{,il}(\Delta^2 u)_{,il}\\ &\quad + \frac{1}{2}u_{,i}(\Delta^3 u)_{,i} - (\Delta u)_{,ij}(\Delta u)_{,ij} - (\Delta u)_{,ijl}u_{,ijl} - (\Delta u)_{,ijl}u_{,ijl}\\ &\quad - (\Delta^2 u)_{,ij}u_{,ij} - \frac{1}{2}(\Delta u)_{,il}(\Delta u)_{,il} - \frac{1}{2}(\Delta^2 u)_{,i}(\Delta u)_{,i}\\ &= 2|\nabla^4 u|^2 - \frac{1}{2}f'|\nabla u|^2 - \frac{3}{2}(\Delta u)_{,il}(\Delta u)_{,il}.\end{aligned}$$

Hence by (9) and the assumption that $f'(u) \leq 0$, it follows that ΔP is nonnegative.

The second class of sixth-order equation that we consider contains linearities in both the Laplacian and Bilaplacian of the solution and has the form

$$\Delta^3 u - a\Delta^2 u + b\Delta u - cf(u) = 0 \text{ in } \Omega. \tag{10}$$

We prove: □

THEOREM 3. *Let $u \in C^7(\Omega) \cap C^5(\bar{\Omega})$ be a solution of (10), where the constants $a, b, c \geq 0$. If $f \in C^1(\mathbf{R})$ satisfies $f'(u) \geq 0$ and $n \leq 3$ then the function*

$$\begin{aligned}R &= |\nabla^3 u|^2 + \frac{1}{2}u_{,i}(\Delta^2 u)_{,i} - (\Delta u)_{,ij}u_{,ij} - \frac{1}{4}(\Delta u)_{,i}(\Delta u)_{,i} + \frac{a}{2}|\nabla^2 u|^2\\ &\quad - \frac{a}{2}u_{,i}(\Delta u)_{,i} + \frac{b}{4}|\nabla u|^2\end{aligned}$$

takes on its maximum value on $\partial\Omega$.

Proof. Using our calculation from Theorem 2 and applying the Laplacian to the last three terms in R yield

$$\begin{aligned}\Delta R &= 2|\nabla^4 u|^2 + \frac{c}{2}f'|\nabla u|^2 - \frac{3}{2}(\Delta u)_{,ij}(\Delta u)_{,ij} + \frac{b}{2}|\nabla^2 u|^2 + \frac{b}{2}u_{,i}(\Delta u)_{,i}\\ &\quad + a|\nabla^3 u|^2 + \frac{a}{2}u_{,i}(\Delta^2 u)_{,i} - \frac{b}{2}u_{,i}(\Delta u)_{,i} - \frac{a}{2}u_{,i}(\Delta^2 u)_{,i} - \frac{a}{2}(\Delta u)_{,i}(\Delta u)_{,i}.\\ &= 2|\nabla^4 u|^2 - \frac{3}{2}(\Delta u)_{,ij}(\Delta u)_{,ij} + a|\nabla^3 u|^2 + \frac{c}{2}f'|\nabla u|^2 + \frac{b}{2}|\nabla^2 u|^2\\ &\quad - \frac{a}{2}(\Delta u)_{,i}(\Delta u)_{,i}.\end{aligned}$$

By (6), (9), the nonnegativity of a, b and c and the assumption that $f'(u) \geq 0$ we observe that $\Delta R \geq 0$. □

5. Eighth-order equations. Before we state and prove results for semilinear eighth-order equations, we begin here by proving a maximum principle for some linear eighth-order equations.

The auxiliary function that we introduce below will provide the basis for more complicated auxiliary functions associated with our semilinear equations.

THEOREM 4. *Suppose* $u \in C^9(\Omega) \cap C^7(\bar{\Omega})$ *is a solution of* $\Delta^4 u = 0$. *If* $n \leq 4$, *then the function*

$$S = |\nabla^2(\Delta^2 u)|^2 - (\Delta^2 u)_{,i}(\Delta^3 u)_{,i}$$

takes its maximum value on the boundary of Ω.

