Canad. Math. Bull.Vol. **36** (4), 1993 pp. 497–506

BLOW ANALYTIC MAPPINGS AND FUNCTIONS

Dedicated to Professor Haruo Suzuki on his sixtieth birthday

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ABSTRACT. Let $\pi\colon \mathcal{M} \longrightarrow R^n$ be the blowing-up of R^n at the origin. Then a continuous map-germ $f\colon (R^n - 0, 0) \longrightarrow R^m$ is called *blow analytic* if there exists an analytic map-germ $\tilde{f}\colon \big(\mathcal{M}, \pi^{-1}(0)\big) \longrightarrow R^m$ such that $f \circ \pi | \mathcal{M} - \pi^{-1}(0) = \tilde{f} | \mathcal{M} - \pi^{-1}(0)$. Then an inverse mapping theorem for blow analytic mappings as a generalization of classical theorem is shown. And the following is shown. Theorem: The analytic family of blow analytic functions with isolated singularities admits an analytic trivialization after blowing-up.

1. **Introduction.** The notion of blow analytic mapping was originally defined by T-C. Kuo ([4]). A continuous mapping of Euclidean spaces is called a *blow analytic mapping* if the mapping after blowing-up is analytic (see the Definition 2-1). He proves that some families of analytic functions with isolated singularities admit analytic trivializations after one point blowing-up. In successive studies, T. Fukui and I prove that some families of non-degenerate analytic functions and mappings admit analytic trivializations after some modifications and some general blowing-ups (see [2, 3, 8]). M. Suzuki ([7]) studies a necessary condition for families of analytic functions to admit a blow analytic trivialization. In these papers, the notion of blow analytic mappings appears only as their tools to study singularity theory.

In this paper, we shall study blow analytic map-germs and function-germs themselves. We shall prove a so called inverse mapping theorem for blow analytic mappings (see Theorem 2-4 in §2) and a blow analytic trivialization theorem of some families of blow analytic functions with isolated singularities (see Theorem 3-7 in §3) as a generalization of Kuo's theorem (see Theorem 3-8). We can see some related studies in [1].

2. **Blow differentiable mappings and an inverse mapping theorem.** Let $(x_1, x_2, \ldots, x_n)$ be a coordinate system of n dimensional Euclidean space $\mathbf{R}^n$ and $[\xi_1 : \xi_2 : \cdots : \xi_n]$ be a homogeneous coordinate system of $n-1$ dimensional real projective space $\mathbf{P}^{n-1}$.

The blowing-up of $\mathbf{R}^n$ at the origin is defined by the following:

$$\mathcal{M}^n = \{(x, \xi) \in \mathbf{R}^n \times \mathbf{P}^{n-1} \mid x_i\xi_j = x_j\xi_i, 1 \le i,j \le n\}$$

Supported in part by the Japan Society for the Promotion of Science.
Received by the editors February 8, 1992.
AMS subject classification: Primary: 58C27; secondary: 58A07.
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https://doi.org/10.4153/CMB-1993-065-1 Published online by Cambridge University Press

and

$$\pi_n\colon \mathcal{M}^n \longrightarrow \mathbf{R}^n, \qquad \pi_n(x,\xi) = x.$$

DEFINITION 2-1. Let $0 \le r \le \infty$ or $r = \omega$.

(1) A continuous map-germ $f\colon (\mathbf{R}^n - 0, 0) \longrightarrow \mathbf{R}^m$ is *blow* C^r if and only if there exists a C^r map-germ $\tilde{f}\colon \big(\mathcal{M}^n, \pi_n^{-1}(0)\big) \longrightarrow \mathbf{R}^m$ such that $f \circ \pi_n | \mathcal{M}^n - \pi_n^{-1}(0) = \tilde{f} | \mathcal{M}^n - \pi_n^{-1}(0)$. Namely, the following diagram is commutative:

$$\begin{array}{ccc}
\big(\mathcal{M}^n, \pi_n^{-1}(0)\big) & \xrightarrow{\tilde{f}} & \mathbf{R}^m \\
\iota \uparrow & & \uparrow \mathrm{id}_{\mathbf{R}^m} \\
\big(\mathcal{M}^n - \pi_n^{-1}(0), \pi_n^{-1}(0)\big) & \xrightarrow{\tilde{f}|} & \mathbf{R}^m \\
\pi_n \downarrow & & \downarrow \mathrm{id}_{\mathbf{R}^m} \\
(\mathbf{R}^n - 0, 0) & \xrightarrow{f} & \mathbf{R}^m
\end{array}$$

where $\tilde{f}| = \tilde{f} | \mathcal{M}^n - \pi_n^{-1}(0)$ and ι is a canonical inclusion map-germ.

