



RESEARCH ARTICLE

Functorial Fast-Growing Hierarchies

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Abstract

We prove an isomorphism theorem between the canonical denotation systems for large natural numbers and large countable ordinal numbers, linking two fundamental concepts in Proof Theory. The first one is fast-growing hierarchies. These are sequences of functions on \mathbb{N} obtained through processes such as the ones that yield multiplication from addition, exponentiation from multiplication, etc. and represent the canonical way of speaking about large finite numbers. The second one is ordinal collapsing functions, which represent the best-known method of describing large computable ordinals.

We observe that fast-growing hierarchies can be naturally extended to functors on the categories of natural numbers and of linear orders. The isomorphism theorem asserts that the categorical extensions of binary fast-growing hierarchies to ordinals are isomorphic to denotation systems given by cardinal collapsing functions. As an application of this fact, we obtain a restatement of the subsystem $\Pi_1^1\text{-CA}_0$ of analysis as a higher-type well-ordering principle asserting that binary fast-growing hierarchies preserve well-foundedness.

1. Introduction

1.1. Motivation

Mathematicians often face the need to express very large quantities for one reason or another. In some extreme occasions, these numbers are incomprehensibly large. Examples include Graham's number g_{64} (see Gardner [10]; see also Graham-Rothschild [14]) or Friedman's tree number $\text{TREE}(3)$ (see Friedman [9]). These immense numbers are completely foreign to our everyday experience, and so we would naturally like to attempt to provide a sense of scale for them. The way this is usually done is via *fast-growing hierarchies*, collections of functions $f_\alpha : \mathbb{N} \rightarrow \mathbb{N}$, where α ranges over elements of \mathbb{N} , or possibly even over transfinite numbers, and each f_α eventually dominates f_β , for all $\beta < \alpha$. One is mainly interested in fast-growing hierarchies obtained through a given recursive construction which usually generalizes the process that produces multiplication as iterated addition, exponentiation as iterated multiplication, etc. A commonly known example of such a hierarchy is given by Knuth's arrow hierarchy (see Knuth [17]):

$$f_n(a, b) = \underbrace{a \uparrow \uparrow \cdots \uparrow}_n b.$$

Fast-growing functions are really the ‘canonical’ way to express the magnitude of large integers and place them into perspective. Knuth’s and other related hierarchies (some of which will be discussed below) have proven to be effective for this purpose and indeed can be used to give bounds for the numbers g_{64} and $TREE(3)$ which illustrate their magnitude to the greatest extent one could perhaps hope for.

For completely different purposes, logicians – and, in particular, proof theorists – often face the need to talk about large countably infinite ordinal numbers (in fact, large *computable* ordinal numbers) such as the *Bachmann-Howard ordinal* $\psi(\varepsilon_{\aleph_{1+1}})$. There is also a ‘canonical’ way to express these via what are known as *ordinal collapsing functions*. These allow us to make use of *uncountable* numbers to express large countably infinite (and indeed computable) numbers. In this article, we will prove an isomorphism theorem (Theorem 14 on p. 12) according to which the two constructions – ordinal collapsing functions and fast-growing hierarchies – can be regarded as one and the same.

1.2. Fast-growing hierarchies

Let us focus our attention on hierarchies of unary functions. Fast-growing hierarchies can be defined in various ways, and doing so often results in one hierarchy being a refinement of another. Another example is the *Hardy hierarchy* (introduced by Wainer [25], though implicit in G. H. Hardy’s [15] construction of a subset of \mathbb{R} of cardinality \aleph_1), the *iterative fast-growing hierarchy* and the *binary fast-growing hierarchy*. Let us focus on the latter one, for the sake of definiteness, though we remark that the constructions in this article could be adapted to the other hierarchies. It is defined by transfinite induction according to the following construction:

$$\begin{aligned} B_0(n) &= n + 1 \\ B_{\alpha+1}(n) &= B_\alpha \circ B_\alpha(n) \\ B_\lambda(n) &= B_{\lambda[n]}(n), && \text{at limit stages,} \end{aligned}$$

where for each limit ordinal λ , $\{\lambda[n] : n \in \mathbb{N}\}$ is a sequence which converges to λ , fixed in advance. The precise functions defined depend on the sequences chosen, but for natural choices of sequences, the functions behave very regularly. The ordinal ε_0 is defined by

$$\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}.$$

B_{ε_0} is a total recursive function whose totality cannot be proved in Peano Arithmetic (or, equivalently, Arithmetical Comprehension, ACA_0). The foundational significance of fast-growing functions was observed by Ackermann and, later, by Kreisel [18].

Theorem 1 (Kreisel 1952, essentially). *There is a canonical well-ordering W of \mathbb{N} of length ε_0 such that the following theories prove the same Π_2^0 theorems:*

1. *Arithmetical Comprehension;*
2. *Primitive Recursive Arithmetic + $\{B_\alpha \text{ is total} : \alpha \in W\}$.*

In particular, the two theories are equiconsistent.

Kreisel’s theorem is a computational analysis of arithmetic in the sense that it gives an explicit characterization of the recursive functions which are provably total.

1.3. Fast-growing functions and ordinal denotation systems

Kreisel’s result is of great foundational significance and has led to a great deal of metamathematical and combinatorial results in the context of arithmetic, as well as generalizations to stronger systems. For instance, one can prove a version of Kreisel’s theorem for the subsystem $\Pi_1^1\text{-CA}_0$ of analysis where B_{ε_0} is replaced by $B_{\psi_0(\aleph_\omega)}$ (see Buchholz [4]). Here, $\psi_0(\aleph_\omega)$ is the *Takeuti ordinal*, a recursive ordinal

that is most easily described by *ordinal collapsing functions* for the uncountable cardinals \aleph_n , where $n \in \mathbb{N}$. $\Pi_1^1\text{-CA}_0$ is a historically and mathematically important theory, and is the strongest of the ‘big five’ subsystems of Second-Order Arithmetic commonly studied in Reverse Mathematics. Ordinal collapsing functions were introduced by Bachmann [3] and are a powerful method in proof theory involved in the description of large recursive ordinals, such as the *Bachmann-Howard* ordinal $\psi(\varepsilon_{\aleph_1+1})$, the Takeuti ordinal and other larger numbers such as the proof-theoretic ordinals of KP_i and $\text{KP} + \Pi_3\text{-reflection}$. These ordinals involve collapsing functions for inaccessible and weakly compact cardinals, respectively; see Rathjen [20] for an overview. Indeed, ordinal collapsing functions are the canonical method of speaking about large countable ordinals.

The first observation we make is that fast-growing hierarchies can be regarded as functors on the category of natural numbers with strictly increasing functions as morphisms. This fact was perhaps underpinning the main result of [2]. Let us say more about this observation.

