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STRESS-GRADIENT COUPLING IN GLACIER FLOW: IV. EFFECTS OF THE "T" TERM*

By BARCLAY KAMB *and* KEITH A. ECHELMEYER†

(Division of Geological and Planetary Sciences, California Institute of Technology, Pasadena, California 91125, U.S.A.)

ABSTRACT. The "T term" in the longitudinal stress-equilibrium equation for glacier mechanics, a double y-integral of $\partial^2\tau_{xy}/\partial x^2$ where x is a longitudinal coordinate and y is roughly normal to the ice surface, can be evaluated within the framework of longitudinal flow-coupling theory by linking the local shear stress τ_{xy} at any depth to the local shear stress τ_B at the base, which is determined by the theory. This approach leads to a modified longitudinal flow-coupling equation, in which the modifications deriving from the T term are as follows: 1. The longitudinal coupling length ℓ is increased by about 5%. 2. The asymmetry parameter σ is altered by a variable but small amount depending on longitudinal gradients in ice thickness h and surface slope α. 3. There is a significant direct modification of the influence of local h and α on flow, which represents a distinct "driving force" in glacier mechanics, whose origin is in pressure gradients linked to stress gradients of the type $\partial\tau_{xy}/\partial x$. For longitudinal variations in h, the "T force" varies as d^2h/dx^2 and results in an in-phase enhancement of the flow response to the variations in h, describable (for sinusoidal variations) by a wavelength-dependent enhancement factor. For longitudinal variations in α, the "force" varies as $d\alpha/dx$ and gives a phase-shifted flow response. Although the "T force" is not negligible, its actual effect on τ_B and on ice flow proves to be small, because it is attenuated by longitudinal stress coupling. The greatest effect is at shortest wavelengths ($\lambda \lesssim 2.5h$), where the flow response to variations in h does not tend to zero as it would otherwise do because of longitudinal coupling, but instead, because of the effect of the "T force", tends to a response about 4% of what would occur in the absence of longitudinal coupling. If an effect of this small size can be considered negligible, then the influence of the T term can be disregarded. It is then unnecessary to distinguish in glacier mechanics between two length scales for longitudinal averaging of τ_B, one over which the T term is negligible and one over which it is not.

Longitudinal flow-coupling theory also provides a basis for evaluating the additional datum-state effects of the T term on the flow perturbations Δu that result from perturbations Δh and $\Delta\alpha$ from a datum state with longitudinal stress gradients. Although there are many small effects at the ~1% level, none of them seems to stand out significantly, and at the 10% level all can be neglected.

The foregoing conclusions apply for long wavelengths $\lambda \gtrsim h$. For short wavelengths ($\lambda \lesssim h$), effects of the T term become important in longitudinal coupling, as will be shown in a later paper in this series.

RÉSUMÉ. *Couplage du gradient de contrainte dans l'écoulement des glaciers: IV. Effets du terme "T".* Dans l'équation qui, en mécanique des glaciers, décrit l'équilibre des contraintes dans le sens longitudinal (intégrale double de $\partial^2\tau_{xy}/\partial x^2$ où x est la coordonnée dans le sens longitudinal, et y est approximativement selon la normale à la surface de la glace) le terme T peut être évalué, dans le cadre de la théorie du couplage longitudinal des écoulements en reliant la contrainte de cisaillement locale τ_{xy} à une profondeur donnée, à la cission à la base τ_B, déterminée par la théorie. Cete approche conduit à une équation du couplage longitudinal modifiée. Les modifications provenant du terme T sont les suivantes: (1) la longueur de couplage longitudinal ℓ est accrue de 5% environ; (2) le paramètre d'asymétrie σ subit un faible changement qui dépend des gradients longitudinaux de l'épaisseur de glace h et de la pente de surface α; (3) il y a une modification significative de l'influence des valeurs locales de h et α sur l'écoulement qui représente une force motrice distincte dans la mécanique du glacier due aux gradients de pression liés aux gradients de contraintes en $\partial\tau_{xy}/\partial x$. Concernant les variations longitudinales de h, la "force T" varie comme d^2h/dx^2 et produit une amplification en phase de la réponse de l'écoulement aux variations d'épaisseur: dans le cas de variations sinusoidales cette amplification peut être décrite par un gain fonction de la longueur d'onde. Quant aux variations longitudinales de α, la "force T" varie comme $d\alpha/dx$ et conclut d'un déphasage des variations de l'écoulement. Bien que la "force T" ne soit pas négligeable, son effet réel sur τ_B et sur l'écoulement demeure faible, car celui-ci est atténué par le couplage des contraintes longitudinales. L'effet le plus important a lieu pour les longueurs d'onde les plus courtes ($\lambda \lesssim 2{,}5h$): la réponse de l'écoulement aux variations de h ne tend pas vers zéro (ce qui serait la conséquence du couplage longitudinal) mais, conséquence de la "force T", représente environ 4% de la réponse qui aurait lieu en l'absence de tout couplage longitudinal. Si l'effet de ces faibles longueurs d'ondes peut être négligé, alors l'influence du terme T peut être omise. Dans ce cas, en vue du calcul du frottement moyen dans le sens longitudinal, il est inutile de distinguer entre une échelle à laquelle le terme T est négligeable, et une à laquelle il ne l'est pas.

La théorie du couplage longitudinal de l'écoulement fournit une base pour l'évaluation des effets additionnels du terme T sur la perturbation de l'écoulement Δu causée par des perturbations Δh et $\Delta\alpha$ à partir d'un état de référence comportant des gradients de contrainte longitudinaux. Bien qu'il y ait de nombreux effets de l'ordre du 1%, aucun ne semble émerger de façon significative, et tous peuvent être négligés à un niveau de précision de l'ordre de 10%.

Les conclusions ci-dessus sont valables aux longueurs d'onde $\lambda \gtrsim h$. Pour les très courtes longueurs d'onde ($\lambda \lesssim h$), les effets du terme T deviennent importants en couplage longitudinal, ceci sera exposé dans un futur article du journal.

ZUSAMMENFASSUNG. *Kopplung von Spannungsgradienten im Gletscherfluss: IV. Auswirkungen des "T-Terms".* Der "T-Term" in der Gleichung des longitudalinen Spannungsgleichgewichts für die Gletschermechanik – ein y-Doppelintegral von $\partial^2\tau_{xy}/\partial x^2$, worin x eine Koordinate in

*Contribution No. 4117, Division of Geological and Planetary Sciences, California Institute of Technology, Pasadena, California 91125, U.S.A.