(2) A homeomorphism map-germ $f\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^n, 0)$ is a *blow* C^r *isomorphism* if and only if there exists a C^r isomorphism map-germ

$$\tilde{f}\colon \big(\mathcal{M}^n, \pi_n^{-1}(0)\big) \longrightarrow \big(\mathcal{M}^n, \pi_n^{-1}(0)\big)$$

such that $f \circ \pi_n = \pi_n \circ \tilde{f}$. Namely, the following diagram is commutative:

$$\begin{array}{ccc}
\big(\mathcal{M}^n, \pi_n^{-1}(0)\big) & \xrightarrow{\tilde{f}} & \big(\mathcal{M}^n, \pi_n^{-1}(0)\big) \\
\pi_n \downarrow & & \downarrow \pi_n \\
(\mathbf{R}^n, 0) & \xrightarrow{f} & (\mathbf{R}^n, 0).
\end{array}$$

EXAMPLE 2-2. (1) Let

$$f(x_1, x_2) = \frac{x_1 x_2}{x_1^2 + x_2^2} \text{ if } (x_1, x_2) \ne (0,0).$$

This analytic function germ $f\colon (\mathbf{R}^2 - 0, 0) \longrightarrow \mathbf{R}$ is blow analytic because there exists an analytic function germ $\tilde{f}\colon \big(\mathcal{M}^2, \pi_2^{-1}(0)\big) \longrightarrow (\mathbf{R}, 0)$ such that $f \circ \pi_2 | \mathcal{M}^2 - \pi_2^{-1}(0) = \tilde{f} | \mathcal{M}^2 - \pi_2^{-1}(0)$. In fact, we may take

$$\tilde{f}(x_1, x_2; \xi_1 : \xi_2) = \frac{\xi_1 \xi_2}{\xi_1^2 + \xi_2^2}.$$

(2) Let

$$f(x_1, x_2) = \begin{cases} \frac{x_1^3 |x_1|}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \ne (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

This continuous function germ $f\colon (\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}, 0)$ is blow C^1 because $f \circ \pi_n = \frac{\xi_1^2}{\xi_1^2 + \xi_2^2} x_1 \, |x_1| \colon \big(\mathcal{M}^2, \pi_2^{-1}(0)\big) \longrightarrow (\mathbf{R}, 0)$ is a C^1 function.

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(3) Let

$$f_1(x_1,x_2) = \begin{cases} \frac{x_1^3}{x_1^2+x_2^2} & \text{if } (x_1,x_2) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f_2(x_1,x_2) = \begin{cases} \frac{x_2^4}{x_1^2+x_2^2} & \text{if } (x_1,x_2) \neq (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

Then the map-germ $f = (f_1,f_2)\colon (\mathbf{R}^2,0) \longrightarrow (\mathbf{R}^2,0)$ and the function germ $g = f_1\colon (\mathbf{R}^2,0) \longrightarrow (\mathbf{R},0)$ are both blow analytic. But the composition germ

$$g \circ f = \frac{x_1^9}{(x_1^6+x_2^8)(x_1^2+x_2^2)}$$

is not blow analytic. In fact,

$$g \circ f \circ \pi_2|_{\mathcal{M}_2^2} = \frac{x\xi^9}{(\xi^6+x^2)(\xi^2+1)}$$

is not analytic at $(x,\xi) = (0,0)$, where $(x,\xi) = (x_2, \xi_1/\xi_2)$ is a local coordinate system of a local chart $\mathcal{M}_2^2 = \{(x_1,x_2;\xi_1 : \xi_2) \in \mathcal{M}^2 \mid \xi_2 \neq 0\}$ in $\mathcal{M}^2$.

REMARK 2-3. (1) f: C^r map-germ $\Longrightarrow f$:blow C^r map-germ.

(2) $f = (f_1,f_2,\ldots,f_m)$:blow C^r map-germ $\Longleftrightarrow$ all components function germs f_i $(1 \leq i \leq m)$ are blow C^r.

(3) In general, the condition that f, g are blow C^r map-germs does not imply that $f \circ g$ is a blow C^r map-germ (see Example 2-2 (3)).

Let

$$\phi_n\colon \mathbf{R}^n \times \mathbf{P}^{n-1} \longrightarrow \mathcal{M}^n$$

be an analytic deformation retract defined by

$$\phi_n(x;\xi) = \left(\frac{\langle x,\xi\rangle}{|\xi|^2}\xi, \xi \right)$$

where $\langle x,\xi\rangle = \sum_{i=1}^n x_i\xi_i$ and $|\xi|^2 = \sum_{i=1}^n \xi_i^2$.

