ANZIAM J. 59(2018), 349–369
doi:10.1017/S1446181117000438

AN APPROPRIATE APPROACH TO PRICING EUROPEAN-STYLE OPTIONS WITH THE ADOMIAN DECOMPOSITION METHOD

ZIWIE KE ✉[1], JOANNA GOARD[1] and SONG-PING ZHU[1]

(Received 4 December, 2016; accepted 31 August, 2017; first published online 26 February 2018)

Abstract

We study the numerical Adomian decomposition method for the pricing of European options under the well-known Black–Scholes model. However, because of the nondifferentiability of the pay-off function for such options, applying the Adomian decomposition method to the Black–Scholes model is not straightforward. Previous works on this assume that the pay-off function is differentiable or is approximated by a continuous estimation. Upon showing that these approximations lead to incorrect results, we provide a proper approach, in which the singular point is relocated to infinity through a coordinate transformation. Further, we show that our technique can be extended to pricing digital options and European options under the Vasicek interest rate model, in both of which the pay-off functions are singular. Numerical results show that our approach overcomes the difficulty of directly dealing with the singularity within the Adomian decomposition method and gives very accurate results.

2010 *Mathematics subject classification*: 35E15.

Keywords and phrases: Adomian decomposition method, European option, digital option, Vasicek model.

1. Introduction

The Adomian decomposition method (ADM) was developed by George Adomian in the 1970s to yield series solutions for ordinary differential equations (ODEs). Since then, the method has been used to determine analytic solutions not only to the linear ODEs [4], but also a wide class of nonlinear ODEs [2, 7, 23] and even partial differential equations (PDEs) [3, 18, 24, 29], which arise from many fields, such as physics, engineering as well as finance. In this paper, we investigate the application of the ADM to the European option pricing problem.

A European option is a contract between the writer and the holder of the contract, which gives the holder the right but not the obligation to buy or sell a prescribed asset

[1]School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia;
e-mail: zk990@uowmail.edu.au, joanna@uow.edu.au, spz@uow.edu.au.

(called the underlying asset) at a specified price (called the strike price or exercise price) on a given date in the future (called the expiry date). The option to buy an asset is known as a *call* option, while the option to sell an asset is known as a *put* option. In general, the underlying asset is a risky stock with price S. By assuming that there is no transaction fee and no arbitrage opportunity, the stock price S follows the process

$$dS = (r - q)S\, d_t + \sigma S\, dW_t,$$

where r is the risk-free bank rate, q is the dividend yield rate, σ is the standard deviation of the return of the asset and W_t is a Wiener process. The value of the European put option with expiry date T and exercise price K can be obtained by solving the well-known Black–Scholes (BS) model [11, 25, 31],

$$P_t + \tfrac{1}{2}\sigma^2 S^2 P_{SS} + (r - q)S P_S - rP = 0, \tag{1.1a}$$
$$P(S, T) = \max(K - S, 0), \tag{1.1b}$$
$$P(0, t) = K e^{-r(T-t)}, \tag{1.1c}$$
$$\lim_{S\to\infty} P(S, t) = 0. \tag{1.1d}$$

Under the same assumptions, the value of a corresponding European call option satisfies the following problem.

$$C_t + \tfrac{1}{2}\sigma^2 S^2 C_{SS} + (r - q)S C_S - rC = 0, \tag{1.2a}$$
$$C(S, T) = \max(S - K, 0), \tag{1.2b}$$
$$C(0, t) = 0, \tag{1.2c}$$
$$\lim_{S\to\infty} C(S, t) = S e^{-q(T-t)}. \tag{1.2d}$$

The connection between a put option and a call option, which have the same expiry T and the same strike price K, is called the *put-call* parity [22]. This relationship allows us to get the solution of (1.2) once we obtain the solution of the PDE system (1.1). The solution to the system (1.1) can be found in a number of ways, for example, using a Mellin transform [28] or Green's function [22, 31], and is generally known as the BS formula [31].

Since the BS formula for European options is in a closed form, one may argue that there is no need to find an approximation to the solution using the ADM. However, we believe that it is an essential step to apply the method to a well-known system in order to gain insight into more complicated option pricing problems. It is also important to investigate how to apply the ADM to this very special PDE problem that is characterized by the presence of singularities in the initial (or terminal) condition, but has a wide application in mathematical finance.

Although some authors have already attempted to apply the ADM to solving the BS model, their methods have some shortcomings and limitations. Bohner and Zheng [14] obtain an explicit price formula for both European puts and calls, which require the terminal condition to have derivatives of all orders. However, as the terminal conditions of the European put and call, respectively, are (1.1b) and (1.2b), both of which are nondifferentiable at $S = K$, their formula is not appropriate to price these options. Bohner et al. [15] try to deal with the nonsmooth condition (1.2b), and they find an approximation for the condition. Using Theorem 2.1 of Bohner and Zheng [14], they obtain the price for European call options. While they give several numerical examples, they only consider the situation where $S \gg K$. Although the approximation of the condition (1.2b) is differentiable, there exists a singular point at $S = K$ in the derivatives of the approximation. In Section 3, we take a closer look at this approach. The other works in the literature consider some differentiable, possibly contrived, pay-off functions. For example, González-Gaxiola et al. [21] use $S + 10(\sqrt{S} + 1/4)$ and $S + 200\sqrt{S} + 100$; El-Wakil et al. [18] use $S + 1/S^{7/5}$; and Eric et al. [10] only consider the linear pay-off, that is, $K - S$ for the puts and $S - K$ for the calls. Unfortunately, for a standard option, such pay-off functions of (1.1b) or (1.2b) already exist as they represent the simplest and yet still meaningful way to design an option financially. However, mathematically, that means we have to deal with the singularity at $S = K$ in (1.1b) or (1.2b) if the ADM is used to price options. Therefore, how to resolve the incompatibility between the suitability of the ADM and the financial reality is a key challenge we face.

With the ADM, one can use Fourier series expansions to deal with the nondifferentiability in the boundary or initial conditions of a problem defined on a finite domain (see, for example, [33, Example 4.1] and [6, Section 1]), or one can, instead, develop a rapidly convergent decomposition series of such conditions [6]. Unfortunately, in our problem, the stock price S is defined in a half-infinity domain, and thus the condition (1.1b) (or (1.2b)) cannot be expanded into a Fourier series. Therefore, for our specific problem, we are required to find another way to deal with the singularity at $S = K$ in the pay-off function.

In this paper, we provide an appropriate way to apply the ADM to the BS model for European options, which can deal with the singularity problem without requiring a differentiable approximation of the terminal condition. We relocate the singular point to $\pm\infty$ through a variable transformation. Through our approach, the solution to (1.1) is obtained, and is, in fact, equivalent to the BS formula. Moreover, we apply our technique to pricing digital options, which also have a nonsmooth pay-off condition at expiry, as well as to the two-dimensional problem of pricing a European option under the stochastic Vasicek interest rate model [27, 30, 31]. Numerical results show that in all of these examples our method is efficient and accurate.

This paper is organized into seven sections. Section 2 gives a brief review of the ADM algorithm, and Section 3 highlights the inaccuracies of Bohner et al. [15] in the approximation of the nonsmooth terminal condition. We detail our approach on the use of the ADM for pricing European options in Section 4 and then, in Section 5, we show how our approach can also be used to price digital options and European options under the stochastic Vasicek interest rate model. In Section 6, we compare the prices obtained by our method with those from the numerical techniques of the binomial method (BM) and the Monte–Carlo (MC) simulation method. Finally, in Section 7, we present our conclusions.