For our purposes, it will be convenient to shift our focus to the norm-based presentation of fast-growing hierarchies. This presentation goes back to [5] and, for our purposes, is described as follows: suppose that we are given a denotation system D for ordinals. We will use D to index a fast-growing hierarchy. The definition of ‘denotation system’ is recalled in the next section, where these are defined as a particular kind of *dilator*. The reader unfamiliar with denotation systems might temporarily think of D as a system for representing ordinals in some way using natural numbers as parameters. For example, we may think of representing ordinals $< \omega^\omega$ as sums of powers of ω using integers as parameters together with the symbol ‘ ω ’; for example,

$$\omega^7 + \omega^3 + 21.$$

The *norm* $N\alpha$ of an ordinal α is the strict supremum of the natural number parameters which occur in the denotation. We define

$$B_\alpha(n) = \sup \{ \{ B_\beta \circ B_\beta(n) : \beta < \alpha \wedge N\beta \leq n \} \cup \{ n + 1 \} \}.$$

Natural numbers can be represented by repeated applications of functions in the hierarchy:

$$m = B_{\alpha_k} B_{\alpha_{k-1}} \dots B_{\alpha_1}(n). \tag{1}$$

If n is fixed and some constraints are imposed on the values $\alpha_1, \dots, \alpha_k$, then the resulting representation is unique. We define

$$B_D(n) = B_{D(\omega)}(n) = \sup \{ B_\beta \circ B_\beta(n) : N\beta \leq n \} \cup \{ n + 1 \}.$$

Making use of the representation of natural numbers in terms of iterations of the fast-growing hierarchy, we can regard B_D as a functor on the category of natural numbers with strictly increasing functions as morphisms. This is proved in §3.

From the fact that B_D is a functor on the category of natural numbers, it follows that B_D can be uniquely extended to a finitary functor on the category of linear orders. In particular, expressions of the form (1) can be given transfinite parameters and construed as denotations for elements of some linear order. In §4, we give an alternative (isomorphic) definition of the functor B_D as a term system performing a formal ordinal collapse of D . Thus, the main contribution of this work is captured by the following:

Fundamental Observation. *Binary fast-growing functors are naturally isomorphic to ordinal denotation systems given by ordinal collapses.*

This insight is made precise by Theorem 14 on p. 12, which, together with Proposition 9 on p. 10, asserts that the ‘canonical’ method used to refer to large countable numbers is derived from a functorialization of the ‘canonical’ method used to refer to large finite numbers. In particular, we have an object B_D which can be simultaneously viewed through the lenses of two fundamental concepts in

Proof Theory (fast-growing functions and ordinal collapses), connecting the two fields of study. For example, we can use fast-growing functors to express the Bachmann-Howard ordinal as

$$\psi(\varepsilon_{\aleph_1+1}) = B_D(0),$$

where

$$D = \sum_{n \in \mathbb{N}} (\exp^n \circ \text{Add}(1))$$

and where \exp is the natural exponential functor mapping a linear order X to ω^X and $\text{Add}(1)$ is the addition functor mapping X to $X + 1$.

1.4. An application

In §5, we use functorial fast-growing hierarchies to prove a result in Reverse Mathematics. We first show that the ordinal collapses given by B_D are well-foundedness-preserving, so that if A is a well-order, then so too is $B_D(A)$. This justifies us calling B_D an ‘ordinal denotation system’ rather than a denotation system for elements of some linear order.

In fact, B_D is a functor on the category of ordinal numbers and indeed a *dilator*. The proof of this fact requires the use of a powerful set-existence principle called $\Pi_1^1\text{-CA}_0$, and this is unavoidable: the fact that B_D is well-foundedness-preserving, in turn, implies $\Pi_1^1\text{-CA}_0$. This way, we obtain a restatement of $\Pi_1^1\text{-CA}_0$ in terms of fast-growing hierarchies and abstract ordinal collapses. We state the theorem; for all undefined terms, we refer the reader to the following section.

Theorem 2. *The following are equivalent over ACA_0 :*

1. $\Pi_1^1\text{-CA}_0$;
2. for every dilator D , B_D is a dilator.

To establish the reversal in Theorem 2, it suffices to consider so-called *weakly finite* dilators. Theorem 2 fits within the family of results known as *well-ordering principles*. Most commonly, these are principles of the form ‘if X is a well-order, then $f(X)$ is a well-order’, where f is some transformation. These principles have been extensively studied in Reverse Mathematics and have led to reformulations of several subsystems of analysis. Some examples are the works of Girard [13], Friedman (unpublished), Hirst [16], Afshari-Rathjen [1], Marcone-Montalbán [19], Rathjen-Valencia Vizcaino [21], Thomson-Rathjen [23] and [22]. Higher-type well-ordering principles were used by Girard [11] and Freund [7] to characterize $\Pi_1^1\text{-CA}_0$. Freund’s theorem is an ingredient of our proof of Theorem 2. Freund’s principle maps dilators to well-orderings, while ours and Girard’s map dilators to dilators. Girard’s characterization of $\Pi_1^1\text{-CA}_0$ asserts the totality of the functor Λ , which itself is reminiscent of fundamental-sequence-based fast-growing hierarchies. Well-ordering principles for theories such as $\Pi_1^1\text{-CA}_0$ which are not Π_2^1 -axiomatizable require the use of higher-type objects such as dilators.

Although throughout this article we focus on the binary fast-growing hierarchy B_D , we mention that the statement of Theorem 2 applies to many of the other usual fast-growing hierarchies as well. The use of B_D is merely for convenience and to simplify the computations involved.

2. Preliminaries

We deal with the category LO. The objects of LO are linear orders whose domains are subsets of \mathbb{N} . The morphisms are strictly increasing maps between these orders. We shall work within the subsystem ACA_0 of second-order arithmetic. This is the system in the language of second-order arithmetic (which contains sorts for natural numbers and sets of natural numbers) whose principal axioms are the induction axiom and the schema asserting that every arithmetically definable set exists. When it leads to no confusion, we shall leave the precise formalization of statements to the reader and work informally.

In the language of second-order arithmetic, one cannot generally speak of functors from LO to LO; thus, we shall restrict to dealing with *finitary* functors (i.e., functors preserving co-limits of upward directed diagrams (direct limits in the model-theoretic sense)). These can be coded by sets of natural numbers:

We consider Nat , the full subcategory of LO where objects are natural numbers n , which we identify with the orders $([0, n], <)$. Since both Ob_{Nat} and Mor_{Nat} are countable, each functor from Nat to LO can be coded by a set of natural numbers. Each functor $D : \text{Nat} \rightarrow \text{LO}$ can be uniquely extended to a finitary functor $D : \text{LO} \rightarrow \text{LO}$ as follows:

Below, we write $A' \subseteq_{\text{fin}} A$ if A' is a finite subset (or substructure) of A . Given a linear order A , we consider an upward directed diagram H_A consisting of all finite suborders $A' \subseteq_{\text{fin}} A$ and all morphisms $\text{id}_{A' \rightarrow A''}$, for $A' \subseteq_{\text{fin}} A'' \subseteq_{\text{fin}} A$, where $\text{id}_{A' \rightarrow A''} : A' \rightarrow A''$ denotes the inclusion map given by $\text{id}_{A' \rightarrow A''}(x) = x$. Let H'_A be the naturally isomorphic diagram in which all objects are orders in Ob_{Nat} . Applying D to all morphisms and objects in H'_A , we obtain a diagram $D[H'_A]$ and define

$$D(A) = \varinjlim D[H'_A],$$

where $\varinjlim D[H'_A]$ denotes the co-limit of the diagram. The value $D(f)$, for an LO-morphism $f : A \rightarrow C$ is recovered in the natural way.