For a blow analytic function $f\colon (\mathbf{R}^n - 0,0) \longrightarrow \mathbf{R}$, we have an analytic functions $\tilde{f}\colon \left(\mathcal{M}^n, \pi_n^{-1}(0)\right) \longrightarrow \mathbf{R}$ such that $f \circ \pi_n = \tilde{f}|_{(\mathcal{M}^n - \pi_n^{-1}(0))}$. Define an analytic function $\hat{f}$ as

$$\hat{f} = \tilde{f} \circ \phi_n\colon (\mathbf{R}^n \times \mathbf{P}^{n-1}, 0 \times \mathbf{P}^{n-1}) \longrightarrow \mathbf{R}.$$

Let $U_\lambda = \{\xi \in \mathbf{P}^{n-1} \mid \xi_\lambda \neq 0\}$, $\lambda = 1,2,\ldots,n$. Then the function $\hat{f}$ restricted to $\mathbf{R}^n \times U_\lambda$ is analytic and so it has a Taylor expansion:

$$\hat{f}(x;\xi) = \sum_k c_{\lambda,k}(\xi)x^k$$

where $k = (k_1,k_2,\ldots,k_n)$ and $x^k = x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$.

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It is clear that $c_{\lambda,k}(\xi) = c_{\mu,k}(\xi)$ for any $\xi \in U_\lambda \cap U_\mu$. So we have the Taylor expansion

$$\hat{f}(x:\xi) = \sum_k c_k(\xi)x^k$$

in a neighbourhood of $0 \times \mathbf{P}^{n-1}$ in $\mathbf{R}^n \times \mathbf{P}^{n-1}$ where $c_k(\xi)$ defined by $c_k(\xi)|U_\lambda = c_{\lambda,k}(\xi)$ are analytic functions on $\mathbf{P}^{n-1}$. So we have the following expression like a "power series"

$$f(x) = \sum_k c_k(x)x^k \text{ if } x \neq 0.$$

We must note that this power expansion may not be unique because $f(x_1,x_2) = \frac{x_1x_2}{x_1^2+x_2^2}x_1 = \frac{x_1^2}{x_1^2+x_2^2}x_2$ for example. But each homogeneous polynomial is uniquely determined in the homogeneous decomposition $f(x) = H_d(x) + H_{d+1}(x) + \cdots$ of the "power series expansion" of $f(x)$ where $H_d(x) \not\equiv 0$. And define $H(f) = H_d(x)$, the principal part of $f(x)$.

Now, we have the following

THEOREM 2-4 (INVERSE MAPPING THEOREM). *Let $f = (f_1, f_2, \ldots, f_n)\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^n, 0)$ be a blow analytic map-germ and $H(f) = \big(H(f_1), H(f_2), \ldots, H(f_n)\big)$. Then the following three statements are equivalent:*

(1) $f\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^n, 0)$ is a blow analytic isomorphism.

(2) $\deg H(f_p) = 1$, $1 \leq p \leq n$ and $H(f)\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^n, 0)$ is a blow analytic isomorphism.

(3) $\deg H(f_p) = 1$, $1 \leq p \leq n$ and $[H(f)] = [H(f_1) : H(f_2) : \cdots : H(f_n)]\colon \mathbf{P}^{n-1} \longrightarrow \mathbf{P}^{n-1}$ is an analytic isomorphism.

PROOF (1) $\Rightarrow$ (3). Since f is a blow analytic isomorphism, there exists an analytic isomorphism $\tilde{f}\colon \big(\mathcal{M}^n, \pi_n^{-1}(0)\big) \longrightarrow \big(\mathcal{M}^n, \pi_n^{-1}(0)\big)$ such that $f \circ \pi_n = \pi_n \circ \tilde{f}$. Namely, the following diagram is commutative:

$$\begin{array}{ccc} \big(\mathcal{M}^n, \pi_n^{-1}(0)\big) & \xrightarrow{\tilde{f}} & \big(\mathcal{M}^n, \pi_n^{-1}(0)\big) \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_n} \\ (\mathbf{R}^n, 0) & \xrightarrow{f} & (\mathbf{R}^n, 0). \end{array}$$

We have

$$\begin{aligned} \tilde{f}(0, [\xi]) &= \lim_{x \to 0} \tilde{f}(x, [\xi]) \\ &= 0 \times \lim_{x \to 0} \big[f \circ \pi_n(x, [\xi])\big] \\ &= 0 \times \lim_{t \to 0} \big[f \circ \pi_n(t\xi, [\xi])\big] \\ &= 0 \times \lim_{t \to 0} [f(t\xi)] \\ &= 0 \times \big[H(f)([\xi])\big]. \end{aligned}$$