2. The Adomian decomposition method

We briefly recall some general notation and formulae of the ADM that will be used frequently henceforth. First, consider the general differential equation

$$\mathcal{F}u = g,$$

where $\mathcal{F}$ is a general differential operator involving both linear and nonlinear terms, so that the above equation can be decomposed as

$$\mathcal{L}u + \mathcal{R}u + \mathcal{N}u = g.$$

Here $\mathcal{L}$ is a linear invertible operator, $\mathcal{R}$ is the remainder of the linear operator and $\mathcal{N}$ is a nonlinear operator. As $\mathcal{L}$ is invertible, the equivalent expression is

$$\mathcal{L}^{-1}\mathcal{L}u = \mathcal{L}^{-1}g - \mathcal{L}^{-1}\mathcal{R}u - \mathcal{L}^{-1}\mathcal{N}u. \tag{2.1}$$

Solving (2.1) for u yields

$$u = \phi + \mathcal{L}^{-1}g - \mathcal{L}^{-1}\mathcal{R}u - \mathcal{L}^{-1}\mathcal{N}u, \tag{2.2}$$

where ϕ is the integration constant and satisfies $\mathcal{L}\phi = 0$. In many PDE problems, there are several invertible linear operators with respect to different variables to choose. For a specific variable direction, one can obtain a solution which is called "partial solution" [5]. With the ADM [4], the unknown function u is decomposed into a sum of components

$$u = \sum_{n=0}^{\infty} u_n, \tag{2.3}$$

and the nonlinear term $\mathcal{N}u$ is also decomposed into a series

$$\mathcal{N}u = \sum_{n=0}^{\infty} A_n, \tag{2.4}$$

where A_n $(n = 0, 1, 2, \cdots)$ are called Adomian polynomials and are obtained by the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\mathcal{N}\Big(\sum_{i=0}^{\infty} u_i\lambda^i\Big)\Big|_{\lambda=0} \quad \text{for all } n = 0, 1, 2, \ldots.$$

A simple algorithm for calculating the Adomian polynomials is provided by Biazar and Shafiof [13]. The programming codes for MATLAB are given by Fatoorehchi and Abolghasemi [20], and the codes for MAPLE are presented by Biazar and Pourabd [12]. We substitute equations (2.3) and (2.4) into (2.2) to obtain

$$\sum_{n=0}^{\infty} u_n = u_0 - \mathcal{L}^{-1}\mathcal{R}\sum_{n=0}^{\infty} u_n - \mathcal{L}^{-1}\sum_{n=0}^{\infty} A_n,$$

where

$$u_0 = \phi + \mathcal{L}^{-1}g,$$
$$u_{n+1} = -\mathcal{L}^{-1}\mathcal{R}u_n - \mathcal{L}^{-1}A_n \quad \text{for all } n \geq 0.$$

All the u_n are calculable, and thus the n-term approximate series solution can be obtained using

$$\Psi_n = \sum_{i=0}^{n-1} u_i \sim \sum_{i=0}^{\infty} u_i = u.$$

The decomposition of the solution series converges, in general, very quickly. The convergence of the ADM was studied by Cherruault [16, 17, 26] and, more recently, by Abdelrazec [8].

3. A review of the paper by Bohner et al. [15]

Bohner et al. [15] apply the ADM directly to the BS model for a European call option (1.2), whereas the terminal condition (1.2b) is replaced by an approximate function

$$g(S) = \tfrac{1}{2}(S - K) + \tfrac{1}{2}\sqrt{(S - K)^2 + (2\sqrt{2} - 1)/n^2},$$

where n is a parameter. It is easy to verify that the above function has the property, $\lim_{n\to\infty} g(S) = \max(S - K, 0)$, which indicates that the larger n will give us more accurate results. With the standard ADM, defining the linear invertible operator $\mathcal{L}_t$ as $\partial/\partial t$ and the remaining operator $\mathcal{L}_S$ as

$$\frac{\sigma^2}{2} S^2 \frac{\partial^2}{\partial S^2} + (r - q) S \frac{\partial}{\partial S} - r,$$

the equation (1.2a) can be written as $\mathcal{L}_t C = -\mathcal{L}_S C$. Then, we apply the inverse operator $\mathcal{L}_t^{-1} = \int_t^T (\cdot)\, dt$ and substitute $C = \sum_{k=0}^{\infty} u_k$ to both sides of $\mathcal{L}_t C = -\mathcal{L}_S C$. After some simplification, Bohner et al. [15] found that each u_k ($k \in \mathbb{N}_0$) satisfies

$$u_k = \frac{(T - t)^k}{k!} \bigg(\sum_{m=0}^{2k} \bigg(\sum_{v=0}^{m} \frac{(-1)^{m-v}}{v!(m - v)!} \rho_v^k \bigg) S^m g^{(m)}(S) \bigg), \tag{3.1}$$

where $\rho_m = (\sigma^2 m/2 + r)(m - 1) - qm$ for all $m \in \mathbb{N}_0$.

Since larger n makes $g(S)$ closer to $\max(S - K, 0)$, a large n will essentially give us more accurate option values. However, as n gets larger, there is a singularity at $S = K$ in the derivatives of $g(S)$. The first order derivative

$$g'(S) = \frac{1}{2}\bigg(1 + \frac{S - K}{\sqrt{(S - K)^2 + (1/n)^2 (2\sqrt{2} - 1)}}\bigg),$$

so

$$\lim_{n\to\infty} g'(S) = \frac{1}{2}\bigg(1 + \frac{S - K}{|S - K|}\bigg).$$

Also, it is simple to verify that the higher-order derivatives of $g(S)$ also have a singularity at $S = K$, as $n \to \infty$. From (3.1), this singularity leads to a large error in the value of the European call option near $S = K$. Moreover, if $S < K$,

$$\lim_{n\to\infty} g(S) = \tfrac{1}{2}(S - K) + \tfrac{1}{2}\sqrt{(S - K)^2} = \tfrac{1}{2}(S - K) - \tfrac{1}{2}(S - K) = 0,$$

TABLE 1. Comparison of the values of European call options on a nondividend-paying asset and a dividend paying asset [15] (denoted by "BMR") with corresponding V^* and the BS values when $S > K$. Model settings are $K = 40$, $S = 65$, $\sigma = 0.324366$, $r = 0.05$, $q = 0$, $q = 0.02$, $n = 100$ and $T = 1/4, 1/6, 1/12$. Relative errors of the results in Bohner et al. [15] are, respectively, measured against the V^* values (the 6th column) and the BS values (the last column).

(T,q)		BMR [15] value	V^* value	BS value	Relative error*	Relative error$_{BS}$
1/4	0	25.49685	25.49689	25.49932	1.66E−06	9.71E−05
1/4	0.02	25.17260	25.17270	25.17541	3.79E−06	1.12E−04
1/6	0	25.33195	25.33195	25.33210	6.36E−08	6.14E−06
1/6	0.02	25.11564	25.11564	25.11581	1.09E−07	6.90E−06
1/12	0	25.16632	25.16632	25.16632	8.30E−08	8.19E−08
1/12	0.02	25.05808	25.05808	25.05808	3.63E−08	3.47E−08

and all the derivatives of $g(S)$ approach zero as n approaches infinity. That is from (3.1), as $n \to \infty$, $\sum_{k=0}^{\infty} u_k$ gives the value zero for the European call option pricing for all S less than the strike price K. If $S > K$, the result of Bohner et al. [15] is, in fact, asymptotic to $Se^{-q(T-t)} - Ke^{-r(T-t)}$,

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=0}^{\infty} u_k(S,t) = S\,(1 - q(T-t) + \tfrac{1}{2}q^2(T-t)^2 - \cdots)\\ - K(1 - r(T-t) + \tfrac{1}{2}r^2(T-t)^2 - \cdots)\\ \sim Se^{-q(T-t)} - Ke^{-r(T-t)}.\end{aligned}$$

Intuitively, we use the same parameter values as used by Bohner et al. [15, Tables 4.1–4.2] to give some numerical examples for various values of S/K. Bohner et al. [15] demonstrate excellent results only when $S \gg K$. However, as from the above analysis, their results are, in fact, approximately $V^* = Se^{-q(T-t)} - Ke^{-r(T-t)}$. The comparison of the results (5 terms of (3.1), equivalent to the equation below equation (2.4) in Bohner et al. [15]) with the corresponding BS values and V^* values is shown in Table 1. From the table, one can see that as time to maturity T becomes smaller, the European call option values obtained by using the method in [15] become more accurate. By comparing the relative errors in the last two columns in Table 1, it is worth noting that the results obtained by Bohner et al. [15] are much closer to V^* than to the BS values.