A finitary functor D on LO coded by a set of natural numbers is called a pre-dilator if it preserves pullbacks. Clearly, D preserves pullbacks if and only if its restriction to Nat preserves pullbacks.

It will be convenient for our purposes to work with inclusion-preserving (or \subseteq -preserving) functors. These are the functors D such that for any order A and any suborder B of A ,

1. $D(B)$ is a suborder of $D(A)$, and furthermore,
2. $D(\text{id}_{B \rightarrow A}) = \text{id}_{D(B) \rightarrow D(A)}$.

Finitary \subseteq -preserving functors were called ω -local functors by S. Feferman [6].¹ For \subseteq -preserving functors, there are convenient reformulations of the conditions of being a finitary functor and a pre-dilator. Namely, a \subseteq -preserving functor D is finitary if and only if for any order A , we have

$$D(A) = \bigcup_{A' \subseteq_{\text{fin}} A} D(A').$$

An \subseteq -preserving finitary functor D is a pre-dilator iff for any order A and its suborders $B, C \subseteq A$ we have $D(B \cap C) = D(B) \cap D(C)$.

Working in ACA_0 , we code \subseteq -preserving finitary functors as follows. We consider the full subcategory FO of LO, where objects are finite linear orders. As in the case of Nat , both Ob_{FO} and Mor_{FO} are countable, so \subseteq -preserving functors D from FO to LO can each be coded naturally by a set of natural numbers. We extend any such D to a functor with domain LO by putting

$$D(A) = \bigcup_{A' \subseteq_{\text{fin}} A} D(A') \quad \text{and} \quad D(f) = \bigcup_{A' \subseteq_{\text{fin}} A} D(f \upharpoonright A').$$

Here, for $f : A \rightarrow C$ and $A' \subseteq A$, $f \upharpoonright A' : A' \rightarrow C$ is the restriction of f .

A pre-dilator is called a dilator if it maps well-orders to well-orders. Dilators were introduced by Girard [12], to whom we refer the reader for further background. The fundamental theorem of dilators states that every pre-dilator D is naturally isomorphic to a *denotation system*, a special kind of \subseteq -preserving predilators.

Each denotation system D consists of a set of terms $t(x_1, \dots, x_n)$ and comparison rules establishing, for each pair of terms $t(\vec{x})$ and $s(\vec{y})$, which one is greater, depending on the relative ordering of the constants \vec{x} and \vec{y} . Note that we do not allow $t(\vec{x}) = s(\vec{y})$ unless $t(\vec{x})$ and $s(\vec{y})$ are syntactically the same term. This allows us to define a binary relation $D(A)$ for each linear order A as follows: the

¹Feferman [6] is an early reference for functorial ordinal notation systems.

domain of $D(A)$ is the set of all expressions $t(a_1, \dots, a_n)$, where $a_1 > \dots > a_n$ are elements of A and $t(x_1, \dots, x_n)$ is a D -term. The relative ordering of elements of the domain of $D(A)$ is determined by the comparison rules for each pair of terms.

Given a strictly increasing map $f: A \rightarrow C$ between finite orderings A and C , we define

$$D(f): D(A) \rightarrow D(C)$$

$$D(f): t(a_1, \dots, a_n) \mapsto t(f(a_1), \dots, f(a_n)).$$

We shall sometimes identify the values of terms $t(x_1, \dots, x_n)$ with the ordinals they denote. If $\alpha = t(x_1, \dots, x_n)$, we call the set $\{x_1, \dots, x_n\}$ the *support* of α and denote it by $\text{supp}(\alpha)$. Given ordinals α, β , we write

$$\text{supp}(\alpha) < \beta \text{ if } x_i < \beta \text{ for all } x_i \in \text{supp}(\alpha).$$

2.1. Conventions

Whenever we deal with a linear order L , we will write $<_L$ to indicate the ordering. If no confusion arises, we may occasionally omit reference to L and simply write $<$.

When dealing with functions $f(x)$, we will sometimes omit brackets and simply write fx . The purpose of this is to avoid cluttering when dealing with nested application of functions.

3. The functoriality of fast-growing hierarchies

In this section, we fix a dilator D and investigate the functorial structure of B_D . All the constructions in this section are formalizable in ACA_0 . Note that the norms $N\alpha$ – and hence the B -hierarchy on $D(\omega)$ – are unaffected if we replace D by a naturally isomorphic denotation system. Hence, in order to simplify notation, we will assume that D is a denotation system (in fact, we will only use that it is \subseteq -preserving). If so, $N\alpha = \min\{n \mid \alpha \in D(n)\}$.

Definition 3. Let n be a natural number and $\alpha \in D(\omega) \cup \{D(\omega)\}$. An (n, α) -normal form term is an expression of one of the following forms:

1. m , where $m < n$ is a natural number, or
2. $B_{\alpha_k} \dots B_{\alpha_1}(n)$, where
 - (a) $\alpha_k <_{D(\omega)} \dots <_{D(\omega)} \alpha_1 <_{D(\omega)} \alpha$; and
 - (b) $N\alpha_{i+1} \leq B_{\alpha_i} \dots B_{\alpha_1}(n)$ for all i with $0 \leq i < k$.

Normal form terms denote natural numbers obtained simply by evaluating the functions. If $m = B_{\alpha_k} \dots B_{\alpha_1}(n)$ is as above, we may omit reference to α and call this expression the n -normal form of m ; if so, we may write

$$m \stackrel{NF}{=} B_{\alpha_k} \dots B_{\alpha_1}(n).$$

For finite sequences of ordinals $(\alpha_1, \dots, \alpha_k) \in (D(\omega))^{<\omega}$, the lexicographical comparison $<_{\text{lex}}$ is defined in the usual manner: by putting

$$(\alpha_1, \dots, \alpha_k) <_{\text{lex}} (\beta_1, \dots, \beta_l)$$

if there exists $0 \leq n \leq \min(k, l)$ such that $\alpha_i = \beta_i$, for $1 \leq i \leq n$ and either $n = k < l$ or $n < \min(k, l)$ and $\alpha_{n+1} <_{D(\omega)} \beta_{n+1}$.