†Present address: Geophysical Institute, University of Alaska, Fairbanks, Alaska 99775, U.S.A.

Längsrichtung und y eine der x-Achse senknechte, ungefähr anfwärts gerichtete Koordinate ist – kann in Rahmen der Theorie der longitudinalen Fluss-Kopplung durch Verbindung der lokalen Scherspannung τ_{xy} in jeder Tiefe mit der lokalen Scherspannung τ_B am Untergrund, die aus der Theorie hervorgeht, ausgewertet werden. Dieses Vorgehen führt zu einer modifizierten Gleichung der longitudinalen Fluss-Kopplung, in der die durch den T-Term verursachten Änderungen die folgenden sind: 1. Die longitudinale Kopplungslänge ℓ wird etwa 5% grösser. 2. Der Asymmetrie-Parameter σ wird durch einen variablen, aber kleinen Betrag verändert, der von den Längsgradienten der Eisdicke h und der Oberflächenneigung α abhängt. 3. Es besteht eine wesentliche direkte Änderung des Einflusses der lokalen Grössen h und α auf den Fluss, die eine deutliche "Triebkraft" in der Gletschermechanik darstellt; sie stammt von Druckgradienten in Verbindung mit Spannungsgradienten des Typs $\partial\tau_{xy}/\partial x$. Für longitudinale Schwankungen in h ändert sich die "T-Kraft" mit d^2h/dx^2 und liefert eine phasengleiche Verstärkung der Flussreaktionen auf die Schwankungen von h, beschreibbar (bei sinusförmigen Schwankungen) durch einen von der Wellenlänge abhängigen Verstärkungsfaktor. Für longitudinale Schwankungen in α ändert sich die Kraft mit $d\alpha/dx$ und ergibt eine phasenversetzte Flussreaktion. Obwohl die "T-Kraft" nicht vernachlässigbar ist, erweist sich ihr tatsächlicher Einfluss auf τ_B und den Eisfluss als gering, weil sie durch longitudinale Spannungskopplung abgeschwächt wird. Die grösste Wirkung tritt bei den kürzesten Wellenlängen ($\lambda \lesssim 2,5h$) ein, wo die Flussreaktion auf Schwankungen in h nicht gegen Null geht, wie sie es sonst infolge longitudinaler Kopplung tun würde, sondern statt dessen – infolge der Wirkung der "T-Kraft" – gegen eine Reaktion tendiert, die etwa 4% von dem beträgt, was bei Fehlen der longitudinalen Kopplung auftreten würde. Wenn eine Wirkung solch geringen Ausmasses vernachlässigbar ist, kann auch der Einfluss des T-Terms ausser Betracht bleiben. Es erübrigt sich dann, in der Gletschermechanik zwischen zwei Längsskalen für die Mittelung von τ_B in Längsrichtung zu unterscheiden, für die der T-Term jeweils vernachlässigbar ist oder nicht.

Die Theorie der longitudinalen Flusskopplung liefert auch eine Grundlage für die Auswirkungen der zusätzlichen Zustandsauswirkungen des T-Terms auf die Flussstörungen Δu, die von Störungen Δh und $\Delta\alpha$ gegenüber einem Ausgangszustand mit longitudinalen Spannungsgradienten herrühren. Obwohl es hier viele kleine Wirkungen auf dem ~1%-Niveau gibt, scheint keine davon signifikant auszuschlagen; auf dem 10%-Niveau können sie alle vernachlässigt werden.

Die vorstehenden Schlussfolgerungen gelten für grosse Wellenlängen $\lambda \gtrsim h$. Für kurze Wellenlängen ($\lambda \lesssim h$) haben die Wirkungen des T-Terms hingegen Bedeutung für die Kopplung in Längsrichtung, wie in einem weiteren Beitrag dieser Serie gezeigt werden wird.

1. INTRODUCTION

In the "vertically" integrated stress-equilibrium equation for a limitlessly wide glacier or ice sheet flowing in plane strain over a basal surface of longitudinally varying slope, as discussed by Kamb (1986), there occurs a term often designated "T" and given by

$$T = \int_{y_B}^{y_S} dy \int_{y}^{y_S} \frac{\partial^2\tau_{xy}}{\partial x^2}\, dy' \tag{1}$$

where x is a longitudinal coordinate, y is generally upward normal to x, and y_B and y_S are the y coordinates of the bed and surface. This term was formulated and discussed in a series of papers by Budd (1968, 1970[a], [b], 1971), who called it the "variational stress" term. Its role in longitudinal stress equilibrium as presently understood has been summarized by Nye (1969, p. 212), Raymond (1978, p. 808; 1980, p. 104), and Paterson (1981, p. 100 and 164), who quoted Budd's conclusion that T is negligible when averaged longitudinally over distances greater than $3h$–$4h$, where h is the ice thickness. On the other hand, Budd (1971, p. 186) stated, on the basis of a calculation of flow over sinusoidal bedrock topography of wavelength λ, that "for wavelengths $\lambda = 4h$ or less the T term cannot be neglected".

In the theory of longitudinal stress-gradient coupling developed by Kamb and Echelmeyer (1986) in Part I it was found that glacier flow is governed by a longitudinal average of ice thickness and surface slope weighted in such a way that most of the contribution to the average comes from a longitudinal interval $2\ell \approx 4h$ or somewhat greater (see Kamb and Echelmeyer, 1986; i.e. Part I, section 5). From the quotations above, it is uncertain whether the T term should have a significant effect on the longitudinal averaging or not. The present paper addresses this issue within the framework of the theory developed by Echelmeyer and Kamb (1986), and Kamb and Echelmeyer (1986), which provides a new way to approach the question of the effects of the T term on longitudinal stress equilibrium and flow. In sections 3 and 4, the flow effects are analyzed at the level of approximation of Part I and interpreted in terms of their consequences for the flow of glaciers and ice sheets generally, while in sections 7 and 8 they are treated at the higher level of approximation of Part II, leading to their implications for the flow response of glaciers to small perturbations in ice thickness and surface slope.