In Table 2, we let $S = 40$ $(= K)$ and $S = 30$ $(< K)$, and keep the other parameters the same. The results obtained using the method of Bohner et al. [15] are significantly different from the BS values. When $S = K$, the results obtained by this method [15] approach infinity, while, when $S < K$, they are very close to zero.

From both the theoretical analysis and the numerical examples, we observe that the ADM application of Bohner et al. [15] does not properly solve the system (1.2) for European call options. An attempt to use a differentiable approximation to the pay-off function leads to failure, as the more accurate the approximation is, the more

TABLE 2. Comparison of the values of European call options on a nondividend-paying asset and a dividend paying asset [15] (denoted by "BMR") with the corresponding BS values when $S = K$ and $S < K$. Model settings are $K = 40$, $\sigma = 0.324366$, $r = 0.05$, $q = 0$, $q = 0.02$, $n = 100$ and $T = 1/4, 1/6, 1/12$.

(T, q)		$S = 40 = K$		$S = 30 < K$	
		BMR [15] value	BS value	BMR [15] value	BS value
1/4	0	−1.167E+31	2.82523	5.34E−04	0.10266
1/4	0.02	−1.98E+35	2.71416	3.77E−04	0.09497
1/6	0	3.694E+36	2.27297	−9.37E−04	0.02873
1/6	0.02	1.082E+41	2.20024	−1.32E−03	0.02674
1/12	0	−4.787E+40	1.57520	−7.58E−05	0.00112
1/12	0.02	−1.082E+40	1.53971	5.63E−05	0.00105

chance that it is singular at $S = K$. It is natural to ask whether the ADM can be used to successfully solve the pricing problem. From the above analysis, a key challenge is thus to find a way to deal with the singularity at $S = K$, when $t = T$ is present in the pay-off function. As we mentioned in the introduction, the method that combines Fourier series [6, 33] is not suitable for the problem on an infinite domain. Therefore, it is necessary to resolve the suitability of the ADM for the European option pricing. Thus, the main contribution of this paper is that it presents one particular approach with the singularity being shifted to infinity through a variable transformation, which is shown in the next section.

4. Our solution approach

Focusing on dealing with the singularity at $S = K$ in the pay-off function at expiry, we consider the European put option pricing problem, since the corresponding problem for call options can be easily obtained from the put–call parity. To simplify and nondimensionalize the system (1.1), we apply the transformation

$$\tau = (T - t)\sigma^2/2, \quad x = \ln(S/K), \quad \varphi = P/K \tag{4.1}$$

to the equations to get

$$\frac{\partial \varphi}{\partial \tau} = \frac{\partial^2 \varphi}{\partial x^2} + (k_1 - 1)\frac{\partial \varphi}{\partial x} - k_2\varphi, \tag{4.2a}$$

$$\varphi(x, 0) = \max(1 - e^x, 0), \tag{4.2b}$$

$$\varphi(-\infty, \tau) = e^{-k_2 \tau}, \tag{4.2c}$$

$$\varphi(\infty, \tau) = 0, \tag{4.2d}$$

where $k_1 = 2(r - q)/\sigma^2$ and $k_2 = 2r/\sigma^2$. Hence, the nondifferentiable final condition (1.1b) becomes an initial condition (4.2b). Further, our problem has been converted to one with an infinite spatial domain. Then, we introduce the transformation

$$y = \frac{x}{\sqrt{\tau}}, \quad z = \sqrt{\tau}, \quad u = \frac{\varphi}{\sqrt{\tau}}, \tag{4.3}$$

https://doi.org/10.1017/S1446181117000438 Published online by Cambridge University Press

in order to shift the singularity in (4.2b) to infinity. Indeed, with the above variable transformation, it is clear that the singular point at $x = 0$ in (4.2b), where the variable $\tau = 0$, has been shifted to infinity, as $\lim_{\tau\to 0} y = \pm\infty$. By doing this, the difficulty that was discussed in the previous section can be overcome without the need to approximate or simplify the pay-off function, and thus apply the ADM properly.

Hence, through the transformation (4.3), we convert (4.2b) to

$$\lim_{z\to 0} u(y,z) = \begin{cases} \dfrac{1-e^{yz}}{z} & \text{for } y \to -\infty, \\ 0 & \text{for } y \to +\infty, \end{cases}$$

and expand the above in a Taylor series to get

$$\lim_{z\to 0} u(y,z) = \begin{cases} -y - \dfrac{y^2}{2!}z - \dfrac{y^3}{3!}z^2 - \cdots - \dfrac{y^n}{n!}z^{n-1} - \cdots & \text{for } y \to -\infty, \\ 0 & \text{for } y \to +\infty. \end{cases} \tag{4.4}$$

Also, the PDE (4.2a) becomes

$$u_z + \frac{u}{z} = \frac{1}{z}(2u_{yy} + yu_y + 2(k_1-1)zu_y - 2k_2z^2u). \tag{4.5}$$

To solve equation (4.5), define the linear differentiable operator L_z [32] and its inverse as

$$L_z(\cdot) = \frac{\partial}{\partial z}(\cdot) + \frac{(\cdot)}{z}, \quad L_z^{-1}(\cdot) = \frac{1}{z}\int_0^z z(\cdot)\,dz. \tag{4.6}$$

It is easy to verify that

$$L_z^{-1}L_z(u) = \frac{1}{z}\int_0^z (zu_z + u)\,dz = \frac{1}{z}\Big([zu]_0^z - \int_0^z u\,dz + \int_0^z u\,dz\Big) = u.$$

We now apply L_z^{-1} to both sides of (4.5) and let $u(y,z) = \sum_{i=0}^{\infty} u_i(y,z)$ to get

$$\begin{aligned} u_0 + u_1 + u_2 + \cdots = \frac{1}{z}\int_0^z \{&2u_{0_{yy}} + yu_{0_y} + 2(k_1-1)zu_{0_y} - 2k_2z^2u_0 \\ &+ 2u_{1_{yy}} + yu_{1_y} + 2(k_1-1)zu_{1_y} - 2k_2z^2u_1 + \cdots\}\,dz. \end{aligned} \tag{4.7}$$

In the standard ADM, u_0 is typically chosen to be the initial (or terminal) condition or a boundary condition [4]. However, no such obvious terms appear in (4.7). To satisfy the conditions at infinity (4.4), we want to find each u_i such that the limit of u_i is equal to the i-th term in (4.4) when y tends to $-\infty$, and 0, when y tends to ∞. Equating terms of order $O(z^0)$ gives $u_0(y,z) = f_0(y)z^0$ as the solution to

$$u_0(y,z) = \frac{1}{z}\int_0^z (2u_{0_{yy}} + yu_{0_y})\,dz,$$

that is,

$$f_0(y) = 2f_0''(y) + yf_0'(y). \tag{4.8}$$

https://doi.org/10.1017/S1446181117000438 Published online by Cambridge University Press

Since equation (4.8) is a linear homogeneous second-order ODE, we can find the general solution

$$f_0(y) = C_1 y + C_2\left\{\frac{e^{-y^2/4}}{\sqrt{\pi}} + \frac{y}{2}\,\mathrm{erf}\left(\frac{y}{2}\right)\right\},$$

where C_1 and C_2 are two arbitrary constants and $\mathrm{erf}(\cdot)$ is the error function [1]. From (4.4), u_0 should satisfy

$$u_0 = \begin{cases} -y & \text{as } y \to -\infty, \\ 0 & \text{as } y \to +\infty. \end{cases}$$

Therefore, we find

$$u_0(y,z) = -\frac{1}{2}y + \frac{e^{-y^2/4}}{\sqrt{\pi}} + \frac{y}{2}\,\mathrm{erf}\left(\frac{y}{2}\right).$$

In our approach, each successive term $(u_1, u_2, \ldots)$ is recursively found by solving an ODE involving previous terms, instead of finding each term algebraically by using the preceding ones in the standard ADM. From (4.7), we assume that $u_i(y,z) = f_i(y)z^i$, in order to find the solution to $f_i(y)$ by solving

$$u_1(y,z) = \frac{1}{z}\int_0^z \{2u_{1_{yy}} + yu_{1_y} + 2(k_1 - 1)zu_{0_y}\}\,dz,$$

and, in general, for $n \geq 2$,

$$u_n(y,z) = \frac{1}{z}\int_0^z \{2u_{n_{yy}} + yu_{n_y} + 2(k_1 - 1)zu_{n-1_y} - 2k_2 z^2 u_{n-2}\}\,dz.$$