Lemma 4 (ACA_0). *Suppose $\alpha \in D(\omega) \cup \{D(\omega)\}$ and $n \in \mathbb{N}$. Then,*

1. *each number smaller than $B_\alpha(n)$ is the value of a unique (n, α) -normal form; and*
2. *for any two (n, α) -normal forms $B_{\alpha_k} \dots B_{\alpha_1}(n)$ and $B_{\beta_l} \dots B_{\beta_1}(n)$, we have*

$$B_{\alpha_k} \dots B_{\alpha_1}(n) < B_{\beta_l} \dots B_{\beta_1}(n) \iff (\alpha_1, \dots, \alpha_k) <_{\text{lex}} (\beta_1, \dots, \beta_l).$$

Proof. We prove the lemma by transfinite induction on α . If there are no $\alpha' <_{D(\omega)} \alpha$ with $N\alpha' \leq n$, then $B_\alpha(n) = n + 1$ and the only (n, α) -normal forms are constants $\leq n$, and hence, the lemma holds.

Otherwise, choose $\alpha' <_{D(\omega)} \alpha$ such that $N\alpha' \leq n$ and $B_\alpha(n) = B_{\alpha'}(B_{\alpha'}(n))$. Let $s = B_{\alpha'}(n)$. By induction hypothesis,

1. any number $< B_{\alpha'}(s)$ is the value of a unique (s, α') -normal form;
2. any number $< s$ is the value of a unique (n, α') -normal form.

Combining these two facts, we observe that each number $< B_\alpha(n)$ is the value of the unique term of one of the following three forms:

1. m , for $m < n$;
2. $B_{\alpha_k} \dots B_{\alpha_1}(n)$, where $\alpha_k <_{D(\omega)} \dots <_{D(\omega)} \alpha_1 <_{D(\omega)} \alpha'$ and $N\alpha_{i+1} \leq B_{\alpha_i} \dots B_{\alpha_1}(n)$ for all i with $0 \leq i < k$;
3. $B_{\alpha_k} \dots B_{\alpha_1}(B_{\alpha'}(n))$, where $\alpha_k <_{D(\omega)} \dots <_{D(\omega)} \alpha_1 <_{D(\omega)} \alpha'$ and $N\alpha_{i+1} \leq B_{\alpha_i} \dots B_{\alpha_1}(B_{\alpha'}(n))$ for all i with $0 \leq i < k$.

Since $\alpha' <_{D(\omega)} \alpha$ and $N\alpha' \leq n$, it follows that any number $< B_\alpha(n)$ is indeed the value of an (n, α) -normal form. To prove that (n, α) -normal forms are compared as prescribed, we simply need to consider the cases of the forms (2) and (3). If both normal forms are of the same form, then we get the comparison property directly by induction hypothesis. The fact that the comparison property holds for terms of different forms follows from the fact that any term of the form (2) has smaller value than any term of the form (3). This is because, by construction, every term of the form (2) has value $< s$ and every term of the form (3) has value $\geq s$. □

Definition 5. Let $f : n \rightarrow n'$ be a strictly increasing function. We define the function $B_D(f) : B_D(n) \rightarrow B_D(n')$ to be the unique strictly increasing function such that

1. if $m < n$, then $B_D(f)(m) = f(m)$;
2. if $n \leq m < B_D(n)$ and $m \stackrel{NF}{=} B_{\alpha_k} \dots B_{\alpha_1}(n)$, then

$$B_D(f)(m) = B_{\alpha'_k} \dots B_{\alpha'_1}(n'),$$

where $\alpha'_i = D(B_D(f))(\alpha_i)$ for each i .

Lemma 6 (ACA₀). Let $f : n \rightarrow n'$ be a strictly increasing function. Then, there is a unique strictly increasing function $B_D(f)$ satisfying Definition 5.

Proof. By induction on $s \leq B_D(n)$, we define a sequence of strictly increasing maps

$$g_s : s \rightarrow B_D(n')$$

such that g_{s+1} extends g_s for each s . The map $g_0 : 0 \rightarrow B_D(n')$ is simply the empty map.

Assume that we already have defined maps g_0, \dots, g_s and established that they are an increasing family of strictly increasing functions. We define the map

$$g_{s+1} : (s + 1) \rightarrow B_D(n')$$

as follows:

1. if $m < n$, then we put $g_{s+1}(m) = m$;
2. if $n \leq m \stackrel{NF}{=} B_{\alpha_k} \dots B_{\alpha_1}(n)$, then we put $g_{s+1}(m) = B_{\alpha'_k} \dots B_{\alpha'_1}(n')$, where $\alpha'_i = D(g_s)(\alpha_i)$ for each i .

Note that in the second clause, the values $D(g_s)(\alpha_i)$ are well defined. This is because, by the condition on n -normal forms, we have

$$N\alpha_i \leq B_{\alpha_{i-1}} \dots B_{\alpha_1}(n') < m \leq s, \tag{2}$$

and hence, $\alpha_i \in D(s)$.

Let us check that the expression $g_{s+1}(m)$ in (2) is an n' -normal form. The fact that $\alpha'_k <_{D(\omega)} \dots <_{D(\omega)} \alpha'_1$ simply follows from the functoriality of D . We show that $N\alpha'_i \leq B_{\alpha'_{i-1}} \dots B_{\alpha'_1}(n')$. By definition, $\alpha'_i = D(g_s)(\alpha_i)$. Thus, α'_i is the value of a D -term where all the constants result from shifting constants $\leq N\alpha_i$ according to g_s . Hence, α'_i is the value of a D -term with constants $\leq g_s(N\alpha_i)$. In other words, $D(g_s)(\alpha_i) \in D(g_s(N\alpha_i))$. By the definition of N , it follows that

$$N\alpha'_i \leq g_s(N\alpha_i) \leq g_s(B_{\alpha_{i-1}} \dots B_{\alpha_1}(n)) = B_{\alpha'_{i-1}} \dots B_{\alpha'_1}(n'),$$

as desired. Hence, all the values produced in (2) are indeed normal forms.

Now, the comparison algorithm for n -normal forms provided by Lemma 4 implies that g_{s+1} is strictly increasing. If $s = 0$, then g_{s+1} extends g_s simply because g_0 is the empty map. If $s > 0$, then we immediately obtain that g_{s+1} extends g_s from their definitions and the fact that g_s extends g_{s-1} .

Consider

$$g : B_D(n) \rightarrow B_D(n')$$

given by

$$g = \bigcup_{m \leq B_D(n)} g_m.$$

From the definition, it is clear that g satisfies properties (1) and (2) of the definition of $B_D(f)$. By a straightforward induction on m , we show that if

$$g' : B_D(n) \rightarrow B_D(n')$$

is any other function satisfying properties (1) and (2) of the definition of $B_D(f)$, then $g(m) = g'(m)$. Therefore, the definition of $B_D(f)$ indeed defines a unique strictly increasing function. \square

Corollary 7. *B_D is a functor on the category Nat .*

Commenting on an earlier draft of this article, Wainer has brought to our attention an earlier result of his [24] in which he establishes functoriality for a version of the binary fast-growing hierarchy in the context of tree-ordinals.