2. EFFECT OF T ON LONGITUDINAL EQUILIBRIUM

The effect of the T term in the longitudinal equilibrium equation (23) of Part III (Kamb, 1986) (hereafter designated (III-23)) can be determined by introducing into Equation (1) the longitudinal variation in τ_{xy} that necessarily arises in the context of the theory in Parts I and II when the basal shear stress τ_B varies with x. Consistent with the assumptions in equations (I-1) and (I-2) (from Part I) as to how flow and stress are related, the longitudinal variation in τ_{xy} is linked to the longitudinal variation in τ_B by assuming that τ_{xy} varies linearly with y from the bed to the surface, as it would exactly if the longitudinal stress gradient $\partial\tau_{xx}/\partial x$ were independent of y. Hence we take

$$\tau_{xy}(x,y) = \tau_B(x)\,\frac{y_S(x) - y}{h(x)}\,. \tag{2}$$

$\tau_B(x)$ is the quantity whose departure from the value $\rho g h\alpha$ (where α is the local slope) the theory in Part I implicitly calculates from the effect of longitudinal stress gradients. By making $\tau_{xy}(y_S) = 0$ in Equation (2) we neglect the small amount, of order δ, by which it differs from 0 because the surface is inclined at angle δ (assumed small) to the x-axis. Figure 1 in Part III shows the flow geometry and the angles $\delta = +dy_S/dx$ and $\theta = +dy_B/dx$, the latter being the slope of the bed relative to the x-axis, also assumed small.

If we now perform on Equation (2) the x differentiations and then the y integrations indicated in Equation (1), and define $dh/dx = \chi \cong +(\delta - \theta)$, we obtain after some manipulation

$$T = \frac{1}{6}h^2\,\frac{d^2\tau_B}{dx^2} - \frac{1}{3}h(3\delta + \chi)\,\frac{d\tau_B}{dx} - \frac{1}{6}h\left[3\,\frac{d\delta}{dx} + \frac{d\chi}{dx}\right]\tau_B + \frac{1}{3}\chi(3\delta + \chi)\tau_B\,. \tag{3}$$

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We then introduce Equation (3) into the equilibrium equation (III-23), obtaining, after re-arrangement

$$2\frac{\mathrm{d}}{\mathrm{d}x}(h\bar{\tau}'_{xx}) + \frac{1}{6}h^2\frac{\mathrm{d}^2\tau_B}{\mathrm{d}x^2} - \frac{1}{3}h(3\delta + \chi)\frac{\mathrm{d}\tau_B}{\mathrm{d}x} + \rho g h\alpha + B + K =$$
$$= \tau_B\left[1 + 2\sin^2\theta - \frac{1}{3}\chi(3\delta + \chi) + \frac{1}{6}h\left[3\frac{\mathrm{d}\delta}{\mathrm{d}x} + \frac{\mathrm{d}\chi}{\mathrm{d}x}\right]\right] \tag{4a}$$

If we make the small-angle approximation $\delta \sim \chi << 1$, then, as discussed in Part III, section 9, the terms B and K can be dropped, and the right-hand side of Equation (4a) reduces to

$$\cong \tau_B\left[1 + \frac{1}{6}h\left[\frac{\mathrm{d}\delta}{\mathrm{d}x} + \frac{\mathrm{d}\chi}{\mathrm{d}x}\right]\right]. \tag{4b}$$

3. EFFECT ON THE LONGITUDINAL FLOW COUPLING EQUATION OF PART I

We now proceed from Equation (4) to the longitudinal flow-coupling equation by the same steps used in Part I, sections 2 and 3, starting from Equation (I-4). First, we calculate the derivatives of τ_B in terms of the flow $\bar{u}$ from Equation (I-1):

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}x} = \frac{\tau_B}{n}\left[\frac{1}{\bar{u}}\frac{\mathrm{d}\bar{u}}{\mathrm{d}x} - \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}\right] \tag{5}$$

$$\frac{\mathrm{d}^2\tau_B}{\mathrm{d}x^2} = \frac{1}{n}\left[\frac{\mathrm{d}\tau_B}{\mathrm{d}x} + \tau_B\frac{\mathrm{d}}{\mathrm{d}x}\right]\left[\frac{1}{\bar{u}}\frac{\mathrm{d}\bar{u}}{\mathrm{d}x} - \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}\right]. \tag{6}$$

Then we apply the perturbation treatment described in Part I, section 3, putting $\nu = (u-u_0)/u_0$. This treatment involves a datum state in which all the derivatives in Equations (5) and (6) are zero. Thus the perturbed forms of Equations (5) and (6) are

$$\frac{\mathrm{d}\tau_B}{\mathrm{d}x} = \frac{\tau_0}{n}\left[\frac{\mathrm{d}\nu}{\mathrm{d}x} - \frac{\chi}{h_0}\right], \tag{7}$$

$$\frac{n}{\tau_0}\frac{\mathrm{d}^2\tau_B}{\mathrm{d}x^2} = \frac{\mathrm{d}^2\nu}{\mathrm{d}x^2} - \frac{n-1}{n}\left[\frac{\mathrm{d}\nu}{\mathrm{d}x}\right]^2 - \frac{2\chi}{nh_0}\frac{\mathrm{d}\nu}{\mathrm{d}x} + \frac{n+1}{n}\frac{\chi^2}{h_0^2} - \frac{1}{h_0}\frac{\mathrm{d}\chi}{\mathrm{d}x} \tag{8}$$

where $\tau_0 = (u_0/c_1h_0)^{1/n}$. When Equations (7) and (8) are substituted into Equation (4), we obtain the longitudinal flow-coupling equation with inclusion of the effects of the T term:

$$-4\bar{u}_0\frac{\mathrm{d}}{\mathrm{d}x}\left[h\bar{\eta}\frac{\mathrm{d}\nu}{\mathrm{d}x}\right] - \frac{h_0^2}{6n}\frac{\mathrm{d}^2\nu}{\mathrm{d}x^2} + \frac{1}{n}\left[\frac{n+1}{3n}\chi + \delta\right]\tau_0h_0\frac{\mathrm{d}\nu}{\mathrm{d}x} + \frac{n-1}{6n^2}\tau_0h_0^2\left[\frac{\mathrm{d}\nu}{\mathrm{d}x}\right]^2 +$$
$$+\frac{\tau_0}{n}\nu = \frac{\rho g}{h_0^{1/n}}\Delta\left[\alpha h^{1+(1/n)}\right] - \frac{\tau_0h_0}{6n^2}\left[3n^2\frac{\mathrm{d}\delta}{\mathrm{d}x} + n(n+1)\frac{\mathrm{d}\chi}{\mathrm{d}x} + \chi^2 + 3n(\chi+2\delta)\chi\right]. \tag{9}$$