Thus the ODEs to solve for $u_1, u_2, \ldots u_i$ are formed by equating terms of $O(z^i)$, and using (4.4) these should be solved subject to

$$\begin{cases} \lim\limits_{y\to\infty} f_i(y) = 0, \\ \lim\limits_{y\to-\infty} f_i(y) = -\dfrac{y^{i+1}}{(i+1)!}. \end{cases}$$

Solving for the first several orders yields

$$u_1(y,z) = z\left[\frac{ye^{-y^2/4}}{2\sqrt{\pi}} + \left(\frac{k_1}{2} + \frac{y^2}{4}\right)\left(\mathrm{erf}\left(\frac{y}{2}\right) - 1\right)\right],$$

$$u_2(y,z) = z^2\left[\frac{1}{12\sqrt{\pi}}(2y^2 + 3(k_1+1)^2 - 4(3k_2+1))e^{-y^2/4} + \frac{y}{12}(y^2 + 6k_1 - 6k_2)\left(\mathrm{erf}\left(\frac{y}{2}\right) - 1\right)\right],$$

https://doi.org/10.1017/S1446181117000438 Published online by Cambridge University Press

$$u_3(y,z) = \frac{z^3 e^{-y^2/4}}{24\sqrt{\pi}}(y^2 + (-k_1^3 + 3k_1^2 + 9k_1 - 12k_2 - 1))y$$
$$+ \frac{z^3(\text{erf}(y/2) - 1)}{48}(y^4 + (12k_1 - 12k_2)y^2 + 12k_1(k_1 - 2k_2)),$$
$$u_4(y,z) = \frac{z^4 e^{-y^2/4}}{240\sqrt{\pi}}\Big[2y^4 + \Big(\frac{5}{4}k_1^4 - 5k_1^3 + \frac{15}{2}k_1^2 + 35k_1 - 40k_2 - \frac{11}{4}\Big)y^2$$
$$- \frac{5}{2}k_1^4 + 30k_1^3 + (-60k_2 + 45)k_1^2 + (-120k_2 - 10)k_1 + 120k_2^2 + 20k_2 + \frac{3}{2}\Big]$$
$$+ \frac{z^4(\text{erf}(y/2) - 1)}{240}(y^5 + (20k_1 - 20k_2)y^3 + 60(-k_2 + k_1)^2 y).$$

We denote the n-term solution as Φ_n, and $\lim_{n\to\infty}\Phi_n$ as Φ_∞. It is easy to verify that Φ_∞ also satisfies boundary conditions (4.2c)–(4.2d). In fact, Φ_∞ is equivalent to the BS formula for European put option, which is in the scaled variables through (4.1) and (4.3). In practice, we only use finite terms to approximate the exact solution to the original problem (see, for example, [4, 24, 29].

5. Extension to other cases

As discussed in the previous section, the nondifferentiable pay-off function at expiry can cause problems, and we have demonstrated an approach so that the ADM can be applied to the option pricing problem under the BS model. However, the versatility of the method needs to be further demonstrated in terms of pricing exotic options as well as options under a stochastic interest rate model subject to a transform that enables a reduction of dimensionality. In this section, we apply our approach to pricing a digital option and a European option under the Vasicek stochastic interest rate model [27, 30, 31]. In both cases, the pay-off functions still display singularities. The first example focuses on a pay-off function that has a singularity which behaves worse than the case presented in the previous section, while the second one focuses on a particular case, in which higher-dimensional option pricing problems can still be dealt with by the ADM through a cleverly constructed transform on independent variables.

5.1. Pricing digital options Digital options are often referred to as “all-or-nothing options”. By definition, the payout is predetermined and fixed at a constant amount $B > 0$. The value of a digital option at expiry can be mathematically represented as

$$\text{for a call:}\quad V_c(S,T) = \begin{cases} B & \text{if } S > K, \\ 0 & \text{otherwise,} \end{cases}$$
$$\text{for a put:}\quad V_p(S,T) = \begin{cases} B & \text{if } S < K, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Under the BS model, solving (1.1a) together with (5.1) gives us the value of the digital put option, and the value of the corresponding digital call option can be found by using

https://doi.org/10.1017/S1446181117000438 Published online by Cambridge University Press

the put–call parity for digital options [31]. We now give details of the derivation of digital put option evaluations via the ADM.

Similarly to the transformations (4.1), we now let

$$\tau = (T - t)\sigma^2/2, \quad x = \ln(S/K), \quad \varphi = V_p/B.$$

Substituting these into (1.1a) and (5.1), we find

$$\frac{\partial \varphi}{\partial \tau} = \frac{\partial^2 \varphi}{\partial x^2} + (k_1 - 1)\frac{\partial \varphi}{\partial x} - k_2\varphi, \tag{5.2}$$

$$\varphi(x, 0) = \mathcal{H}(1 - e^x), \tag{5.3}$$

where $k_1 = 2(r - q)/\sigma^2$, $k_2 = 2r/\sigma^2$, and $\mathcal{H}$ is a Heaviside step function. Again, there is a singularity in (5.3) at $x = 0$. In order to shift this singularity to infinity, we let $y = x/\sqrt{\tau}$ and $z = \sqrt{\tau}$. Further, we use the linear differentiable operator $L_z(\cdot)$ as in (4.6) and let $\varphi = \sum_{n=0}^{\infty} \varphi_n$. After following the same procedure as for the European option case from (5.2)–(5.3),

$$\begin{aligned}\varphi_0 + \varphi_1 + \varphi_2 + \cdots = \frac{1}{z}\int_0^z &(2\varphi_{0_{yy}} + y\varphi_{0_y} + \varphi_0 + 2(k_1 - 1)z\varphi_{0_y} - 2k_2z^2\varphi_0 \\ &+ 2\varphi_{1_{yy}} + y\varphi_{1_y} + \varphi_1 + 2(k_1 - 1)z\varphi_{1_y} - 2k_2z^2\varphi_1 + \cdots)\,dz,\end{aligned}$$

with the condition

$$\lim_{z\to 0}\sum_{n=0}^{\infty}\varphi_n(y, z) = \begin{cases} 1 & \text{if } y \to -\infty, \\ 0 & \text{if } y \to \infty. \end{cases}$$

Equating terms of order $O(z^n)$ gives $\varphi_n(y, z) = f_n(y)z^n$ as the solution to

$$\begin{cases} \varphi_0(y, z) = \dfrac{1}{z}\displaystyle\int_0^z (2\varphi_{0_{yy}} + y\varphi_{0_y} + \varphi_0)\,dz & \text{for } n = 0, \\ \varphi_1(y, z) = \dfrac{1}{z}\displaystyle\int_0^z (2\varphi_{1_{yy}} + y\varphi_{1_y} + \varphi_1 + 2(k_1 - 1)z\varphi_{0_y})\,dz & \text{for } n = 1, \\ \varphi_n(y, z) = \dfrac{1}{z}\displaystyle\int_0^z (2\varphi_{n_{yy}} + y\varphi_{n_y} + \varphi_n + 2(k_1 - 1)z\varphi_{n-1_y} - 2k_2z^2\varphi_{n-2})\,dz & \text{for } n \geq 2, \end{cases}$$

together with the conditions

$$\lim_{z\to 0}\varphi_0(y, z) = \begin{cases} 1 & \text{if } y \to -\infty \\ 0 & \text{if } y \to \infty \end{cases} \text{ for } n = 0;$$

$$\lim_{z\to 0}\varphi_n(y, z) = \begin{cases} 0 & \text{if } y \to -\infty \\ 0 & \text{if } y \to \infty \end{cases} \text{ for } n \geq 1.$$

https://doi.org/10.1017/S1446181117000438 Published online by Cambridge University Press

Solving the first several orders yields

$$\begin{aligned}
\varphi_0 &= \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\Big(\frac{y}{2}\Big),\\
\varphi_1 &= -\frac{(k_1-1)e^{-y^2/4}}{2\sqrt{\pi}}z,\\
\varphi_2 &= \Big[\frac{(k_1-1)^2 y e^{-y^2/4}}{8\sqrt{\pi}} - \frac{k_2}{2}\,\mathrm{erfc}\Big(\frac{y}{2}\Big)\Big]z^2,\\
\varphi_3 &= -\frac{[(k_1-1)^3(y^2-2) - 24k_2(k_1-1)]e^{-y^2/4}}{48\sqrt{\pi}}z^3,\\
\varphi_4 &= \Big[\Big(-\frac{k_2(k_1-1)^2}{8\sqrt{\pi}}y - \frac{(k_1-1)^4}{384\sqrt{\pi}}(y^2-6)y\Big)e^{-y^2/4} + \frac{k_2^2}{4}\,\mathrm{erfc}\Big(\frac{y}{2}\Big)\Big]z^4,
\end{aligned}$$

where $\mathrm{erfc}(\cdot)$ is the complementary error function [1]. It should be noted that the series solutions for the digital options in this section and European options in Section 4 can be used to derive solutions for other options that can be replicated in terms of these options, for example asset-or-nothing digital options.