4. Fast-growing hierarchies as ordinal collapses

In this section, we give an alternate definition of B_D in terms of formal ‘ordinal collapses’ which will directly result in a \subseteq -preserving dilator. These are orderings defined formally using the notation of ordinal collapsing functions in a way that results in them having similar properties. This will become clearer in the following section; cf. for example, Lemma 20; see also the comment following Definition 8.

We will show that the new construction, as a functor, is naturally isomorphic to the one constructed in the previous section. It will be clear from the construction that this definition coincides with the previous one when applied to (finite) natural numbers with the usual ordering. As a consequence of this and Corollary 7, it follows that the new definition coincides with the previous one for arbitrary linear orderings A . Again, all the constructions will be formalizable in ACA_0 .

Recall that for a linear order A , 2^A is the linear order consisting of formal base-2 Cantor normal forms

$$2^{a_1} + \dots + 2^{a_n}$$

with $a_1 >_A \dots >_A a_n$. These formal sums are compared in the natural way – that is, $2^{a_1} + \dots + 2^{a_n} <_{2^A} 2^{b_1} + \dots + 2^{b_m}$ if $(a_1, \dots, a_n) <_{\text{lex}} (b_1, \dots, b_m)$. Here, the empty sum is allowed and identified with the term 0. This construction naturally extends to a dilator: given a strictly increasing function $f : A \rightarrow B$, we define

$$2^f : 2^A \rightarrow 2^B$$

$$2^f : 2^{a_1} + \dots + 2^{a_n} \mapsto 2^{f(a_1)} + \dots + 2^{f(a_n)}.$$

Below, if A is a linear ordering and $a \in A$, we denote by $A \upharpoonright a$ the initial segment $\{x \in A : x < a\}$, viewed as a linear order.

Definition 8. Let A be a linear ordering. $B_D(A)$ is defined to be the (inclusion-)least linear ordering C such that the following hold:

1. C contains the term a^* , for any $a \in A$.
2. C contains the term $\psi(0)$.
3. Suppose that
 - (a) $\alpha_1, \dots, \alpha_k, \alpha_{k+1} \in D(C)$,
 - (b) $\psi(2^{\alpha_1} + \dots + 2^{\alpha_k}) \in C$,
 - (c) $\alpha_{k+1} <_{D(C)} \alpha_k$, and
 - (d) $\alpha_{k+1} \in D(C \upharpoonright \psi(2^{\alpha_1} + \dots + 2^{\alpha_k}))$;
 then, $\psi(2^{\alpha_1} + \dots + 2^{\alpha_k} + 2^{\alpha_{k+1}}) \in C$.
4. if $a <_A b$, then $a^* <_C b^*$.
5. $a^* <_C \psi(t)$, for any $a^*, \psi(t) \in C$.
6. if $\psi(t), \psi(u) \in C$, $t, u \in 2^{D(C)}$, and $t <_{2^{D(C)}} u$, then $\psi(t) <_C \psi(u)$.

We shall refer to B_D as the *collapsing functor* built over D . This is because Definition 8 could be regarded as a term system performing formal ordinal collapsing which admit a variation of the usual semantics for ordinal collapsing functions familiar from Ordinal Analysis.

Proposition 9 (ZFC). Let α be an ordinal and D be an \subseteq -preserving dilator. Let Ω be a regular cardinal such that $\max(|\alpha|, |D|) < \Omega$. We define a partial function

$$\tilde{\psi}^\alpha : \Omega + 2^{D(\Omega)} + 1 \rightarrow \Omega$$

by recursion on $2^{D(\Omega)}$ as follows:

Suppose $\alpha_n < \alpha_{n-1} < \dots < \alpha_1$. Then, $\Omega + 2^{\alpha_1} + \dots + 2^{\alpha_n} < 2^{D(\Omega)}$ is in the domain of $\tilde{\psi}^\alpha$ if for each $i < n$, $\alpha_{i+1} \in D(\tilde{\psi}^\alpha(\Omega + 2^{\alpha_1} + \dots + 2^{\alpha_i}))$. Additionally, we add $\Omega + 2^{D(\Omega)}$ to the domain of $\tilde{\psi}^\alpha$. For $\beta \in \text{dom}(\tilde{\psi}^\alpha)$, we put

$$\tilde{\psi}^\alpha(\beta) = \min \left\{ \Omega \setminus \left(\alpha \cup \{ \tilde{\psi}^\alpha(\beta') \mid \beta' \in \beta \cap \text{dom}(\tilde{\psi}^\alpha) \} \right) \right\}.$$

Then, $\tilde{\psi}^\alpha(\Omega + 2^{D(\Omega)})$ is a well-order isomorphic to $B_D(\alpha)$.

Proof. First, using the fact that $\max(|\alpha|, |D|) < \Omega$, a simple induction shows that

$$\forall \beta \in \text{dom}(\tilde{\psi}^\alpha) \left| \alpha \cup \{ \tilde{\psi}^\alpha(\beta') \mid \beta' \in \beta \cap \text{dom}(\tilde{\psi}^\alpha) \} \right| < \Omega,$$

and hence, $\tilde{\psi}^\alpha(\beta)$ is defined.

Given this, the isomorphism is essentially immediate from the definition of $B_D(\alpha)$. Note that the conditions on the domain of $\tilde{\psi}^\alpha$ correspond to the term-formation rules of B_D in Definition 8(1)–8(3); that the range of the function $\tilde{\psi}^\alpha$ takes values in (α, Ω) , corresponding to Definition 8(4) and 8(5); and

indeed that $\tilde{\psi}^\alpha(\beta)$ is precisely the least ordinal greater than $\tilde{\psi}^\alpha(\beta')$ for $\beta' < \beta$ in the domain of $\tilde{\psi}^\alpha$, corresponding to Definition 8(6). From this, it follows that the function

$$\begin{aligned} \iota_\alpha : B_D(\alpha) &\rightarrow \tilde{\psi}^\alpha(\Omega + 2^{D(\Omega)}) \\ \gamma &\mapsto \gamma, & (\gamma < \alpha) \\ \psi(2^{\alpha_1} + \dots + 2^{\alpha_n}) &\mapsto \tilde{\psi}^\alpha(\Omega + 2^{\alpha_1} + \dots + 2^{\alpha_n}) \end{aligned}$$

is an isomorphism. □

The function $\tilde{\psi}^\alpha$ in Proposition 9 is a ‘pure’ variant of collapsing function, where the ordinals $< \Omega$ are not closed under any ordinal functions (e.g., addition or the Veblen functions) as part of the construction; these are represented either as constants or values of $\tilde{\psi}^\alpha$. The ordinals $\geq \Omega$ are represented according to the dilator D .

While Proposition 9 gives a natural semantics for the collapsing functor B_D , it requires access to uncountable cardinals. We now establish the totality of B_D in the weak system ACA_0 .