In the bracketed coefficient on the right in Equation (9) the second-order terms in χ and δ can be neglected at our level of approximation, but the curvature terms have to be retained at this level. If we assume that since ν is treated as a small perturbation, the term in $(\mathrm{d}\nu/\mathrm{d}x)^2$ can be neglected, and if we multiply through by n/τ_0 in Equation (9) and gather terms, we obtain the equivalent of the longitudinal flow-coupling equation (I-10), corrected by addition of terms coming from T:

$$-\left[\ell^2 + \frac{1}{6}h_0^2\right]\frac{\mathrm{d}^2\nu}{\mathrm{d}x^2} - \left[\ell^2\frac{\mathrm{d}\ln(h\bar{\eta})}{\mathrm{d}x} - h_0\left[\frac{n+1}{3n}\frac{\mathrm{d}h}{\mathrm{d}x} + \delta\right]\right]\frac{\mathrm{d}\nu}{\mathrm{d}x} + \nu =$$
$$= \Delta\ln(\alpha^n h^{n+1}) - \frac{1}{6}h_0\left[3n\frac{\mathrm{d}\delta}{\mathrm{d}x} + (n+1)\frac{\mathrm{d}^2h}{\mathrm{d}x^2}\right] \tag{10}$$

where ℓ is as defined in equation (I-11). Here we have replaced χ by $\mathrm{d}h/\mathrm{d}x$.

For a convenient overview we can rewrite Equation (10) in the condensed form

$$-(\ell')^2\frac{\mathrm{d}^2\nu}{\mathrm{d}x^2} - 2\sigma'\ell'\frac{\mathrm{d}\nu}{\mathrm{d}x} + \nu = F(x) + C(x) \tag{11}$$

where

$$\ell' = \sqrt{\ell^2 + \frac{1}{6}h_0^2}, \tag{12}$$

$$2\sigma'\ell' = \frac{\mathrm{d}\ln(h\bar{\eta})}{\ell^2\,\mathrm{d}x} - \frac{n+1}{3n}h_0\frac{\mathrm{d}h}{\mathrm{d}x} - h_0\delta \tag{13}$$

$$F(x) = \Delta\ln(\alpha^n h^{n+1}), \tag{14}$$

$$C(x) = -\frac{n+1}{6}h_0\frac{\mathrm{d}^2h}{\mathrm{d}x^2} - \frac{n}{2}h_0\frac{\mathrm{d}\alpha}{\mathrm{d}x}. \tag{15}$$

4. EVALUATION OF EFFECTS OF THE T TERM

The effects of the T term on the longitudinally coupled flow are exhibited by Equation (11), which is similar in form to equation (I-10), with modifications as follows:

1. The longitudinal coupling length ℓ is increased to ℓ' as given by Equation (12). In view of the ℓ/h_0 values

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given in Part I, table I, this increase is a small effect, amounting at most to a few per cent.

2. The asymmetry parameter σ, given by equation (I-12), is changed to σ' in Equation (13). To assess easily the effect of this change, it is helpful to define by σ_h the contribution to σ from the longitudinal gradient of ice thickness in equation (I-12),

$$\sigma_h = \frac{1}{2}\ell \frac{\mathrm{d}\ln h}{\mathrm{d}x} = \frac{1}{2}\frac{\ell}{h}\frac{\mathrm{d}h}{\mathrm{d}x} \tag{16a}$$

which is what was used in calculating σ values for Part II, table I, and also to define a similarly scaled quantity deriving from the surface topography

$$\sigma_\alpha = \frac{1}{2}\frac{\ell}{h}\delta = -\frac{1}{2}\frac{\ell}{h}\frac{\mathrm{d}y_s}{\mathrm{d}x}. \tag{16b}$$

With Equations (16a) and (16b), σ' from Equation (13) can be expressed as

$$\sigma' = \frac{\ell}{\ell'}\left[\sigma - \left(\frac{h_0}{\ell}\right)^2\left(\frac{n+1}{3n}\sigma_h + \sigma_\alpha\right)\right]. \tag{17}$$

Since σ, σ_h, and σ_α are of the same order of size, and since $\ell/\ell' \approx 1$ while h_0/ℓ is a fraction of unity (Part I, table I), σ' differs from σ by only a modest fraction, in general.

3. The $C(x)$ term on the right in Equation (11) is a forcing term that affects the flow via longitudinal averaging in a way similar to the effects of slope and thickness from the $F(x)$ term on the right, already treated in Part I. We may call it the "T force". It arises from longitudinal curvature of the surface and/or bed. The form of $C(x)$ in Equation (15) is obtained from Equation (10) by noting that $\alpha = \delta + \gamma$ and $\mathrm{d}\gamma/\mathrm{d}x = 0$, γ being the slope of the x-axis. To assess the magnitude of the effects of $C(x)$ it is helpful to expand Equation (14) and re-group the terms on the right side of Equation (11) as follows:

$$F(x) + C(x) = (n+1)\left[\frac{\Delta h}{h_0} - \frac{1}{6}h_0\frac{\mathrm{d}^2\Delta h}{\mathrm{d}x^2}\right] + n\left[\frac{\Delta\alpha}{\alpha_0} - \frac{1}{2}h_0\frac{\mathrm{d}\Delta\alpha}{\mathrm{d}x}\right]$$
$$= \qquad H(x) \qquad + \qquad A(x) \tag{18}$$

where $H(x)$ is a forcing term due to thickness variations and $A(x)$ is a forcing term due to slope variations. Equation (18) utilizes the fact that for the perturbation considered, $\mathrm{d}h/\mathrm{d}x = \mathrm{d}\Delta h/\mathrm{d}x$ and $\mathrm{d}\alpha/\mathrm{d}x = \mathrm{d}\Delta\alpha/\mathrm{d}x$.