5.2. Pricing European options depend on a Vasicek interest rate

As in the BS framework, we again assume that the risk-neutral model for the stock price is of the form

$$dS = rS\,dt + \sigma_1 S\,dW_1, \tag{5.4}$$

with W_1 as a Brownian motion. Unlike the vanilla European option case where the interest rate is treated as a constant, we now suppose that the interest rate is stochastic and that the risk-neutral rate follows the Ornstein–Uhlenbeck process

$$\begin{aligned}
dr &= a(b-r)\,dt + \sigma_2\,dW_2,\\
r(0) &= r_0.
\end{aligned} \tag{5.5}$$

Here a is a constant speed of reversion, b is a constant long-term mean level, σ_2 is constant volatility and W_2 is a Brownian motion. This model is commonly known as the Vasicek model [27, 30, 31]. Under this model, the zero-coupon bond, denoted as $B(r,t)$, can be shown to follow the stochastic differential equation

$$\frac{dB(r,t)}{B(r,t)} = r\,dt + \sigma_B\,dW_2,$$

where $\sigma_B = \sigma_B(t)$ is the volatility and is time-dependent. From equations (9) and (27) of Vasicek [30], σ_B is given by $\sigma_B = \sigma_2[1-\exp\{-a(T-t)\}]/a$. Since the bond is a traded security, the drift rate of the bond price under the risk-neutral measure is simply given by the risk-free rate r. We suppose that the changes in W_1 and W_2 are correlated with coefficient ρ, that is, $dW_1dW_2 = \rho dt$. It can be shown that when the underlying

https://doi.org/10.1017/S1446181117000438 Published online by Cambridge University Press

asset and interest rate follow (5.4) and (5.5), respectively, the value of a European option satisfies the PDE

$$V_t + \tfrac{1}{2}\sigma_1^2 S^2 V_{SS} + \tfrac{1}{2}\sigma_B^2 B^2 V_{BB} + rSV_S + rBV_B + \rho\sigma_1\sigma_B SBV_{SB} - rV = 0. \tag{5.6}$$

Here, $V(S, B, t)$ represents the value of the European option [19], and B is the value of a zero-coupon bond that matures at the same time T as the option and pays one dollar at expiry (that is, $B(r, T) = 1$). The derivation of (5.6) can be obtained by considering a riskless portfolio containing one aforementioned option $V(S, B, t)$, selling some underlying stock and zero-coupon bonds and assuming that the portfolio earns the risk-free interest rate. For more details on the derivation, we refer the reader to the work of Fang [19]. At expiry,

$$V(S, B, T) = \begin{cases} \max(KB - S, 0) & \text{for a put,} \\ \max(S - KB, 0) & \text{for a call.} \end{cases} \tag{5.7}$$

The put–call parity for the aforementioned European options is given by Abudy and Izhakian [9]. Therefore, it suffices to give details of the derivation for the put option alone. Introducing these transformations,

$$x = \ln\frac{S}{K}, \quad \tau = T - t, \quad \varphi = \frac{V}{KB}, \quad w = \ln B,$$

into (5.6) and (5.7) yields

$$\varphi_\tau = \frac{1}{2}\sigma_1^2\varphi_{xx} + \frac{1}{2}\sigma_B^2\varphi_{ww} + \Big(r - \frac{\sigma_1^2}{2} + \rho\sigma_1\sigma_B\Big)\varphi_x + \Big(r + \frac{\sigma_B^2}{2}\Big)\varphi_w + \rho\sigma_1\sigma_B\varphi_{xw}, \tag{5.8}$$

and

$$\varphi(x, w, 0) = \max(1 - e^{x-w}, 0). \tag{5.9}$$

Similarly to the vanilla European option case, we also use the transformation,

$$y = \frac{x}{\sqrt{\tau}}, \quad z = \sqrt{\tau}, \quad u = \frac{\varphi}{\sqrt{\tau}}, \quad v = \frac{w}{\sqrt{\tau}},$$

in (5.8). Then

$$\begin{aligned} u_z + \frac{u}{z} = \frac{1}{z}\Big\{&\sigma_1^2 u_{yy} + \sigma_B^2 u_{vv} + yu_y + vu_v + 2\rho\sigma_1\sigma_B u_{yv} + 2\Big(r + \frac{\sigma_B^2}{2}\Big)zu_v \\ &+ 2\Big(r - \frac{\sigma_1^2}{2} + \rho\sigma_1\sigma_B\Big)zu_y\Big\}. \end{aligned} \tag{5.10}$$

We now let $\xi = y - v$, so that the singularity in (5.9) is shifted to ∞. Hence, from (5.9),

$$\lim_{z\to 0} u(y, v, z) = \begin{cases} -\displaystyle\sum_{n=1}^{\infty}\frac{(y-v)^n}{n!} & \text{if } \xi \to -\infty, \\ 0 & \text{if } \xi \to +\infty. \end{cases}$$

https://doi.org/10.1017/S1446181117000438 Published online by Cambridge University Press

In the following, we make use of the fact (see [30]) that $\sigma_1^2 + \sigma_B^2 - 2\rho\sigma_1\sigma_B$ can be treated as a constant. This term can be replaced by its average value, that is,

$$\bar{\sigma}^2 = \frac{1}{\tau}\int_0^\tau (\sigma_1^2 + \sigma_B^2 - 2\rho\sigma_1\sigma_B)\, ds.$$

From equations (7) and (8) of Vasicek [30],

$$\bar{\sigma}^2 = \sigma_1^2 + \left[1 - \frac{2A}{\tau} + \frac{\{1 - \exp(-2a\tau)\}}{2a\tau}\right]\left(\frac{\sigma_2}{a}\right)^2 - \frac{2\rho\sigma_1(\tau - A)\sigma_2}{a\tau},$$

where $A = (1 - \exp(-a\tau))/a$. Following the ADM, we use the the linear differentiable operator $L_z(\cdot)$ as in (4.6) and let $u = \sum_{n=0}^{\infty} u_n$. Applying the inverse linear differentiable operator $L_z^{-1}(\cdot)$ to (5.10), equating terms of order $O(z^n)$ and letting $u_n(y, v, z) = f_n(y, v)z^n$, we get equations for each f_n $(n = 0, 1, 2, \ldots)$. For example,

$$\begin{cases} f_0 = \bar{\sigma}^2 f_0'' + \xi f_0' & \text{for } n = 0, \\ f_1 = \dfrac{\bar{\sigma}^2}{2} f_1'' + \dfrac{1}{2}\xi f_1' - \dfrac{\bar{\sigma}^2}{2} f_0' & \text{for } n = 1, \\ f_n = \dfrac{\bar{\sigma}^2}{n+1} f_n'' + \dfrac{1}{n+1}\xi f_n' - \dfrac{\bar{\sigma}^2}{n+1} f_{n-1}' & \text{for } n \geq 2. \end{cases} \tag{5.11}$$