Lemma 10 (ACA_0). *Suppose D is an \subseteq -preserving pre-dilator and A is a linear ordering. Then, $B_D(A)$ exists.*

Proof. The idea is to construct $B_D(A)$ as the limit of an inductive definition of length ω . We construct linear orders

$$B_{D,0}(A) \subseteq B_{D,1}(A) \subseteq \dots \subseteq B_{D,n}(A) \subseteq \dots$$

Recursively, we denote by $B_{D,<n}(A)$ the already defined linear order $\bigcup_{m<n} B_{D,m}(A)$ and define $B_{D,n}(A)$ to contain the following terms:

1. a^* , for $a \in A$;
2. $\psi(0)$;
3. $\psi(2^{\alpha_1} + \dots + 2^{\alpha_k} + 2^{\alpha_{k+1}})$, whenever $\alpha_1, \dots, \alpha_k, \alpha_{k+1} \in D(B_{D,<n}(A))$ are such that
 - (a) $\psi(2^{\alpha_1} + \dots + 2^{\alpha_k}) \in B_{D,<n}(A)$,
 - (b) $\alpha_{k+1} <_{D(B_{D,<n}(A))} \alpha_k$, and
 - (c) $\alpha_{k+1} \in D(B_{D,<n}(A) \upharpoonright \psi(2^{\alpha_1} + \dots + 2^{\alpha_k}))$.

The order on these terms is defined in the same way as above (i.e., $a^* <_{B_{D,<n}(A)} b^*$ if and only if $a <_A b$); we always have $a^* <_{B_{D,<n}(A)} \psi(t)$; and $\psi(t) <_{B_{D,n}(A)} \psi(u)$ if and only if $t <_{2^{D(B)}} u$. It is straightforward to check that, provably in ACA_0 , the order $B_D(A)$ constructed as the union of the chain $\{B_{D,<n}(A) : n \in \mathbb{N}\}$ satisfies the definition of $B_D(A)$, for any linear ordering A . □

Definition 11. Let A and A' be linear orders and $f : A \rightarrow A'$ be a morphism. We define

$$B_D(f) : B_D(A) \rightarrow B_D(A')$$

to be the unique map $g : B_D(A) \rightarrow B_D(A')$ such that

1. $g(a^*) = (f(a))^*$, and
2. $g(\psi(t)) = \psi(2^{D(g)}(t))$.

Lemma 12 (ACA_0). *Suppose that A and A' are linear orders and $f : A \rightarrow A'$ is a morphism. Then, $B_D(f)$ exists.*

Proof. This is similar to the previous proof. By induction, we define a sequence of embeddings $B_{D,n}(f) : B_{D,n}(A) \rightarrow B_{D,n}(A')$. Letting

$$B_{D,<n}(f) = \bigcup_{m<n} B_{D,m}(f) : B_{D,<n}(A) \rightarrow B_{D,<n}(A'),$$

we put

1. $B_{D,n}(f)(a^\star) = (f(a))^\star$, and
2. $B_{D,n}(f)(\psi(t)) = \psi(2^{D(B_{D,<n}(f))}(t))$.

Clearly, $B_{D,<\omega}(f)$ satisfies the definition of $B_D(f)$. By induction on n , one shows that for any $g: B_D(A) \rightarrow B_D(B)$ satisfying the definition of $B_D(f)$, we have

$$g \upharpoonright_{B_{D,n}(A)} = B_{D,n}(f).$$

Thus, indeed there is a unique mapping satisfying the definition of $B_D(f)$. □

Lemma 13 (ACA₀). *Suppose D is a \subseteq -preserving pre-dilator. Then, so too is B_D .*

Proof. The functoriality and \subseteq -preservation of B_D are immediate from the definition. In order to check that B_D is finitary, it is enough to show that for any A , we have

$$B_D(A) = \bigcup_{A' \subseteq_{\text{fin}} A} B_D(A').$$

This is done by showing, via a straightforward induction on n , that

$$B_{D,n}(A) = \bigcup_{A' \subseteq_{\text{fin}} A} B_{D,n}(A').$$

By a straightforward induction on n , we also show that for any order A and any suborders A', A'' of A , we have

$$B_{D,n}(A') \cap B_{D,n}(A'') = B_{D,n}(A' \cap A'').$$

This implies that for any order A and its suborders A', A'' , we have

$$B_D(A') \cap B_D(A'') = B_D(A' \cap A'')$$

(i.e., that B_D preserves pullbacks). This concludes the proof that B_D is a \subseteq -preserving pre-dilator. □

For an arbitrary pre-dilator D , we may define the pre-dilator B_D to be $B_{D'}$, where D' is some fixed \subseteq -preserving pre-dilator naturally isomorphic to D . Below, a pre-dilator D is *weakly finite* if it maps finite orders to finite orders.

Together with Proposition 9, Theorem 14 below asserts (in particular) that if D is a denotation system for ordinals, then the binary fast-growing functor given by D is isomorphic to an ordinal collapsing function; hence, it can be regarded as a precise formulation of the fundamental observation on p. 4.

Theorem 14 (ACA₀). *Suppose D is a weakly finite dilator. Let B_D be the collapsing functor over D as in Definition 8 and let \hat{B}_D be the fast-growing functor over D defined in §3. Then, there is a natural isomorphism*

$$\eta : B_D \cong \hat{B}_D.$$

Proof. Let $\theta_D : D \rightarrow D'$ be the natural isomorphism given by the definition of B_D for arbitrary pre-dilators D . For each $n \in \mathbb{N}$, let

$$\eta_n : \hat{B}_D(n) \rightarrow B_D(n)$$

be the function given by

$$\begin{aligned}
 m &\mapsto m^*, && \text{if } m < n, \\
 B_{\alpha_k} \dots B_{\alpha_1}(n) &\mapsto \psi(2^{\alpha'_1} + \dots + 2^{\alpha'_k}), \text{ where } \alpha'_i = \theta_{B_D(n)}(\alpha_i), && \text{otherwise.}
 \end{aligned}$$

It is clear that the collection of η_n for $n \in \mathbb{N}$ forms a natural isomorphism between $\hat{B}_D : \text{Nat} \rightarrow \text{LO}$ to $B_D \upharpoonright \text{Nat}$, the restriction of B_D to the category Nat . Indeed, suppose that $f : n \rightarrow n'$ is strictly increasing. Then, for each number $B_{\alpha_k} \dots B_{\alpha_1}(n) < \hat{B}_D(n)$, and letting α'_i be as above, we have

$$\begin{aligned}
 B_D(f) \circ \eta_n(B_{\alpha_k} \dots B_{\alpha_1}(n)) &= B_D(f)(\psi(2^{\alpha'_1} + \dots + 2^{\alpha'_k})) \\
 &= \psi(2^{D(B_D(f))(\alpha'_1)} + \dots + 2^{D(B_D(f))(\alpha'_k)}) \\
 &= \eta_{n'}(B_{D(\hat{B}_D(f))(\alpha_k)} \dots B_{D(\hat{B}_D(f))(\alpha_1)}(n')) \\
 &= \eta_{n'} \circ \hat{B}_D(f)(B_{\alpha_k} \dots B_{\alpha_1}(n)).
 \end{aligned}$$

Since both \hat{B}_D and B_D are finitary, this natural transformation extends to a natural isomorphism between the extension of \hat{B}_D to LO and B_D , as desired. \square

Remark 15. We note, without giving details, that, in fact, the operation

$$D \mapsto B_D$$

could be extended to a pullback-preserving functor on the category of pre-dilators; in other words, B is a *preptyx* and – as we will see in the next section – indeed it is a *ptyx*. Furthermore, if we treat the category of pre-dilators as a category of structures (denotation systems can be regarded as structures), then, in fact, the definition of B in terms of formal ‘ordinal collapses’ will be a \subseteq -preserving functor.