If we take harmonic variations of slope and thickness

$$\Delta\alpha = \alpha_1 \sin\frac{2\pi}{\lambda}x, \qquad \Delta h = h_1 \sin\left[\frac{2\pi}{\lambda}x - \phi\right] \tag{19}$$

(where the phase shift ϕ is arbitrary), then

$$H(x) = \left[1 + \frac{1}{6}\left(\frac{2\pi h_0}{\lambda}\right)^2\right](n+1)\frac{h_1}{h_0}\sin\left[\frac{2\pi}{\lambda}x - \phi\right], \tag{20a}$$

$$A(x) = n\frac{\alpha_1}{\alpha_0}\left[\sin\frac{2\pi}{\lambda}x - \pi\frac{\alpha_0 h_0}{\lambda}\cos\frac{2\pi}{\lambda}x\right]. \tag{20b}$$

From Equation (20a) we see that the effect of the T term is to enhance $H(x)$ by the factor $1 + (2\pi h_0/\lambda)^2/6$. This enhancement of the influence of longitudinal variations in ice thickness on flow is a new effect, not previously recognized in glacier-flow mechanics. Its physical basis is subtle and not immediately obvious but it can be understood in detail by tracing its origin back to the terms in the equilibrium equations from which it springs. The enhancement factor is 2 for $\lambda = 2.6h$, while for $\lambda = 8.1h$ the factor has fallen to 1.1. The enhancement cannot be considered negligible at the 10% level except for $\lambda \gtrsim 8h$. However, the effect of longitudinal stress-gradient coupling in attenuating the flow variations at short wavelengths works to suppress strongly the flow variations that tend to be enhanced by the "T force". Thus, from Part I, section 3, longitudinal averaging causes attenuation of the flow response to Equation (20) by a factor $(1 + (2\pi\ell/\lambda)^2)^{-1}$, so that the response to $H(x)$ in Equation (20a) is

$$v(x) = \frac{1 + \frac{1}{6}\left(\frac{2\pi h_0}{\lambda}\right)^2}{1 + \left(\frac{2\pi\ell}{\lambda}\right)^2}(n+1)\frac{h_1}{h_0}\sin\left[\frac{2\pi}{\lambda}x - \phi\right]. \tag{21a}$$

Since $\ell \approx 2h$ (see Part I), Equation (21a) indicates that when the enhancement factor due to the T term is 1.4, the attenuation factor due to longitudinal averaging is 0.1, which nearly eliminates the response wave.

In the short-wavelength limit, as $\lambda \to 0$, the response in Equation (21a) tends to

$$v(x) \to \frac{1}{6}\left(\frac{h_0}{\ell}\right)^2 (n+1)\frac{h_1}{h_0}\sin\left[\frac{2\pi}{\lambda}x - \phi\right]. \tag{21b}$$

This short-wavelength response is $(h_0/\ell)^2/6 \approx 0.04$ times the flow response that would occur if there were no longitudinal coupling or if the thickness perturbation h_1 were at very long wavelength. Although the treatment cannot be considered to remain valid as $\lambda \to 0$, there is an indication here of a small but fundamental effect of the "T force" in counteracting longitudinal coupling's otherwise complete suppression of flow response at short wavelengths. If, however, a flow response at the 5% level is neglected, then this effect of the "T force" can be disregarded.

A second, more practically oriented reason for disregarding this effect is that the percentage variations in ice thickness at relatively short wavelengths are typically small compared to the percentage variations in slope, so that most of the longitudinal variation in the input function $F(x) + C(x)$ comes from the slope function $A(x)$ rather than from the thickness function $H(x)$. Thus, for Variegated Glacier (Part I, fig. 8), the variations in $h(x)$ at wavelengths $\lesssim$ 1000 m are less than about 5%, whereas the variations in $\alpha(x)$ are 20–100%. A large variation in $h(x)$ occurs over a half wavelength equal to the length of the glacier, of course, but for this variation $\lambda/h \sim 100$ and the T-term correction to $H(x)$ in Equations (18) or (20a) is completely negligible. Blue Glacier (Part I, fig. 10) is similar except that there is a large variation in h connected with the ice fall, over a length scale ~1 km.

The T-term correction to the slope-input function $A(x)$ in Equation (20b), contained in the second term on the right, is a wave shifted 90° in phase relative to the direct input from $F(x)$. This flow effect is again new and has a physical basis very similar to the T-term enhancement of $H(x)$ in Equation (20a). Again, from Part I, section 3, the flow response to $A(x)$ in Equation (20b) will be

$$v(x) = \frac{\sqrt{1 + \left[\pi\alpha_0 \frac{h_0}{\lambda}\right]^2}}{1 + \left[\frac{2\pi\ell}{\lambda}\right]^2} \, n \, \frac{\alpha_1}{\alpha_0} \sin \left[\frac{2\pi}{\lambda}x - \psi\right] \tag{22}$$

where $\psi = \tan^{-1} (\pi\alpha_0 h_0/\lambda)$. The "$T$ force" causes enhancement of the response by the square-root factor in the numerator, and longitudinal coupling superimposes an attenuation factor, the reciprocal of the quantity in the denominator. For Variegated Glacier, with $\alpha_0 \approx 0.1$ and $\lambda \approx 950$ m $\approx 2.7h$ for the typical longitudinal oscillations in α (Part I, fig. 8), the enhancement factor is only 1.01, and the attenuation factor is 1/23. For $\lambda = 1.3h$, corresponding to the shortest wavelength oscillations in α, the enhancement factor is still only 1.03, while the attenuation factor is 1/94. Thus the effects of enhancement of $A(x)$ due to the "T force" appear quite negligible. Even for a large mean slope $\alpha_0 \sim 0.5$, the effects are probably not large enough to be recognizable in an observed flow response.

The above conclusions, based on the theoretical results for harmonic perturbations in Equations (20)–(22), are tested in a practical way by direct calculation of the effect of the forcing term $C(x)$ on the flow curves of Variegated Glacier and Blue Glacier that were obtained in Part I, sections 8 and 9, with $C(x)$ omitted. From the functions $h(x)$ and $\alpha(x)$ used there, we calculate $C(x)$ from Equation (15), add it to the input function $F(x)$ used previously, and carry out the longitudinal averaging by the same method and with the same parameter values used in Part I, sections 8 and 9. For Variegated Glacier, the flow as so recalculated differs by at most 1% from the calculated flow curve for $4\ell = 2.4$ km in Part I, figure 9b. For Blue Glacier, exclusive of the ice-fall reach, the recalculated flow differs by up to 6% from the calculated flow curve in Part I, figure 11c, with no significant alteration in the overall longitudinal pattern of flow. Near the base of the ice fall, where the longitudinal derivations of h and α are particularly large, $C(x)$ is large and the resulting effect on the flow is as much as 15%. Although this correction is appreciable, it is small compared to the effects due to high sliding velocities in the ice fall, discussed in Part I, section 7.