Solving each equation in (5.11) together with the conditions for $n \geq 0$,

$$\begin{cases} \lim\limits_{\xi\to\infty} f_n = 0, \\ \lim\limits_{\xi\to-\infty} f_n = -\dfrac{\xi^{n+1}}{(n+1)!}, \end{cases}$$

and then rewriting in terms of the variables y, v, z gives

$$u_0 = \frac{\bar{\sigma}}{\sqrt{2\pi}} \exp\left(\frac{-(y-v)^2}{2\bar{\sigma}^2}\right) - \frac{1}{2}(y - v)\,\mathrm{erfc}\left(\frac{y-v}{\sqrt{2}\bar{\sigma}}\right),$$

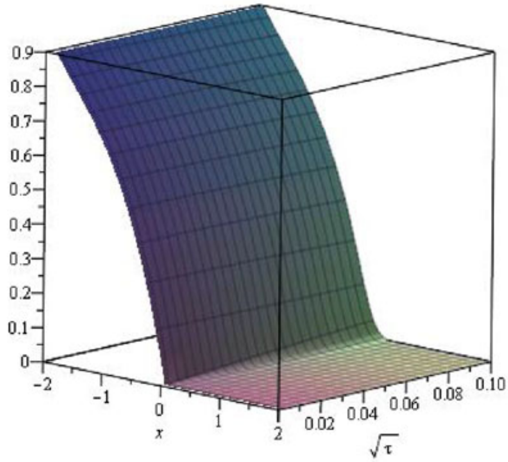
$$u_1 = z\left[\frac{\sqrt{2}\bar{\sigma}}{4\sqrt{\pi}}(y - v)\exp\left(\frac{-(y-v)^2}{2\bar{\sigma}^2}\right) - \frac{1}{4}(y - v)^2\,\mathrm{erfc}\left(\frac{y-v}{\sqrt{2}\bar{\sigma}}\right)\right],$$

$$u_2 = z^2\left[\frac{\sqrt{2}}{48}(4\bar{\sigma}(y - v)^2 - \bar{\sigma}^3)\frac{\exp(-(y-v)^2/2\bar{\sigma}^2)}{\sqrt{\pi}} - \frac{1}{12}(y - v)^3\,\mathrm{erfc}\left(\frac{y-v}{\sqrt{2}\bar{\sigma}}\right)\right],$$

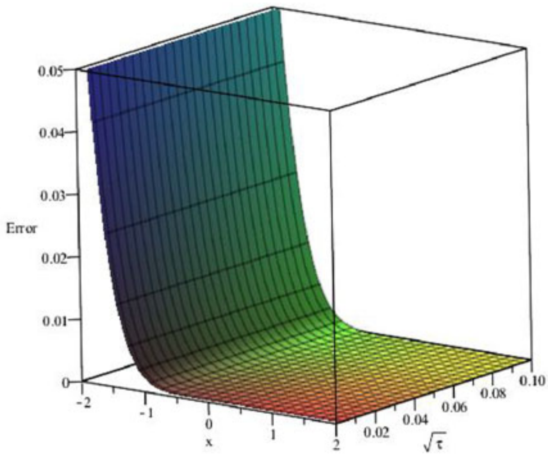
$$\begin{aligned} u_3 =& z^3\left[\left(\frac{\sqrt{2}}{48}\bar{\sigma}(y - v)^3 - \frac{\sqrt{2}}{96}\bar{\sigma}^3(y - v)\right)\frac{\exp(-(y-v)^2/2\bar{\sigma}^2)}{\sqrt{\pi}}\right. \\ &\left. - \frac{1}{48}(y - v)^4\,\mathrm{erfc}\left(\frac{y-v}{\sqrt{2}\bar{\sigma}}\right)\right], \end{aligned}$$

$$\begin{aligned} u_4 = z^4&\left[\left(\frac{\sqrt{2}}{240}\bar{\sigma}(y - v)^4 - \frac{11\sqrt{2}}{3840}\bar{\sigma}(y - v)^2 + \frac{\sqrt{2}}{1280}\bar{\sigma}^3\right)\frac{\exp(-(y-v)^2/2\bar{\sigma}^2)}{\sqrt{\pi}}\right. \\ &\left. -\frac{1}{240}(y - v)^5\,\mathrm{erfc}\left(\frac{y-v}{\sqrt{2}\bar{\sigma}}\right)\right]. \\ \ldots& \end{aligned}$$

https://doi.org/10.1017/S1446181117000438 Published online by Cambridge University Press



(a) Φ_5 for European put option



(b) The difference between Φ_5 and the BS formula

FIGURE 1. Comparison of our result and the BS formula for the European put option.

6. Numerical results

In this section, we compare our results for vanilla European options, digital options and European options under stochastic interest rates with those obtained using other numerical methods. First, for the vanilla European option, we plot a five-term solution Φ_5 obtained through our approach and the difference between our Φ_5 and the BS formula in the scaled variables (x, τ and φ) in Figures 1(a) and 1(b), respectively, using the parameter values $r = 0.05$, $q = 0$ and $\sigma = 0.3$. Figure 1 gives us a rough

https://doi.org/10.1017/S1446181117000438 Published online by Cambridge University Press

TABLE 3. Comparison of our approach with other solutions for short-term European put options: $K = 40, r = 0.05, \sigma = 0.324336$.

S	T	q	The BS formula	Our results (Φ_5)	Binomial method ($n = 3000$)	Monte–Carlo ($n = 10^6$)
30	1/4	0.02	9.74770	9.74771	9.60577	9.60163
		0	9.60578	9.60582	9.74770	9.74691
	1/6	0.02	9.79463	9.79463	9.69677	9.70032
		0	9.69678	9.69681	9.79462	9.79115
	1/12	0.02	9.88469	9.88470	9.83480	9.83300
		0	9.83480	9.83483	9.88469	9.88469
40	1/4	0.02	2.41677	2.41678	2.32814	2.33022
		0	2.32834	2.32835	2.41656	2.41958
	1/6	0.02	2.00140	2.00141	1.94086	1.93713
		0	1.94102	1.94103	2.00123	1.99854
	1/12	0.02	1.44000	1.44000	1.40876	1.41109
		0	1.40888	1.40888	1.43988	1.44284
50	1/4	0.02	0.25047	0.25047	0.23442	0.23371
		0	0.23440	0.23440	0.25049	0.25066
	1/6	0.02	0.10165	0.10165	0.09566	0.09623
		0	0.09569	0.09569	0.10161	0.10278
	1/12	0.02	0.01102	0.01102	0.01044	0.01071
		0	0.01045	0.01045	0.01101	0.01085
AAE				0.00001	0.05469	0.05479
CPU (second)				0.053	4.641	7.078

comparison to check the accuracy of our method. The figure is showing that our approach converges fast and gives highly accurate results.

Moreover, as high computational efficiency together with high accuracy is crucial in the financial industry, we compare our results for a five-term solution (Φ_5) and a ten-term solution (Φ_{10}) for the European put option prices with those obtained via the BS formula, the BM with 3000 time steps and the MC method with 10^6 samples, with regard to accuracy and efficiency. Table 3 shows the results for short times to expiry (1, 3, 6 months), while Table 4 gives the results for longer times to expiry (1, 3, 5 years). Taking the BS solution as the true solution, we give the average absolute error (AAE) for each method. The significantly small absolute errors indicate that our five-term solution gives very accurate approximate results in all cases: $S > K$, $S = K$ and $S < K$. All the experiments were performed using MATLAB R2014a on an Intel Core(TM) 2, 3.16 GHz machine.

TABLE 4. Comparison of our approach with other solutions for long-term European put options: $K = 40$, $r = 0.05$, $\sigma = 0.324336$.