Remark 16. The construction of $B_D(A)$, in fact, works for all \subseteq -preserving finitary functors $D : \text{LO} \rightarrow \text{LO}$. We decided not to cover the extension to finitary functors in this paper for the following reason: The fact that any dilator is naturally isomorphic to a \subseteq -preserving one is a known fact following from Girard’s fundamental theorem of dilators. There is an unpublished result by the second author that establishes an analog of Girard’s theorem for finitary functors on LO that, in particular, implies that any finitary functor on LO is naturally isomorphic to a \subseteq -preserving one. This result shall appear in a forthcoming article by the second and third authors.

In order to make sense of B_D in the case when D is not \subseteq -preserving, it suffices to choose a denotation system D' and put $B_D = B_{D'}$.

5. Π_1^1 -Comprehension from fast-growing hierarchies

In this section, we present a characterization of the system $\Pi_1^1\text{-CA}_0$ in terms of fast-growing hierarchies.

Theorem 17. *The following are equivalent over ACA_0 :*

1. $\Pi_1^1\text{-CA}_0$;
2. for every weakly finite dilator D , B_D is a weakly finite dilator;
3. for every dilator D , B_D is a dilator.

Lemma 18 ($\Pi_1^1\text{-CA}_0$). *For any dilator D and any well-ordering A , $B_D(A)$ is well-ordered.*

Proof. By the definition of B_D , it suffices to consider the case that D is \subseteq -preserving.

Let Ω be the largest well-ordered initial segment of $B_D(A)$. This exists by $\Pi_1^1\text{-CA}_0$. By hypothesis, D is a dilator, so $D(\Omega)$ is well-ordered. By induction on ordinals $\alpha \in D(\Omega)$, we show that for every

$\psi(2^{\alpha_1} + \dots + 2^{\alpha_k}) \in \Omega$, if $\psi(2^{\alpha_1} + \dots + 2^{\alpha_k} + 2^\alpha) \in B_D(A)$, then

$$a := \psi(2^{\alpha_1} + \dots + 2^{\alpha_k} + 2^\alpha) \in \Omega.$$

Suppose that $x <_{B_D(A)} a$. Then, either

$$x \leq \psi(2^{\alpha_1} + \dots + 2^{\alpha_k})$$

or else there are $l \in \mathbb{N}$ and β_1, \dots, β_l with $\beta_l <_{D(B_D(A))} \dots <_{D(B_D(A))} \beta_1 < \alpha$ such that

$$x \leq \psi(2^{\alpha_1} + \dots + 2^{\alpha_k} + 2^{\beta_1} + \dots + 2^{\beta_l}).$$

We show by induction on $i \leq l$ that

1. $\beta_i \in D(\Omega)$, and
2. $\psi(2^{\alpha_1} + \dots + 2^{\alpha_k} + 2^{\beta_1} + \dots + 2^{\beta_i}) \in \Omega$.

By the definition of B_D , we have

$$\beta_{i+1} \in D(B_D(A) \upharpoonright \psi(2^{\alpha_1} + \dots + 2^{\alpha_k} + 2^{\beta_1} + \dots + 2^{\beta_i})).$$

By the induction hypothesis on i , $\psi(2^{\alpha_1} + \dots + 2^{\alpha_k} + 2^{\beta_1} + \dots + 2^{\beta_i}) \in \Omega$ (this is immediate by the assumption on $\alpha_1, \dots, \alpha_k$ in the case $i = 0$) and Ω is an initial segment of $B_D(A)$, so the first claim follows. The second follows from the induction hypothesis on α . We have shown that every element $x <_{B_D(A)} a$ belongs to Ω , and thus, $a \in \Omega$, as desired.

By a straightforward induction on k , it follows that

$$\psi(2^{\alpha_1} + \dots + 2^{\alpha_k}) \in \Omega$$

for all $\psi(2^{\alpha_1} + \dots + 2^{\alpha_k}) \in B_D(A)$ – that is, that $B_D(A) = \Omega$, and thus, $B_D(A)$ is well ordered. □

Lemma 19 (ACA₀). *Suppose that for every weakly finite dilator D , B_D is a weakly finite dilator. Then, for every dilator D , B_D is a dilator.*

Proof. If D and D' are naturally isomorphic pre-dilators, then so too are B_D and $B_{D'}$. Hence, it suffices to prove the theorem for denotation systems D . Let D be a denotation system and enumerate all ordinal terms of D by d_0, d_1, \dots ; write n_i for the arity of d_i . We define a new dilator \hat{D} consisting of terms \hat{d}_i of arity $n_i + i$. Clearly, \hat{D} will be weakly finite. The comparison rules for \hat{D} are given by

$$\hat{d}_i(x_1, \dots, x_{n_i}, \hat{x}_1, \dots, \hat{x}_i) <_{\hat{D}} \hat{d}_j(y_1, \dots, y_{n_j}, \hat{y}_1, \dots, \hat{y}_j)$$

if and only if one of the following holds:

1. $d_i(x_1, \dots, x_{n_i}) <_D d_j(y_1, \dots, y_{n_j})$, or else
2. $i = j, x_1 = y_1, \dots, x_{n_i} = y_{n_i}$, and $(\hat{x}_1, \dots, \hat{x}_i) <_{lex} (\hat{y}_1, \dots, \hat{y}_j)$.

It is easy to check that $<_{\hat{D}}$ is a linear order. And it is easy to see that for any A , we have an embedding of $\hat{D}(A)$ into $2^A \cdot D(A)$:

$$\hat{d}_i(a_1, \dots, a_{n_i}, b_1, \dots, b_i) \mapsto 2^A \cdot d_i(a_1, \dots, a_{n_i}) + 2^{b_1} + \dots + 2^{b_i}.$$

Thus, \hat{D} is a dilator. Observe that for each finite order A , the order $\hat{D}(A)$ contains only terms of the form $\hat{d}_i(\dots)$, where $i \leq |A|$, and hence, $\hat{D}(A)$ is finite. Therefore, \hat{D} is a weakly finite dilator. Thus, $B_{\hat{D}}$ is a dilator.