From the foregoing considerations, we conclude that in the first-approximation treatment of Part I the effects of the T term contained in the function $C(x)$ are essentially negligible. In a higher approximation it might be appropriate to use the modified form of $H(x)$ in Equation (18) and to seek observational evidence for effects of the enhancement factor in Equation (20a). In cases of high surface slope ($\alpha \gtrsim 0.5$), the enhancement of $A(x)$ indicated by Equation (20b) might possibly need to be taken into consideration.

5. EFFECT ON THE BASAL SHEAR STRESS

Because of the direct linkage between flow velocity and basal shear stress via equation (I–1), the conclusions of the last section are tantamount to the conclusion that the T term has an essentially negligible effect on τ_B. The maximum effect on τ_B is for short-wavelength variations in ice thickness h; in this case the T term gives a variation in τ_B that is about 4% of the variation of the local "slope stress" $\rho g\alpha h$, according to Equation (21b) with $\ell \approx 2h$. (As noted in section 4, the accuracy of this estimate is limited by the validity of the underlying assumption in Equation (2), which declines at short wavelengths.) Variations in slope α have a smaller effect via the T term, as long as the slopes are relatively small ($\alpha_0 \sim 0.1$.) If effects at the $\lesssim 5\%$ level can be neglected, the relation between slope stress and τ_B remains as given by equation (I–34) without modification from the T term.

This means that it is not necessary to distinguish between two longitudinal length scales for averaging of τ_B – a short scale over which the effects of T are significant and an intermediate scale over which they are negligible – as has often been done in the literature (for summary see Raymond (1978, p. 809)). There still remains the need to distinguish between a short scale over which longitudinal stress gradients significantly affect τ_B and a long scale over which they do not, as discussed in Part I, section 3.

6. COMPARISON WITH OTHER EVALUATIONS OF T

The evaluation of the T term given by Budd (1968, p. 64) and recapitulated by Hutter ([^{c}1983], p. 265) is based on the simple assumption that $\tau_{xy}(y)$ in Equation (1) is given by $\rho g\alpha(y_S - y)$. This is incorrect. Because of the effect of longitudinal stress gradients, $\tau_{xy}(x,y)$ is everywhere modified from the value $\rho g\alpha(y_S - y)$ that it would have in the absence of these gradients. The mistaken asusmption leads to the conclusion that T varies as $d^2\alpha/dx^2$, whereas we see from Equation (10) that the "direct" contribution from T to the input function $C(x)$ varies instead as $d\alpha/dx$ and $d(\alpha - \beta)/dx$. The source of this contribution is from the ratio $(y_S - y)/h$ in Equation (2), and is physically quite distinct from the effects of the factor $\tau_B(x)$ in Equation (2), which is, according to the present theory, what must replace $\rho gh\alpha$ in Budd's evaluation. The second-derivative effect, from $d^2\tau_B/dx^2$ in Equation (3), appears ultimately in the contribution $h_0^2/6$ in Equation (12), which is the principal "indirect" effect of the T term on the flow. We see here how the flow coupling modulates in a significant and subtle way the flow response to $\alpha(x)$ and $h(x)$ via the T term. These conclusions are, of course, limited by the validity of the assumption in Equation (2) on which evaluation of the T term is based, but this treatment, which takes into consideration in a first approximation the effect of longitudinal stress coupling on the T term, is obviously better than a treatment that does not, as is the case with Budd's.

A second evaluation of the T term was given by Budd (1971, p. 185) along entirely different lines, based on a treatment of flow over sinusoidal bedrock topography by a linear-viscous analysis that in principle can be made exact for topographic undulations of infinitesimal amplitude. The basic idea is good, but, as pointed out by Hutter and others (1981, p. 252), Budd's analysis is flawed by a fundamental error, which makes the results invalid. The magnitude of T given by Budd (1971, fig. 3) would indicate a contribution comparable to what we evaluate from Equation (15), although the detailed form of the dependence on λ is different and there is no indication of the phase shift relative to $\alpha(x)$ shown in Equation (21).

7. EFFECT OF THE T TERM ON THE LONGITUDINAL FLOW-COUPLING EQUATION OF PART II

We now consider the effects of the T term on the flow perturbation that arises as a result of a small perturbation in ice thickness and surface slope from the realistic datum state of a glacier for which h, α, and u are known functions of x. This type of flow perturbation is the subject treated in Part II.

To find the effects of the T term, we need to evaluate the perturbation term T_1 in equation (II–4), to find how it depends on the datum-state functions h_0, α_0, and u_0 and on the perturbations h_1, α_1, and β_1 introduced in Part II, section 2. If we continue to make the basic assumption contained in Equation (2), then the T term is given by Equation (3). To find the perturbation T_1, we introduce into Equation (3) the perturbation relations $\tau_B = \tau_0 + \tau_1$, $h = h_0 + h_1$, $\delta = \delta_0 + \alpha_1$, $\theta = \theta_0 + \beta_1$, $\chi = \chi_0 + \chi_1 = dh_0/dx + dh_1/dx$, $u = u_0 + u_1$, and $T = T_0 + T_1$, and then subtract from Equation (3) the value of the unperturbed term T_0, obtained by setting all the perturbations to zero. Keeping only the first-order perturbation terms, we find

$$T_1 = \frac{1}{3}h_0\,\frac{d^2\tau_0}{dx^2}h_1 + \frac{1}{6}h_0^2\,\frac{d^2\tau_1}{dx^2} - \frac{1}{3}\,\frac{d\tau_0}{dx}\,[h_0(3\alpha_1 + \chi_1) + (3\delta_0 + \chi_0)h_1]$$

$$- \frac{1}{3}\,(3\delta_0 + \chi_0)\,\frac{d\tau_1}{dx} + \left[\frac{1}{3}\chi_0(3\delta_0 + \chi_0) - \frac{1}{6}h_0\left(3\,\frac{d\delta_0}{dx} + \frac{d\chi_0}{dx}\right)\right]\tau_1 \tag{23}$$

$$+ \frac{1}{6}\,\tau_0\left[2\chi_0(3\alpha_1 + \chi_1) + 2(3\delta_0 + \chi_0)\chi_1 - h_0\left(3\,\frac{d\alpha_1}{dx} + \frac{d\chi_1}{dx}\right) - \left(3\,\frac{d\delta_0}{dx} + \frac{d\chi_0}{dx}\right)h_1\right].$$