S	T	q	The BS formula	Our results (Φ_5)	Our results (Φ_{10})	Binomial method ($n = 1000$)	Monte–Carlo ($n = 10^6$)
30	1	0	9.5188	9.5196	9.5188	9.5190	9.5224
		0.02	9.9502	9.9511	9.9502	9.9503	9.9536
	3	0	9.5476	9.5642	9.5476	9.5473	9.5463
		0.02	10.4329	10.4543	10.4329	10.4327	10.4386
	5	0	9.2443	9.3159	9.2443	9.2449	9.2441
		0.02	10.3980	10.4895	10.3981	10.3984	10.3974
40	1	0	4.1115	4.1120	4.1115	4.1112	4.1140
		0.02	4.4185	4.4190	4.4185	4.4182	4.4124
	3	0	5.7301	5.7418	5.7301	5.7299	5.7342
		0.02	6.4531	6.4655	6.4531	6.4527	6.4460
	5	0	6.2122	6.2659	6.2122	6.2122	6.2214
		0.02	7.2072	7.2662	7.2072	7.2070	7.2067
50	1	0	1.5657	1.5659	1.5657	1.5658	1.5618
		0.02	1.7293	1.7294	1.7293	1.7293	1.7283
	3	0	3.4607	3.4682	3.4607	3.4610	3.4638
		0.02	4.0031	4.0091	4.0031	4.0032	3.9968
	5	0	4.2728	4.3117	4.2728	4.2742	4.2728
		0.02	5.0910	5.1265	5.0910	5.0923	5.0871
	AAE			0.02381	0.000005	0.00036	0.02460
	CPU (second)			0.053	0.171	4.726	6.671

For short-term options, the numerical examples show that our approach gives the most accurate results (with an AAE 10^{-5}) of the numerical methods tested. With regard to the speed in calculating the 18 different option values given in Table 3, our approach took 0.053 seconds, which is approximately 100 times faster than the BM with 3000 steps and 150 times faster than the MC with 10^6 samples. Comparison of our results with those in [15], which are listed in Table 1 of Section 3, demonstrates that our approach proposed in this paper successfully deals with the singularity at $S = K$ in the pay-off of European options.

For long-term options, the results are given in Table 4. Our five-term solution provides more accurate option values (with an AAE of the order of 10^{-2}) compared with those obtained via the MC with 10^6 samples. Our ten-term solution provides the most accurate results (with an AAE of the order of 10^{-6}) compared with the BM and MC. This suggests that the more terms that are added into our solution, the more accurate the option values will be. With regard to efficiency, comparison of the CPU times for each method shows that our method takes the shortest time to calculate the 18 different option values: 0.053 seconds for Φ_5 and 0.171 seconds for Φ_{10}.

TABLE 5. Comparison of our approach with other solutions for digital put options: $K = 40, q = 0$.

S	T	r	σ	True value	Our results (Φ_5)	Our results (Φ_{10})	Monte–Carlo ($n = 10^6$)
30	0.25	0.05	0.30	0.9598	0.9598	0.9598	0.9598
	0.50	0.05	0.30	0.8881	0.8881	0.8881	0.8879
	0.75	0.05	0.30	0.8310	0.8310	0.8310	0.8315
	0.25	0.10	0.20	0.9717	0.9717	0.9717	0.9716
	0.50	0.10	0.20	0.9132	0.9132	0.9132	0.9130
	0.75	0.10	0.20	0.8402	0.8403	0.8402	0.8402
	1.00	0.05	0.20	0.8572	0.8573	0.8572	0.8571
	3.00	0.05	0.20	0.6162	0.6175	0.6162	0.6166
40	0.25	0.05	0.30	0.4905	0.4905	0.4905	0.4913
	0.50	0.05	0.30	0.4831	0.4831	0.4831	0.4832
	0.75	0.05	0.30	0.4761	0.4761	0.4761	0.4756
	1.00	0.05	0.20	0.4189	0.4190	0.4189	0.4190
	3.00	0.05	0.20	0.3421	0.3437	0.3421	0.3414
50	0.25	0.05	0.30	0.0665	0.0665	0.0665	0.0666
	0.50	0.05	0.30	0.1402	0.1402	0.1402	0.1397
	0.75	0.05	0.30	0.1842	0.1842	0.1842	0.1836
	1.00	0.05	0.20	0.0978	0.0978	0.0978	0.0976
	3.00	0.05	0.20	0.1575	0.1585	0.1575	0.1577
AAE					0.0002	1.33E-08	0.0003
CPU (second)					0.063	0.203	65.148

Table 5 shows the results of a comparison of digital put option values obtained using the explicit solution [31], our method with Φ_5 and Φ_{10}, as well as the MC with 10^6 samples. In these examples, both the long-term and short-term options depend on a nondividend-paying stock, that is, $q = 0$. Comparison of our results (Φ_5) with the MC with 10^6 samples demonstrates that although they give the same accurate level of option values (with an AAE of the order of 10^{-4}), our approach (Φ_5) is much faster than the MC, with only 0.063 seconds to calculate 18 option values compared with 65.148 seconds by the MC. In addition, the AAE of our ten-term solution (of the order of 10^{-8}) shows the accuracy that can be achieved with our method.

The numerical examples for the European options under the Vasicek interest rate model are listed in Table 6. The true values are calculated using the analytic formula [9, 19]. Our seven-term solution (Φ_7) gives option values with an AAE of the order of 10^{-7}. Further, it only takes 0.203 seconds to calculate 24 options listed in Table 6. Comparison of the AAEs and CPU times of our solution (Φ_5, Φ_7) with the MC with 10^6 samples shows that, for the short- and long-term options considered, our approach is superior with regard to accuracy and efficiency.

TABLE 6. Comparison of our approach with other solutions for European put options under the Vasicek interest rate model: $K = 40, b = 0.1, q = 0$.

S	r_0	a	σ_1	σ_2	ρ	Vasicek model	Our results (Φ_5)	Our results (Φ_7)	Monte–Carlo ($n = 10^6$)
						$T = 0.25$			
30	0.05	0.1	0.2	0.03	0	9.50018	9.4996	9.5002	9.4998
	0.05	0.4	0.2	0.03	−0.8	9.48202	9.4814	9.4820	9.4809
40	0.05	0.1	0.2	0.03	0	1.34664	1.3466	1.3466	1.3515
	0.1	0.4	0.3	0.05	−0.8	1.86371	1.8637	1.8637	1.8495
50	0.05	0.1	0.2	0.03	0	0.01372	0.0137	0.0137	0.0132
	0.05	0.4	0.2	0.03	−0.8	0.01201	0.0120	0.0120	0.0123
						$T = 0.5$			
30	0.1	0.1	0.3	0.05	0	8.52307	8.5224	8.5231	8.5188
	0.05	0.4	0.2	0.03	−0.8	8.97454	8.9739	8.9745	8.9768
40	0.1	0.1	0.3	0.05	0	2.41619	2.4162	2.4162	2.4201
	0.15	0.4	0.4	0.1	−0.8	2.89367	2.8936	2.8937	2.8887
50	0.05	0.1	0.2	0.03	0	0.09909	0.0991	0.0991	0.1001
	0.15	0.4	0.4	0.1	−0.8	0.80181	0.8018	0.8018	0.7983
						$T = 1$			
30	0.1	0.1	0.3	0.05	0	7.80747	7.8069	7.8075	7.7987
	0.1	0.4	0.3	0.05	−0.8	7.61176	7.6112	7.6118	7.6016
40	0.05	0.1	0.2	0.03	0	2.20185	2.2018	2.2018	2.2111
	0.15	0.4	0.4	0.1	−0.8	3.21130	3.2112	3.2113	3.2238
50	0.1	0.1	0.3	0.05	0	0.92397	0.9240	0.9240	0.9365
	0.1	0.4	0.3	0.05	−0.8	0.75658	0.7566	0.7566	0.7483
						$T = 3$			
30	0.05	0.1	0.2	0.03	0.5	7.07033	7.0700	7.0703	7.0745
	0.15	0.4	0.3	0.1	−0.8	3.51237	3.5119	3.5124	3.5024
40	0.15	0.1	0.3	0.1	0.5	3.80184	3.8019	3.8018	3.7965
	0.05	0.4	0.2	0.03	−0.8	1.59110	1.5911	1.5911	1.5952
50	0.15	0.1	0.3	0.1	0.5	2.40757	2.4061	2.4076	2.4294
	0.05	0.4	0.2	0.03	−0.8	0.42926	0.4292	0.4293	0.4307
AAE							0.00026	3.20E–07	0.00599
CPU (second)							0.052	0.203	24.328

7. Conclusion

In this paper, we provide a proper way to solve the BS model for European options using ADM, and then highlight the errors in some of the current literature on the application of the ADM to solving the BS model for European options. Through the

https://doi.org/10.1017/S1446181117000438 Published online by Cambridge University Press

analysis, we find that the singularity at $S = K$ in the pay-off function at expiry appears to be the major issue in the ADM, dealing with the nondifferentiability in the boundary or initial conditions. Further, the existing method in the literature that has been used in finite domains (that is, combining the Fourier series with the ADM [33]) is not suitable for this most fundamental problem in quantitative finance. Therefore, it is essential to find a proper way to resolve this issue. We adopt a different approach, in which the singularity is shifted to infinity through a variable transformation. In such a way, the ADM can be successfully applied not only to solving the BS model for European options, but also to pricing digital options with a nondifferentiable pay-off and European options with a stochastic interest rate (a (2 + 1)-dimensional problem). For all these options, numerical tests in Section 6 show that our Φ_5, Φ_7 and Φ_{10} series solutions outperformed the BM with 3000 time steps and the MC simulation with 10^6 samples, indicating that our solution is extremely accurate and efficient. Note that our series solution can be truncated at any order, and so can achieve higher-order accuracy if necessary.