Since $\tilde{f}(0, [\xi])\colon 0 \times \mathbf{P}^{n-1} \longrightarrow 0 \times \mathbf{P}^{n-1}$ is an analytic isomorphism, $\big[H(f)([\xi])\big]\colon \mathbf{P}^{n-1} \longrightarrow \mathbf{P}^{n-1}$ is an analytic isomorphism. Moreover $\deg H(f_p) = 1$, $1 \leq p \leq n$ because $\tilde{f}$ is

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isomorphic and t in the above equation can be one of coordinate functions of local coordinate system on the manifold $\mathcal{M}^n$. This completes the proof of (3).

(3) $\Rightarrow$ (2). Define $\tilde{H}(f)\colon \big(\mathcal{M}^n, \pi_n^{-1}(0)\big) \longrightarrow \big(\mathcal{M}^n, \pi_n^{-1}(0)\big)$ by

$$\begin{aligned}\tilde{H}(f)(x,[\xi]) &= \big(H(f)(x), [H(f)][\xi]\big)\\ &= \big(H(f)(t\xi), [H(f)][\xi]\big)\\ &= \big(tH(f)(\xi), [H(f)][\xi]\big).\end{aligned}$$

This is clearly an analytic isomorphism and $H(f) \circ \pi_n = \pi_n \circ \tilde{H}(f)$. This shows that $H(f)\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^n, 0)$ is a blow analytic isomorphism.

(2) $\Rightarrow$ (1). Since f is blow analytic, there exists an analytic map-germ $\tilde{f}\colon \big(\mathcal{M}^n, \pi_n^{-1}(0)\big) \longrightarrow \big(\mathcal{M}^n, \pi_n^{-1}(0)\big)$ such that $f \circ \pi_n = \pi_n \circ \tilde{f}$ and $\tilde{f}(x,[\xi]) = \big(f(x), \big[H([\xi])\big]\big)$. An easy calculation implies $d\tilde{f}(0,[\xi]) = d\tilde{H}(f)(0,[\xi])$. So, $\tilde{f}\colon \big(\mathcal{M}^n, \pi_n^{-1}(0)\big) \longrightarrow \big(\mathcal{M}^n, \pi_n^{-1}(0)\big)$ is locally isomorphic.

To prove that $\tilde{f}$ is injective implies the statement (1). Now suppose that $\tilde{f}$ is not injective near $\pi_n^{-1}(0)$. Then there exist two sequences $\{a_n\}$, $\{b_n\}$ of points in $\mathcal{M}^n$ tending to $\pi_n^{-1}(0)$ such that $a_n \neq b_n$ and $\tilde{f}(a_n) = \tilde{f}(b_n)$. Since $\pi_n^{-1}(0)$ is compact, we may suppose, choosing subsequences if necessary, that both sequences $\{a_n\}$, $\{b_n\}$ are convergent to $a_0, b_0 \in \pi_n^{-1}(0)$ respectively. If $a_0 \neq b_0$, then $\tilde{H}(f)(a_0) = \tilde{f}(a_0) = \tilde{f}(b_0) = \tilde{H}(f)(b_0)$. This implies that $\tilde{H}(f)$ is not injective, a contradiction to (2). If $a_0 = b_0$, then $\tilde{f}$ is not locally injective, a contradiction. This proves that $\tilde{f}$ is injective and completes the proof of (2) $\Rightarrow$ (1). ■

EXAMPLE 2-5. Let $f\colon (\mathbf{R}^2, 0) \longrightarrow (\mathbf{R}^2, 0)$ defined by

$$f(x) = \left(\frac{x_1^3 + x_2^3}{x_1^2 + x_2^2}, \frac{(x_1 - x_2)(x_1^2 - x_1x_2 + x_2^2)}{x_1^2 + x_2^2}\right)$$

be a blow analytic map-germ. Then, by the definition, $[H(f)] = [(\xi_1 + \xi_2) : (\xi_1 - \xi_2)]$ and this is an analtyic isomorphism of one dimensional real projective space $\mathbf{P}^1$. By Theorem 2-5, $f(x)$ is a blow analytic isomorphism.