Now we express the perturbation τ_1 in terms of the perturbations u_1 and h_1 by means of equation (I–9), which can be rewritten

$$\tau_1 = \frac{\tau_0}{n}\left(\frac{u_1}{u_0} - \frac{h_1}{h_0}\right) \tag{24}$$

(overlooking the distinction between n and n' discussed in Part II, section 2). From Equation (24) we can also obtain the derivatives $d\tau_1/dx$ and $d^2\tau_1/dx^2$. When these are put into Equation (23) and related terms grouped (noting that $\chi = dh/dx$), we obtain

$$\frac{n}{\tau_0}\,T_1 = \frac{1}{6}\,\frac{h_0{}^2}{u_0}\,\frac{d^2u_1}{dx^2} - \left[\left[\frac{n + 1}{3n}\,\chi_0 + \delta_0\right]\frac{h_0}{u_0} + \frac{n - 1}{3n}\,\frac{h_0{}^2}{u_0{}^2}\,\frac{du_0}{dx}\right]\frac{du_1}{dx}$$

$$+ t_u\,\frac{u_1}{u_0} + t_h\,\frac{h_1}{h_0} + t_\alpha\alpha_1 + t_\beta\beta_1 + \frac{n + 1}{6}\,h_0\,\frac{d^2h_1}{dx^2} - \frac{n}{2}\,h_0\,\frac{d\alpha_1}{dx} \tag{25}$$

in which the coefficients t_u, t_h, etc. are the following functions of the datum-state variables:

$$t_u = \frac{(2n + 1)(n + 1)}{6n^2}\,\chi_0{}^2 + \frac{(n + 1)}{n}\,\delta_0\chi_0 - \frac{n + 1}{6n}\,h_0\,\frac{d^2h_0}{dx^2} - \frac{h_0}{2}\,\frac{d\delta_0}{dx}$$

$$+ \left[\frac{n^2 - 1}{3n^2}\,\chi_0 + \frac{n - 1}{n}\,\delta_0\right]h_0\,\frac{d\ln u_0}{dx} - \frac{(2n + 1)(n - 1)}{6n^2}\,\frac{h_0^2}{u_0{}^2}\left(\frac{du_0}{dx}\right)^2 - \frac{n - 1}{6n}\,\frac{h_0{}^2}{u_0}\,\frac{d^2u_0}{dx^2}, \tag{26}$$

$$t_h = \frac{h_0}{6n}\left\{(n^2 + 1)\,\frac{dh_0}{dx} + 3n(n + 1)\,\frac{d\delta_0}{dx} - \frac{6(n - 1)}{n}\,\frac{\delta_0}{u_0}\,\frac{du_0}{dx} - \frac{2}{n^2}(n^2 + n + 1)\,\frac{\chi_0}{u_0}\,\frac{du_0}{dx}\right.$$

$$\left. - \frac{n - 1}{h_0}\left[\frac{n - 1}{n}\chi_0 + 6\delta_0\right]\frac{dh_0}{dx} + (2n - 1)\,\frac{h_0}{u_0}\,\frac{d^2u_0}{dx^2} - \frac{(2n - 1)(n - 1)}{n}\,\frac{h_0}{u_0{}^2}\left(\frac{du_0}{dx}\right)^2\right\}, \tag{27}$$

$$t_\alpha = n\left[\frac{1}{3}\chi_0 - \frac{n + 1}{n}\,\delta_0 - \frac{2n - 1}{3n^2}\,\frac{h_0}{u_0}\,\frac{du_0}{dx}\right], \tag{28}$$

$$t_\beta = n\left[-\frac{1}{3}\,\chi_0 + \frac{n + 1}{n}\,\theta_0 - \frac{n + 1}{3n^2}\,\frac{h_0}{u_0}\,\frac{du_0}{dx}\right]. \tag{29}$$

If we now introduce T_1 from Equation (25) into equation (II–6), we get a longitudinal coupling equation that contains the effects of the T_1 term. When the terms are grouped and designated as in equation (II–13), this reads

$$-(\ell'')^2\,\frac{d^2u_1}{dx^2} - 2\sigma''\ell''\,\frac{du_1}{dx} + u_1 = u_0\left[\phi'_h\,\frac{h_1}{h_0} + \right.$$

$$\left. + \phi'_\alpha\,\frac{\alpha_1}{\sin\alpha_0^*} - \phi'_\beta\beta_1 - \phi'_H h_0\frac{d^2h_1}{dx^2} - \phi'_A\,h_0\,\frac{d\alpha_1}{dx}\right] \tag{30}$$

where

$$\ell'' = \sqrt{\ell^2 + \frac{1}{6}\,\frac{h_0{}^2}{b'}}, \tag{31}$$

$$2\sigma''\ell'' = \ell^2\,\frac{d\ln(h_0\bar{\eta}_0)}{dx} - \frac{n + 1}{3n}\,\frac{h_0}{b'}\,\frac{dh_0}{dx} -$$

$$- \frac{\delta_0 h_0}{b'} - \frac{n - 1}{3n}\,\frac{h_0{}^2}{b'u_0}\,\frac{du_0}{dx}, \tag{32}$$

$$b' = b - t_u \tag{33}$$

and where ℓ is defined in equation (II–11), b in Equation (II–7), and t_u in Equation (26). Equation (30) has a form similar to Equation (II–13). The modified coefficients ϕ'_h, ϕ'_α, etc., on the right side of Equation (30) are given explicitly below, in Equations (36)–(40).

8. EVALUATION OF EFFECTS ON FLOW RESPONSE TO A PERTURBATION IN ICE THICKNESS AND SLOPE

Of the numerous terms in Equation (30) that are not present in the longitudinal flow-coupling equation (II–13), the magnitudes of some can be judged from their order in the angle δ_0 and the quantity $\chi_0 = \mathrm{d}h_0/\mathrm{d}x$, while for others a specific evaluation based on the characteristics of actual datum states is required. For this purpose, we take the geometry and flow field of Blue Glacier as a representative example (Echelmeyer, unpublished; Part II, table II).