Proof. Note that

$$S_{,l} = 2(\Delta^2 u)_{,ij}(\Delta^2 u)_{,ijl} - (\Delta^2 u)_{,il}(\Delta^3 u)_{,i} - (\Delta^2 u)_{,i}(\Delta^3 u)_{,il}.$$

Hence,

$$\begin{aligned}\Delta S &= 2(\Delta^2 u)_{,ijl}(\Delta^2 u)_{,ijl} + 2(\Delta^2 u)_{,il}(\Delta^3 u)_{,il} - (\Delta^3 u)_{,i}(\Delta^3 u)_{,i} \\ &\quad - 2(\Delta^2 u)_{,il}(\Delta^3 u)_{,il} - (\Delta^2 u)_{,i}(\Delta^4 u)_{,i} \\ &= 2(\Delta^2 u)_{,ijl}(\Delta^2 u)_{,ijl} - (\Delta^3 u)_{,i}(\Delta^3 u)_{,i} - (\Delta^2 u)_{,i}(\Delta^4 u)_{,i}.\end{aligned}$$

Using the equation $\Delta^4 u = 0$ and (6) (in which we replace u by $\Delta^2 u$), we see that $\Delta S \geq 0$.

Next, we consider the linear eighth-order equation,

$$\Delta^4 u = cu \text{ in } \Omega, \text{ where the constant } c < 0 \tag{11}$$

and deduce the following theorem. □

THEOREM 5. *Let* $u \in C^9(\Omega) \cap C^7(\bar{\Omega})$ *be a solution of (11). Suppose that* $n \leq 4$. *Then the function*

$$Q = |\nabla^2(\Delta^2 u)|^2 - (\Delta^2 u)_{,i}(\Delta^3 u)_{,i} - c|\nabla^2 u|^2 + cu_{,i}(\Delta u)_{,i}$$

takes its maximum value on the boundary of Ω.

Here, we note that the auxiliary function in Theorem 5 is a combination of the terms in the auxiliary functions for equation (1) and the homogeneous equation of Theorem 4.

Proof. A calculation similar to that of Theorem 4 above yields:

$$\begin{aligned}\Delta Q &= 2(\Delta^2 u)_{,ijk}(\Delta^2 u)_{,ijk} - (\Delta^3 u)_{,i}(\Delta^3 u)_{,i} - c(\Delta^2 u)_{,i}u_{,i} - 2c|\nabla^3 u|^2 \\ &\quad + cu_{,i}(\Delta^2 u)_{,i} + c(\Delta u)_{,i}(\Delta u)_{,i} \\ &= 2(\Delta^2 u)_{,ijk}(\Delta^2 u)_{,ijk} - (\Delta^3 u)_{,i}(\Delta^3 u)_{,i} - 2c|\nabla^3 u|^2 + c(\Delta u)_{,i}(\Delta u)_{,i}.\end{aligned}$$

Finally, we use (6) twice to conclude that $\Delta Q \geq 0$.

Now we consider a semilinear eighth-order equation for which our next theorem is stated. □

THEOREM 6. *Suppose* $u \in C^9(\Omega) \cap C^7(\bar{\Omega})$ *is a solution of the equation* $\Delta^4 u = f(u)$ *and* $n \leq 3$. *Furthermore, suppose that for constants* γ *and* β *and for a function* $f \in C^1(\mathbf{R})$, *the following conditions are satisfied:*

$$(i)\ \ 1 + 2\beta \leq \gamma \leq 1, \ -\frac{1}{5} \leq \frac{\beta}{\gamma} < 0$$

$$(ii)\ \ f' < 0, \ \frac{\gamma}{-(1+2\beta)} \leq f' \leq -(2\beta + \gamma).$$

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Then, the function

$$T = |\nabla^2(\Delta^2 u)|^2 - (\Delta^2 u)_{,i}(\Delta^3 u)_{,i} + \gamma|\nabla^2 u|^2 - \gamma u_{,i}(\Delta u)_{,i} - \frac{\beta}{2}(\Delta^2 u)^2 + \beta(\Delta u)(\Delta^3 u) - \beta\int_0^u f(s)ds$$

takes its maximum value on the boundary of Ω.