In this case, we can represent explicitly the inverse map-germ f^{-1} of f from an easy calculation:

$$f^{-1}(y) = \left(\frac{(y_1 + y_2)(y_1^2 + y_2^2)}{y_1^2 + 3y_2^2}, \frac{(y_1 - y_2)(y_1^2 + y_2^2)}{y_1^2 + 3y_2^2}\right).$$

3. Blow analytic trivialization.

DEFINITION 3-1. A C^r vector field germ V at the origin of a vector field defined in a punctured neighbourhood of the origin in $\mathbf{R}^n$ is a *blow C^r vector field* if and only if there exists a C^r vector field germ $\tilde{V}$ on $\mathcal{M}^n$ at $\pi_n^{-1}(0)$ such that

$$d\big(\pi|\mathcal{M}^n - \pi_n^{-1}(0)\big)\big(\tilde{V}|\mathcal{M}^n - \pi_n^{-1}(0)\big) = V.$$

Let $U_\lambda = \{\xi \in \mathbf{P}^{n-1} \mid \xi_\lambda \neq 0\}$ and $\mathcal{M}_\lambda^n = \mathcal{M}^n \cap (\mathbf{R}^n \times U_\lambda)$. Let $u^\lambda = x_\lambda$ and $u^\mu = \frac{\xi_\mu}{\xi_\lambda}$ if $\mu \neq \lambda$. Then $u = (u^1, u^2, \ldots, u^n)$ is a local coordinate system of $\mathcal{M}_\lambda^n$.

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Suppose that $V = \sum_{p=1}^{n} a_p(x)\frac{\partial}{\partial x_p}$ is a blow C^r vector field germ at the origin and $\tilde{V}$ is a C^r vector field on $\mathcal{M}^n$ such that $d\pi'\tilde{V}' = V$ where $(\ )' = (\ )|\mathcal{M}^n - \pi_n^{-1}(0)$. Let us put $\tilde{V}^\lambda = \tilde{V}|\mathcal{M}_\lambda^n = \sum_{\mu=1}^{n} b_\mu(u)\frac{\partial}{\partial u^\mu}$.

Then we have $b'_\lambda = a_\lambda \circ \pi'$ and

$$b'_\mu = \frac{(a_\mu \circ \pi' - u^\mu a_\lambda \circ \pi')}{u^\lambda} \text{ if } \mu \neq \lambda.$$

And so,

$$\tilde{V}^\lambda = \tilde{a}'_\lambda \frac{\partial}{\partial u^\lambda} + \sum_{\mu \neq \lambda} \frac{(\tilde{a}'_\mu - u^\mu \tilde{a}'_\lambda)}{u^\lambda}\frac{\partial}{\partial u^\mu}.$$

Here we denote $\tilde{a}'_\mu = a_\mu \circ \pi'$. Thus we have the following

LEMMA 3-2. *A vector field $V = \sum_{p=1}^{n} a_p(x)\frac{\partial}{\partial x_p}$ is a blow C^r vector field germ at the origin if and only if the functions*

$$\tilde{a}'_\lambda = a_\lambda \circ \pi' \text{ and } \frac{(\tilde{a}'_\mu - u^\mu \tilde{a}'_\lambda)}{u^\lambda}$$

are extended to C^r functions in $\mathcal{M}_\lambda^n$ for any λ, μ: $1 \leq \mu \neq \lambda \leq n$.

COROLLARY 3-3. *If $a_p: (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$, $1 \leq p \leq n$ are blow analytic function germs, then $V = \sum_{p=1}^{n} a_p(x)\frac{\partial}{\partial x_p}$ is a blow analytic vector field germ at the origin.*

PROOF. Since $\tilde{a}_p = a_p \circ \pi(u^1, u^2, \ldots, u^n) = a_p(u^1u^\lambda, \ldots, u^\lambda, \ldots, u^nu^\lambda) = u^\lambda d_p(u^1, u^2, \ldots, u^n)$ where d_p are analytic functions, the condition in Lemma 3-2 is satisfied. This completes the proof of 3-3. ∎

LEMMA 3-4. *If $f: (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$ is a blow analytic function germ at the origin of $\mathbf{R}^n$, then $\frac{\partial f}{\partial x_p}$, $1 \leq p \leq n$ is also a blow analytic function germ at the origin.*

PROOF. Let $f: (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$ be a blow analytic function germ at the origin of $\mathbf{R}^n$. Let $u = (u^1, u^2, \ldots, u^n)$ be a local coordinate system of $\mathcal{M}_\lambda^n$ as above. Then we have the following equations:

$$\begin{cases} \dfrac{\partial f \circ \pi_n(u)}{\partial u^\lambda} = \left(\frac{\partial f}{\partial x_\lambda} + \sum_{\mu \neq \lambda} \frac{x_\mu}{x_\lambda}\frac{\partial f}{\partial x_\mu}\right) \circ \pi_n(u), \\ \dfrac{\partial f \circ \pi_n(u)}{\partial u^\mu} = \left(x_\lambda \frac{\partial f}{\partial x_\mu}\right) \circ \pi_n(u) \qquad (\mu \neq \lambda). \end{cases}$$