As indicated in Equation (33), the quantity b', which is present as the factor $1/b'$ in all coefficients on the right-hand side of Equation (30), is modified from b by the quantity t_u in Equation (26). If χ_0 is small, as is δ_0 by assumption, then the first two terms on the right in Equation (26), which are second order in these angle quantities, are negligible, and b itself is negligibly different from 1. If $h_0\mathrm{d}\chi_0/\mathrm{d}x$ and $h_0\mathrm{d}\ln u_0/\mathrm{d}x$ are small of order δ_0, as may often be the case in practice, then the third, fifth, and sixth terms in Equation (26) are negligible on the same basis. The remaining terms are best evaluated numerically. Using the Blue Glacier datum state, we calculate values of b' in the range 0.98 to 1.03. At the level of approximation sought in Part II, b' can thus be taken as unity.

In view of this, the modified longitudinal coupling length ℓ'' given by Equation (31) differs inappreciably from the ℓ' in Equation (12), except for the effect of the n'' in equation (II–11), which was already noted in Part II, section 2.

The modified symmetry parameter σ'' in Equation (32) can be expressed, in terms of Equations (16a), (16b), and the additional similar definition

$$\sigma_u = \frac{1}{2}\ell\,\frac{\mathrm{d}\ln u_0}{\mathrm{d}x} \tag{34}$$

as follows:

$$\sigma'' = \frac{\ell}{\ell'}\left[\sigma - \left[\frac{h_0}{\ell}\right]^2\left[\frac{n+1}{3n}\sigma_h + \sigma_\alpha + \frac{n-1}{3n}\sigma_u\right]\right]. \tag{35}$$

Equation (35) is an augmented version of Equation (17). If σ_u is comparable to σ, σ_h, and σ_α, then the discussion of σ' based on Equation (17) in section 4 applies equally well to σ'' in Equation (35).

The influence coefficients on the right-hand side of Equation (30) are modified from those given by Equations (II–8)–(II–10) as follows:

$$\phi'_h = (n+1)\left[\frac{\sin\alpha_0}{\sin\alpha_0^*} + j_h - t_h\right], \tag{36}$$

$$\phi'_\alpha = n(\cos\alpha_0 + j_\alpha - t_\alpha\sin\alpha_0^*), \tag{37}$$

$$\phi'_\beta = n(2\sin 2\theta_0 + t_\beta) \tag{38}$$

where t_h, t_α, and t_β are given by Equations (27)–(29), j_h and j_α by Equations (II–27) and (II–28), and j_β by

$$j_\beta = -\frac{4\bar{\eta}_0}{b'\tau_0}\frac{\mathrm{d}u_0}{\mathrm{d}x}. \tag{39}$$

In Equations (36)–(38) the quantity b' has been set equal to 1.

The magnitude of the correction term t_h in Equation (36) is found, from Equation (27) evaluated with the Blue Glacier data, to be $|t_h| \leqslant 0.04$. ϕ'_h therefore is dominated by the other terms in Equation (36). However, for longitudinal gradients of h_0, δ_0, or u_0 rather larger than those in the Blue Glacier datum state, the size of t_h could be large enough to have an appreciable influence.

While j_α in Equation (37) is small (as found in Part II, section 5), t_α is not particularly small, being of order δ_0 or θ_0 according to Equation (28). However, the factor $\sin\alpha_0^*$ in Equation (37) reduces the magnitude of the product to $\leqslant 0.02$, for the Blue Glacier datum state. The effect can thus be neglected in general for typical valley glaciers with $\sin\alpha_0^* \sim 0.1$, but for steep glaciers or ice falls it may be non-negligible. The correction term t_β in Equation (38) does not have the $\sin\alpha_0^*$ factor, and it appears that the β_1 response may be appreciably affected by the T term. From Equation (29) this could be evaluated for an actual situation in which a β_1 perturbation came into consideration.

The final two terms in Equation (30), with coefficients

$$\phi'_H = \frac{1}{6}(n+1) \tag{40}$$

and

$$\phi'_A = \tfrac{1}{2}n \tag{41}$$

are the same as the terms given by $C(x)$ in Equation (15). Their role in a flow perturbation here will be quite the same as already discussed in section 4 above. From the Blue Glacier perturbation data the quantity $-(h_0/6)(\mathrm{d}^2h_1/\mathrm{d}x^2)$ has extreme values 1.1%, −0.8%, and −0.6%, at points where h_1/h_0 has values 4.2%, 4.0%, and 7.0%, respectively, indicating that the "thickness–curvature term" can amount to as much as 25% of the direct h_1/h_0 term on the right side of Equation (30). More typically, however, it amounts to only about 3%. Longitudinal averaging tends to suppress the effect of the localized higher values, as discussed in section 4. The quantity $(h_0/2)\mathrm{d}\alpha_1/\mathrm{d}x$ is found, for the Blue Glacier data, to be smaller by a factor of about 10, and therefore negligible.

From the foregoing evaluation, we conclude that the effects of the T_1 term on flow perturbations can in general be neglected, but that we should be on the lookout for unusual datum-state situations in which one or more of the many quantities that the T_1 term generates in Equations (26)–(35) would happen to be large enough to need to be taken into consideration.

9. ROLE OF THE T TERM IN SHORT-WAVELENGTH LONGITUDINAL FLOW VARIATIONS

The simple assumption in Equation (2) that τ_{xy} is a linear function of y, which leads to the conclusions of this paper as to the essentially negligible role of the T term in longitudinal stress coupling, is a reasonable first approximation for long wavelengths, but breaks down at short wavelengths $\lambda \lesssim h$, as has been repeatedly pointed out in sections 4–6. In Part III, section 6, it is noted that in longitudinal flow variations at very short wavelengths ($\lambda \ll h$) the T term plays a major role in the longitudinal equilibrium equation, as expected from the presence of the second derivative in Equation (1). The role of the T term in longitudinal coupling at short wavelengths will be explored further in a later paper, and it will be shown there that longitudinal coupling theory can be extended in an approximate but useful way to short wavelengths by taking effects of the T term into consideration.

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