Proof. We show that T is subharmonic in Ω. Hence, by a straight-forward calculation, we have

$$\begin{aligned}
\Delta T &= 2(\Delta^2 u)_{,ijk}(\Delta^2 u)_{,ijk} - (\Delta^3 u)_{,i}(\Delta^3 u)_{,i} + 2\gamma|\nabla^3 u|^2 \\
&\quad - \gamma(\Delta u)_{,i}(\Delta u)_{,i} - \gamma u_{,i}(\Delta^2 u)_{,i} - \beta(\Delta^2 u)_{,i}(\Delta^2 u)_{,i} - \beta(\Delta^2 u)(\Delta^3 u) \\
&\quad + \beta(\Delta^2 u)(\Delta^3 u) + \beta(\Delta u)(\Delta^4 u) + 2\beta(\Delta u)_{,i}(\Delta^3 u)_{,i} - \beta f'|\nabla u|^2 \\
&\quad - \beta f(\Delta u) - (\Delta^2 u)_{,i}(\Delta^4 u)_{,i} \\
&= 2|\nabla^3(\Delta^2 u)|^2 - (\Delta^3 u)_{,i}(\Delta^3 u)_{,i} - (f' + \gamma)u_{,i}(\Delta^2 u)_{,i} \\
&\quad + 2\gamma|\nabla^3 u|^2 - \gamma(\Delta u)_{,i}(\Delta u)_{,i} - \beta(\Delta^2 u)_{,i}(\Delta^2 u)_{,i} + 2\beta(\Delta u)_{,i}(\Delta^3 u)_{,i} \\
&\quad - \beta f'|\nabla u|^2 \\
&\geq 2|\nabla^3(\Delta^2 u)|^2 - (1-\beta)(\Delta^3 u)_{,i}(\Delta^3 u)_{,i} \\
&\quad - \left(\frac{f'}{2} + \frac{\gamma}{2} + \beta f'\right)|\nabla u|^2 + \left[-\left(\frac{f'}{2} + \frac{\gamma}{2}\right) - \beta\right](\Delta^2 u)_{,i}(\Delta^2 u)_{,i} \\
&\quad + 2\gamma|\nabla^3 u|^2 - (\gamma - \beta)(\Delta u)_{,i}(\Delta u)_{,i}.
\end{aligned}$$

Using assumptions (i), (ii) and inequality (6), $\Delta T \geq 0$. □

REMARK. We note that functions of the form $f(u) = c_1 tan^{-1}(u) + c_2 u$, for suitable choices of the constants c_1, c_2, γ, β , satisfy the conditions of Theorem 6.

Finally, we state our last theorem for an eighth-order semilinear equation of the form

$$\Delta^4 u - a\Delta^3 u + b\Delta^2 u - c\Delta u + df(u) = 0 \text{ in } \Omega. \tag{12}$$

THEOREM 7. *Let* $u \in C^9(\Omega) \cap C^7(\bar{\Omega})$ *be a solution of (12), where the constants* $a, c \geq 0$, *the constant* $d > 0$, $b \geq \alpha > 0$ *(*α *is a constant),* $f \in C^1(\mathbf{R})$, *and* $n \leq 3$.

Define the function

$$W = |\nabla^2(\Delta^2 u)|^2 - (\Delta^2 u)_{,i}(\Delta^3 u)_{,i} + \frac{a}{2}(\Delta^2 u)_{,i}(\Delta^2 u)_{,i} + \frac{c}{2}(\Delta u)_{,i}(\Delta u)_{,i} + \alpha|\nabla^2 u|^2 - \alpha u_{,i}(\Delta u)_{,i} + g(x)|\nabla u|^2,$$

where $g \in C^2(\Omega) \cap C(\bar{\Omega})$ *is a positive function bounded below by* α. *Then* W *takes its maximum value on the boundary of* Ω, *if the following additional conditions hold:*

$$(i)\ \ 0 \leq f' \leq \frac{\alpha}{d}$$

$$(ii)\ \ \Delta g - \frac{10}{\alpha}g^2 - 4\frac{|\nabla g|^2}{g} \geq 0.$$

We leave the details of this proof to the reader.