This implies the next equations.

$$(*) \qquad \begin{cases} \dfrac{\partial f}{\partial x_\lambda} \circ \pi_n(u) = \frac{\partial f \circ \pi_n(u)}{\partial u^\lambda} - \sum_{\mu \neq \lambda} \frac{u^\mu}{u^\lambda}\frac{\partial f \circ \pi_n(u)}{\partial u^\mu}, \\ \dfrac{\partial f}{\partial x_\mu} \circ \pi_n(u) = \frac{1}{u^\lambda}\frac{\partial f \circ \pi_n(u)}{\partial u^\mu} \qquad (\mu \neq \lambda). \end{cases}$$

Now, since $f \circ \pi_n(u)$ is analytic and $f \circ \pi(u^1, \ldots, u^{\lambda-1}, 0, u^{\lambda+1}, \ldots, u^n) = f(0) = 0$, we have $f \circ \pi_n(u) = u^\lambda f'(u)$. So the right hand side of the equations $(*)$ are analytic and this completes the proof of 3.4. ∎

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DEFINITION 3-5. Let $f\colon (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}, 0)$ be a blow analytic germ.

(1) We say that f has an *isolated singularity (at worst)* at the origin if

$$\left\{x \in \mathbf{R}^n - 0 \;\middle|\; \frac{\partial f}{\partial x_1}(x) = \frac{\partial f}{\partial x_2}(x) = \cdots = \frac{\partial f}{\partial x_n}(x) = 0\right\} = \emptyset.$$

(2) Let $f(x) = h_k(x) + h_{k+1}(x) + \cdots$ be a homogeneous decomposition of the "power series expansion" of f. Then we say that f has an *initially isolated singularity* at the origin if h_k has an isolated singularity (at worst) at the origin.

(3) Let $F(x;t)\colon (\mathbf{R}^n \times \mathbf{R}^m, 0 \times I) \longrightarrow (\mathbf{R}, 0)$ be a real analytic family of blow analytic functions, namely $F \circ (\pi \times \mathrm{id}_I)\colon (\mathcal{M}^n \times \mathbf{R}^m, \pi_n^{-1}(0) \times I) \longrightarrow (\mathbf{R}, 0)$ is analytic where $I = \times_{i=1}^m [a_i, b_i]$ a compact cube in $\mathbf{R}^m$. Let $F(x;t) = H_k(x;t) + H_{k+1}(x;t) + \cdots$ be a homogeneous decomposition with respect to x of the Taylor series of $F(x;t)$ and $k \geq 1$. Then we say that F has *(simultaneously) initially isolated singularities* at the origin along I if $H_k(x;t)$ has an isolated singularity for any $t \in I$.

DEFINITION 3-6. A real analytic family $F(x;t)$ with $F(0;t) = 0$, of blow analytic functions admits a *blow analytic trivialization* along I if there exist two neighbourhoods $\tilde{U}_1, \tilde{U}_2$ of $\pi_n^{-1}(0) \times I$ in $\mathcal{M}^n \times \mathbf{R}^m$ and t-level preserving analytic isomorphism $\tilde{H}\colon \tilde{U}_1 \longrightarrow \tilde{U}_2$ such that $F \circ (\pi_n \times \mathrm{id}_I) \circ \tilde{H}$ is independent of t. Here id_I is the identity map of I.

Then $\tilde{H}$ induces automatically a t-level preserving homeomorphism $H\colon U_1 \longrightarrow U_2$ such that $F \circ H$ is independent of t where $U_k = (\pi_n \times \mathrm{id}_I)(\tilde{U}_k)$, $k = 1, 2$ are neighbourhoods of $0 \times I$ in $\mathbf{R}^n \times \mathbf{R}^m$. Namely, the following diagram is commutative:

$$\begin{array}{ccc}
(\mathbf{R}^m, I) & \xrightarrow{\mathrm{id}} & (\mathbf{R}^m, I) \\
{\scriptstyle p_1}\uparrow & & \uparrow{\scriptstyle p_1} \\
\left(\mathcal{M}^n \times \mathbf{R}^m, \pi_n^{-1}(0) \times I\right) & \xrightarrow{\tilde{H}} & \left(\mathcal{M} \times \mathbf{R}^m, \pi_n^{-1}(0) \times I\right) \\
{\scriptstyle \pi_n \times \mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle \pi_n \times \mathrm{id}_I} \\
(\mathbf{R}^n \times \mathbf{R}^m, 0 \times I) & \xrightarrow{H} & (\mathbf{R}^n \times \mathbf{R}^m, 0 \times I) \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle F} \\
(\mathbf{R}^n, 0) & \xrightarrow{f_a} & (\mathbf{R}, 0)
\end{array}$$

where $f_a(x) = F(x;a)$ for $a \in I$ fixed and p_1, p_2 are canonical projections.