Acknowledgement

The first author would like to thank China Scholarship Council (CSC) and the University of Wollongong (UOW) for providing funding support that allowed her to carry out a PhD study at UOW.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables* (Dover Publications, New York, 1964).

[2] G. Adomian, “Convergent series solution of nonlinear equations”, *J. Comput. Appl. Math.* **11** (1984) 225–230; doi:10.1016/0377-0427(84)90022-0.

[3] G. Adomian, “Nonlinear stochastic dynamical systems in physical problems”, *J. Math. Anal. Appl.* **111** (1985) 105–113; doi:10.1016/0022-247X(85)90203-3.

[4] G. Adomian, “Review of the decomposition method in applied mathematics”, *J. Math. Anal. Appl.* **135** (1988) 501–544; doi:10.1016/0022-247X(88)90170-9.

[5] G. Adomian and R. Rach, “Equality of partial solutions in the decomposition method for linear or nonlinear partial differential equations”, *Comput. Math. Appl.* **19** (1990) 9–12; doi:10.1016/0898-1221(90)90246-G.

[6] G. Adomian and R. Rach, “A further consideration of partial solutions in the decomposition method”, *Comput. Math. Appl.* **23** (1992) 51–64; doi:10.1016/0898-1221(92)90080-2.

[7] K. I. Al-Dosary, N. K. Al-Jubouri and H. K. Abdullah, “On the solution of Abel differential equation by Adomian decomposition method”, *Appl. Math. Sci.* **2** (2008) 2105–2118; http://www.m-hikari.com/ams/ams-password-2008/ams-password41-44-2008/aldosaryAMS41-44-2008.pdf

[8] A. Abdelrazec and D. Pelinovsky, “Convergence of the Adomian decomposition method for initial-value problems”, *Numer. Methods Partial Differ. Equ.* **27** (2011) 749–766; doi:10.1002/num.20549.

[9] M. Abudy and Y. Izhakian, “Pricing stock options with stochastic interest rate”, *Intl J. Portfolio Anal. Manag.* **1** (2013) 250–277; doi:10.1504/IJPAM.2013.054408.

[10] E. Avila, A. Estrella and L. Blanco, “Solution of the Black–Scholes equation via the Adomian decomposition method”, *Intl J. Appl. Math. Res.* **2** (2013) 486–494; doi:10.14419/ijamr.v2i4.871.

[11] F. Black and M. Scholes, "The pricing of options and corporate liabilities", *J. Political Economy* **81** (1973) 637–654; http://www.jstor.org/stable/1831029.
[12] J. Biazar and M. Pourabd, "A Maple program for computing Adomian polynomials", *Intl Math. Forum* **1** (2006) 1919–1924; doi:10.13140/RG.2.1.3473.4565.
[13] J. Biazar and S. M. Shafiof, "A simple algorithm for calculating Adomian polynomials", *Intl J. Contemp. Math. Sci.* **2** (2007) 975–982; http://www.m-hikari.com/ijcms-password2007/17-20-2007/biazarIJCMS17-20-2007-1.pdf.
[14] M. Bohner and Y. Zheng, "On analytical solutions of the Black–Scholes equation", *Appl. Math. Lett.* **22** (2009) 309–313; doi:10.1016/j.aml.2008.04.002.
[15] M. Bohner, F. Marín and S. Rodríguez, "European call option pricing using the Adomian decomposition method", *Adv. Dyn. Syst. Appl.* **9** (2014) 75–85; http://campus.mst.edu/adsa/contents/v9n1p5.pdf.
[16] Y. Cherruault, "Convergence of Adomian's method", *Kybernetes* **18** (1989) 31–38; doi:10.1108/eb005812.
[17] Y. Cherruault and G. Adomian, "Decomposition methods: a new proof of convergence", *Math. Comput. Model.* **18** (1993) 103–106; doi:10.1016/0895-7177(93)90233-O.
[18] S. A. El-Wakil, M. A. Abdou and A. Elhanbaly, "Adomian decomposition method for solving the diffusion–convection–reaction equations", *Appl. Math. Comput.* **177** (2006) 729–736; doi:10.1016/j.amc.2005.09.105.
[19] H. Fang, "European option pricing formula under stochastic interest rate", *Program Appl. Math.* **4** (2012) 14–21; doi:10.3968/j.pam.1925252820120401.Z0619.
[20] H. Fatoorehchi and H. Abolghasemi, "On calculation of Adomian polynomials by MATLAB", *J. Appl. Comput. Sci. Math.* **5** (2011) 85–88; https://jacsm.ro/view/?pid=11_13.
[21] O. González-Gaxiola, J. Ruíz de Chávez and J. A. Santiago, "A nonlinear option pricing model through the Adomian decomposition method", *Intl J. Appl. Comput. Math.* **2** (2016) 453–467; doi:10.1007/s40819-015-0070-6.
[22] J. C. Hull, *Options, futures, and other derivatives* (Prentice Hall, Upper Saddle River, NJ, 2006).
[23] S. A. Khuri, "A new approach to Bratu's problem", *Appl. Math. Comput.* **147** (2004) 131–136; doi:10.1016/S0096-3003(02)00656-2.
[24] D. Lesnic, "The decomposition method for linear, one-dimensional, time-dependent partial differential equations", *Intl J. Math. Math. Sci.* **2006** (2006) 1–29; doi:10.1155/IJMMS/2006/42389.
[25] R. C. Merton, "Theory of rational option pricing", *Bell J. Econ. Manag. Sci.* **4** (1973) 141–183; doi:10.2307/3003143.
[26] T. Mavoungou and Y. Cherruault, "Convergence of Adomian's method and applications to non-linear partial differential equations", *Kybernetes* **21** (1992) 13–25; doi:10.1108/eb005942.
[27] R. S. Mamon, "Three ways to solve for bond prices in the Vasicek model", *J. Appl. Math. Decision Sci.* **8** (2004) 1–14; doi:10.1155/S117391260400001X.
[28] R. Panini and R. P. Srivastav, "Option pricing with Mellin transforms", *Math. Comput. Model.* **40** (2004) 43–56; doi:10.1016/j.mcm.2004.07.008.
[29] M. Tatari, M. Dehghan and M. Razzaghi, "Application of the Adomian decomposition method for the Fokker–Planck equation", *Math. Comput. Model.* **45** (2007) 639–650; doi:10.1016/j.mcm.2006.07.010.
[30] O. Vasicek, "An equilibrium characterization of the term structure", *J. Financial Econ.* **5** (1977) 177–188; doi:10.1016/0304-405X(77)90016-2.
[31] P. Wilmott, S. Howison and J. Dewynne, *The mathematics of financial derivatives: a student introduction* (Cambridge University Press, 1995).
[32] A.-M. Wazwaz, "A new method for solving singular initial value problems in the second-order ordinary differential equations", *Appl. Math. Comput.* **128** (2002) 45–57; doi:10.1016/S0096-3003(01)00021-2.
[33] S.-P. Zhu and J. Lee, "On the Adomian decomposition method for solving PDEs", *Commun. Math. Res.* **32** (2016) 151–166; doi:10.13447/j.1674-5647.2016.02.08.