6. Applications. Here we give an application of Theorem 2 by considering the following boundary value problem for functions $u \in C^7(\Omega) \cap C^5(\bar{\Omega})$:

$$\Delta^3 u + f(u) = 0 \text{ in } \Omega,$$
$$u = \frac{\partial u}{\partial n} = \frac{\partial^2 u}{\partial n^2} = 0 \text{ on } \partial\Omega.$$

Using Theorem 2 we easily deduce that

$$P \le \max_{\partial\Omega} \left[|\nabla^3 u|^2 - \frac{1}{4}(\Delta u)_{,i}(\Delta u)_{,i} \right].$$

Now we deduce integral bounds on $|\nabla^3 u|^2$. We integrate in a term by term manner the left side of this equation, using integration by parts and apply the boundary conditions above:

$$\int_\Omega u_{,ijk}u_{,ijk}dx = \int_{\partial\Omega} u_{,ij}n_k u_{,ijk}ds - \int_\Omega u_{,ij}(\Delta u)_{,ij}dx$$

and so

$$\int_\Omega u_{,ijk}u_{,ijk}dx - \int_\Omega u_{,ij}(\Delta u)_{,ij}dx = -2\int_\Omega u_{,ij}(\Delta u)_{,ij}dx$$
$$= -2\int_{\partial\Omega} u_{,i}n_j(\Delta u)_{,ij}ds + 2\int_\Omega u_{,i}(\Delta^2 u)_{,i}dx.$$

Thus,

$$\int_\Omega u_{,ijk}u_{,ijk}dx - \int_\Omega u_{,ij}(\Delta u)_{,ij}dx + \frac{1}{2}\int_\Omega u_{,i}(\Delta^2 u)_{,i}dx = \frac{5}{2}\int_\Omega u_{,i}(\Delta^2 u)_{,i}dx$$
$$= \frac{5}{2}\int_{\partial\Omega} u_{,i}n_i(\Delta^2 u)ds - \frac{5}{2}\int_\Omega (\Delta u)(\Delta^2 u)dx$$
$$= -\frac{5}{2}\int_\Omega (\Delta u)(\Delta^2 u)dx = -\frac{5}{2}\int_{\partial\Omega} (\Delta u)(\Delta u)_{,i}n_i ds + \frac{5}{2}\int_\Omega (\Delta u)_{,i}(\Delta u)_{,i}dx$$
$$= \frac{5}{2}\int_\Omega (\Delta u)_{,i}(\Delta u)_{,i}dx.$$

Gathering all terms in P together and integrating, we see that

$$\int_\Omega u_{,ijk}u_{,ijk}dx - \int_\Omega u_{,ij}(\Delta u)_{,ij}dx + \frac{1}{2}\int_\Omega u_{,i}(\Delta^2 u)_{,i}dx - \frac{1}{4}\int_\Omega (\Delta u)_{,i}(\Delta u)_{,i}dx$$
$$= \frac{3}{2}\int_\Omega (\Delta u)_{,i}(\Delta u)_{,i}dx.$$

A further computation using integration by parts shows that

$$\int_\Omega |\nabla^3 u|^2 dx = \int_\Omega (\Delta u)_{,i}(\Delta u)_{,i}dx$$

and so finally we deduce the following integral bound on the square of the third gradient of u:

$$\int_\Omega |\nabla^3 u|^2 dx \le \frac{2}{3}\left[\max_{\partial\Omega}\left(|\nabla^3 u|^2 dx - \frac{1}{4}(\Delta u)_{,i}(\Delta u)_{,i}\right)\right]\text{Area}(\Omega).$$

Here, we just briefly mention an application of say Theorem 5 in which some integral bounds can be obtained.

We consider the following boundary value problem for functions $u \in C^9(\Omega) \cap C^7(\bar{\Omega})$:

$$\begin{aligned}\Delta^4 u &= cu \text{ in } \Omega, \text{ where } c < 0,\\ u &= \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,\\ \Delta^2 u &= \frac{\partial(\Delta^2 u)}{\partial n} = 0 \text{ on } \partial\Omega.\end{aligned}$$

Using integration by parts we obtain the following integral inequality:

$$\int_\Omega (|\nabla^2(\Delta^2 u)|^2 - c|\nabla^2 u|^2)dx \le \frac{1}{2}\max_{\partial\Omega}(|\nabla^2(\Delta^2 u)|^2 - c|\nabla^2 u|^2)\text{Area}(\Omega).$$

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