THEOREM 3-7. *Suppose that an analytic family F of blow analytic functions has simultaneously initially isolated singularities. Then F admits a blow analytic trivialization along the parameter space I where $I = \times_{i=1}^m [a_i, b_i]$ is a compact cube in $\mathbf{R}^m$.*

PROOF. To prove Theorem 3-7, it is sufficient to show the existence of analytic vector fields $\tilde{V}_q$ tangent to $\left(\mathcal{M}^n \times \mathbf{R}^m, \pi_n^{-1}(0) \times I\right)$ with the properties:

($\tilde{V}_q1$) $\tilde{V}_q$ is tangent to each level set of the mapping $F \circ (\pi_n \times \mathrm{id}_I)$ at its regular point,

($\tilde{V}_q2$) the t component of $\tilde{V}_q$ is $\frac{\partial}{\partial t_q}$ for $1 \leq q \leq m$.

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In fact, if there are analytic vector fields $\tilde{V}_q$ satisfying the previous conditions, then there are trajectories $\phi_q(t_q; x, \xi, c)$ of vector fields $\tilde{V}_q$ with $\phi_q(0; x, \xi, c) = (x, \xi, c)$. Define the analytic isomorphism $\tilde{H}$ of $\big(\mathcal{M}^n \times \mathbf{R}^m, \pi_n^{-1}(0) \times I\big)$ by

$$\tilde{H}(x, \xi, t) = \phi_m\Big(t_m - a_m; \phi_{m-1}\big(\cdots; \phi_1(t_1 - a_1; x, \xi, a)\cdots\big)\Big)$$

where $a = (a_1, a_2, \ldots, a_m)$. Then the two properties $(\tilde{V}_q 1), (\tilde{V}_q 2)$ imply that $F, \tilde{H}$ satisfy the condition of Definition 3-6. And so, F admits a blow analytic trivialization along I.

Now, we show the existence of the vector field $\tilde{V}_q$. Recall a Kuo vector field $V(x; t_q)$ defined in $(\mathbf{R}^n \times \mathbf{R}^m - 0 \times I, 0 \times I)$ (see [3, 5]):

$$\begin{aligned} V(x; t_q) &= \frac{|\operatorname{grad}_{x,t_q} F|^2}{|\operatorname{grad}_x F|^2}\left(\frac{\partial}{\partial t_q} - \left\langle \frac{\partial}{\partial t_q}, \frac{\operatorname{grad}_{x,t_q} F}{|\operatorname{grad}_{x,t_q} F|}\right\rangle \frac{\operatorname{grad}_{x,t_q} F}{|\operatorname{grad}_{x,t_q} F|}\right) \\ &= \frac{-\frac{\partial F}{\partial t_q}}{|\operatorname{grad}_x F|^2}\operatorname{grad}_x F + \frac{\partial}{\partial t_q} \end{aligned}$$

where

$$\operatorname{grad}_x F = \sum_{p=1}^{n} \frac{\partial F}{\partial x_p}\frac{\partial}{\partial x_p},$$

$$\operatorname{grad}_{x,t_q} F = \sum_{p=1}^{n} \frac{\partial F}{\partial x_p}\frac{\partial}{\partial x_p} + \frac{\partial F}{\partial t_q}\frac{\partial}{\partial t_q}.$$

This vector field $V(x; t_q)$ is tangent to the level set of $F(x; t)$ at its regular points, by definition. Moreover, it is a blow analytic vector field for any $t \in I$ fixed.