In order to show that B_D preserves well-foundedness, it is enough to find a strictly increasing map

$$e : B_D(A) \rightarrow B_{\hat{D}}(\omega + A)$$

for an arbitrary well-order A . We fix a well-order A and define, by recursion on n ,

$$e_n : B_{D,n}(A) \rightarrow B_{\hat{D},n}(\omega + A)$$

1. $e_n(a^*) = (\omega + a)^*$;
2. $e_n(\psi(2^{\alpha_1} + \dots + 2^{\alpha_k})) = \psi(2^{\alpha'_1} + \dots + 2^{\alpha'_k})$, where if $\alpha_i = d_j(a_1, \dots, a_{n_j})$, then $\alpha'_i = \hat{d}_j(e_{n-1}(a_1), \dots, e_{n-1}(a_{n_j}), (j-1)^*, \dots, 0^*)$.

By a straightforward induction on n , we show that e_n 's form a sequence of expanding strictly increasing maps. We finish the proof by defining $e = \bigcup_{n < \omega} e_n$. □

In order to complete the proof, we have to show that if B_D is a dilator for every dilator D , then $\Pi_1^1\text{-CA}_0$ holds. This will be done by appealing to a theorem of Freund [7] whereby $\Pi_1^1\text{-CA}_0$ is equivalent to a higher-order fixed-point principle. Using the terminology of Freund [7], this is the statement that every dilator D has a well-founded Bachmann-Howard fixed point (we recall this definition during the course of the forthcoming proof).

Lemma 20 (ACA_0). *Suppose that B_D is a dilator for every dilator D . Then, every dilator has a well-founded Bachmann-Howard fixed point.*

Proof. It suffices to restrict to the case where D is a denotation system and, in particular, \subseteq -preserving. We need to find a Bachmann-Howard fixed point: a well-ordering A and a function

$$\theta : D(A) \rightarrow A$$

such that the following hold:

1. whenever we have $\alpha <_{D(A)} \beta$ and $\text{supp}(\alpha) <_A \theta(\beta)$, then we have $\theta(\alpha) <_A \theta(\beta)$, and
2. $\text{supp}(\alpha) <_A \theta(\alpha)$ for every $\alpha \in D(A)$.

Consider the dilator $F = (\omega + 1)D + \omega$. This is defined as follows: Given a linear order B , the order $F(B)$ consists of terms of two types:

3. $(\omega + 1)\alpha + a$, where $\alpha \in D(B)$ and $a \leq \omega$;
4. $\Lambda + n$, where $n < \omega$.

The terms are compared according to the following rules:

5. terms of the first type are always smaller than the terms of the second type;
6. $(\omega + 1)\alpha + a$ is smaller than $(\omega + 1)\beta + b$ if and only if either $\alpha < \beta$ or $\alpha = \beta$ and $a < b$;
7. $\Lambda + n$ is smaller than $\Lambda + m$ if and only if $n < m$.

For a strictly increasing function $f : B \rightarrow C$, we put

$$\begin{aligned} F(f) : F(B) &\rightarrow F(C) \\ (\omega + 1)\alpha + a &\mapsto (\omega + 1)(D(f)(\alpha)) + a; \\ \Lambda + n &\mapsto \Lambda + n. \end{aligned}$$

It is easy to see that F is indeed a dilator.

We put $A = B_F(0)$. This is a well-ordering since B_F is a dilator by hypothesis. Note that all elements of A are of the form $\psi(2^{\alpha_1} + \dots + 2^{\alpha_n})$, where $\alpha_i \in F(A)$. We define

$$\theta : D(A) \rightarrow A$$

to be the function that maps $\alpha \in D(A)$ to the least element of A of the form

$$\psi(2^{\alpha_1} + \dots + 2^{\alpha_n} + 2^{(\omega+1)\alpha+\omega}).$$

This function is well defined (i.e., for any $\alpha \in D(A)$, there is some element of A as above). Indeed, the elements of the form $\psi(2^{\Lambda+n})$ are cofinal in A . Thus, for any $\alpha \in D(A)$, one can find n large enough so that $\alpha \in D(A \upharpoonright \psi(2^{\Lambda+n}))$, and hence,

$$\psi(2^{\Lambda+n} + 2^{(\omega+1)\alpha+\omega}) \in A.$$

Let us verify that θ satisfies property (1): we suppose $\alpha <_{D(A)} \beta$ and $\text{supp}(\alpha) <_A \theta(\beta)$ and claim that $\theta(\alpha) <_A \theta(\beta)$. Let β_1, \dots, β_n be such that

$$\theta(\beta) = \psi(2^{\beta_1} + \dots + 2^{\beta_n} + 2^{(\omega+1)\beta+\omega}).$$

Observe that all elements of the form

$$\psi(2^{\beta_1} + \dots + 2^{\beta_n} + 2^{(\omega+1)\beta+m})$$

are in A and are cofinal below $\theta(\beta)$. It follows that, since

$$\text{supp}(\alpha) = \text{supp}((\omega + 1)\alpha + \omega)$$

is a finite set, we can find $m \in \mathbb{N}$ large enough so that

$$\text{supp}(\alpha) <_A \psi(2^{\beta_1} + \dots + 2^{\beta_n} + 2^{(\omega+1)\beta+m}). \tag{3}$$

For such an m , $\psi(2^{\beta_1} + \dots + 2^{\beta_n} + 2^{(\omega+1)\beta+m} + 2^{(\omega+1)\alpha+\omega})$ is in A , by (3) and since

$$(\omega + 1)\alpha + \omega <_{F(A)} (\omega + 1)\beta + m$$

follows from $\alpha <_{D(A)} \beta$. Therefore,

$$\begin{aligned} \theta(\alpha) &\leq \psi(2^{\beta_1} + \dots + 2^{\beta_n} + 2^{(\omega+1)\beta+m} + 2^{(\omega+1)\alpha+\omega}) \\ &< \psi(2^{\beta_1} + \dots + 2^{\beta_n} + 2^{(\omega+1)\beta+\omega}) \\ &= \theta(\beta). \end{aligned}$$

The fact that θ satisfies (2) is immediate from the construction. Indeed, for any $\alpha \in D(A)$, the value $\theta(\alpha)$ is of the form $\psi(2^{\alpha_1} + \dots + 2^{\alpha_n} + 2^{(\omega+1)\alpha+\omega}) \in A$ and by the definition of $B_F(A)$, we have

$$\text{supp}(\alpha) = \text{supp}((\omega + 1)\alpha + \omega) <_A \psi(2^{\alpha_1} + \dots + 2^{\alpha_n}) <_A \theta(\alpha),$$

as desired. □

Putting together the lemmata in this section, the proof of Theorem 17 is now complete.

Remark 21. By appealing to the main theorem of Freund [8] and carefully formalizing the definition of B_D in Section 4, the equivalence in Theorem 17 can be proved in RCA_0 .

Competing interest. The authors have no competing interest to declare.

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