In fact, let

$$a_p = -\frac{\left(\frac{\partial F}{\partial t_q}\right)\left(\frac{\partial F}{\partial x_p}\right)}{|\operatorname{grad}_x F|^2}$$

be the coefficient of $\frac{\partial}{\partial x_p}$ in the vector field $\tilde{V}(x; t_q)$. Then, using the local coordinate $u = (u^1, u^2, \ldots, u^n)$ in $\mathcal{M}^n_\lambda$, we have:

$$\begin{aligned} a_p \circ \pi_n(u) &= -\frac{\left(\frac{\partial F}{\partial t_q}\right)\left(\frac{\partial F}{\partial x_p}\right)}{|\operatorname{grad}_x F|^2}(u^\lambda u^1, \ldots, u^\lambda, \ldots, u^\lambda u^n) \\ &= -\frac{u^\lambda\left(\frac{\partial H_k}{\partial t_q}(\tilde{u}) + u^\lambda \frac{\partial H_{k+1}}{\partial t_q}(\tilde{u}) + \cdots\right)\left(\frac{\partial H_k}{\partial x_p}(\tilde{u}) + u^\lambda \frac{\partial H_{k+1}}{\partial x_p}(\tilde{u}) + \cdots\right)}{\sum_{r=1}^n \left(\frac{\partial H_k}{\partial x_r}(\tilde{u}) + u^\lambda \frac{\partial H_{k+1}}{\partial x_r}(\tilde{u}) + \cdots\right)^2} \end{aligned}$$

where $\tilde{u} = (u^1, \ldots, u^{\lambda-1}, 1, u^{\lambda+1}, \ldots, u^n)$. By the assumption of Theorem 3-7,

$$\sum_{r=1}^{n}\left(\frac{\partial H_k}{\partial x_r}(u^1, \ldots, u^{\lambda-1}, 1, u^{\lambda+1}, \ldots, u^n)\right)^2 \neq 0.$$

Hence, $a_p \circ \pi_n$, $1 \le p \le n$ are analytic functions and $a_p \circ \pi_n(0; \xi) = 0$. So, by Corollary 3-3, the vector field $V(x; t_q)$, $1 \le q \le m$ are blow analytic and there exist analytic vector fields $\tilde{V}_q$ tangent to $\big(\mathcal{M}^n \times \mathbf{R}^m, \pi_n^{-1}(0) \times I\big)$ with $d\pi_n(\tilde{V}_q) = V(x; t_q)$. It is clear that the vector field $\tilde{V}_q$ satisfies the conditions $(\tilde{V}_q 1), (\tilde{V}_q 2)$ because $V(x; t_q)$ is tangent to the level set of $F(x; t)$ at its any regular point. This completes the proof of Theorem 3-7. ∎

Theorem 3-7 is a generalization of the following theorem.

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THEOREM 3-8 (T-C. KUO [4]). *Suppose that an analytic family F of analytic functions has simultaneously initially isolated singularities. Then F admits a blow analytic trivialization along the parameter space I where* $I = \times_{i=1}^{m}[a_i, b_i]$ *a compact cube in* $\mathbf{R}^m$.

EXAMPLE 3-9. Let

$$F(x;t) = \frac{(x_1 + x_2)(x_1^2 - tx_1x_2 + x_2^2)}{x_1^2 + x_2^2} : (\mathbf{R}^2 \times \mathbf{R}, 0 \times I) \longrightarrow (\mathbf{R}, 0),$$

$I = [0,1]$, be an analytic family of blow analytic functions. This family F has initially isolated singularities at the origin along I. So, F admits a blow analytic trivialization along I. In particular, $F(x;0) = x_1 + x_2$ and $F(x;1) = \frac{(x_1+x_2)^3}{x_1^2+x_2^2}$ are blow analytic equivalent each other.

PROOF. We have:

$$\begin{cases} \frac{\partial F}{\partial x_1} = \frac{x_1^4+(2+t)x_1^2x_2^2-2tx_1x_2^3+(1-t)x_2^4}{(x_1^2+x_2^2)^2} \\ \frac{\partial F}{\partial x_2} = \frac{x_2^4+(2+t)x_1^2x_2^2-2tx_2x_1^3+(1-t)x_1^4}{(x_1^2+x_2^2)^2} \end{cases}$$

and so

$$\frac{\partial F}{\partial x_1} - \frac{\partial F}{\partial x_2} = \frac{t(x_1^3 - x_2^3)(x_1 + x_2)}{(x_1^2 + x_2^2)^2}.$$

Thus we know that the equations:

$$\frac{\partial F}{\partial x_1} = 0, \quad \frac{\partial F}{\partial x_2} = 0$$

imply $x_1 = x_2 = 0$ if $t^2 \neq 4$. So, the family F has simultaneously initially isolated singularities at the origin along the interval $I = [0,1]$. This shows that the family F satisfies the assumption of Theorem 3-7. This completes the proof of 3-9. ∎

ACKNOWLEDGEMENTS. I would like to express my thanks to Prof. Li Pei Xin for his continuous encouragement during my ten months stay at the Institute of Mathematics, Academia Sinica, Beijing in 1988–1989 when I started to study this work.

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https://doi.org/10.4153/CMB-1993-065-1 Published online by Cambridge University Press