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SETS OF DIFFERENTIALS AND SMOOTHNESS OF CONVEX FUNCTIONS

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Approximation by smooth convex functions and questions on the Smooth Variational Principle for a given convex function f on a Banach space are studied in connection with majorising f by C^1-smooth functions.

It is known that a weakly compactly generated (WCG) Banach space admits an equivalent Fréchet differentiable norm if it admits a Fréchet differentiable bump function (see, for example [**5**]). However, there are nonseparable spaces that admit Fréchet differentiable bump functions but admit no equivalent Fréchet differentiable norm (see, for example [**3**, Chapter VII]). If the space X admits an equivalent norm with modulus of smoothness of power type 2, then every convex continuous function on X has points of Lipschitz smoothness (see, for example [**3**, Chapter IV]). The purpose of this note is to localise these results. We prove that any convex Lipschitz function f that is defined on a WCG Banach space X can be uniformly approximated by Fréchet differentiable convex functions if f is majorised on X by a Fréchet smooth convex function. If, moreover, $\overline{span}^{\|\cdot\|}\{\partial f(x) : x \in X\}$ is a subspace of X^* that admits a norm with modulus of rotundity of power type 2, then there is a convex function ψ with ψ' Lipschitz on X such that $\psi \geqslant f$ on X and $\psi(x) = f(x)$ for some $x \in X$. Thus in particular, f has points of Lipschitz smoothness. A separable version of these problems was studied in [**10**]. We use standard notation in this note (see for example [**3**]), and refer to [**6**, **7**, **9**] and [**3**] for some unexplained notions and results used in this note.

THEOREM 1. *Let f be a convex Lipschitz function defined on a WCG Banach space X. Then the following are equivalent.*

(1) *The function f can be uniformly approximated on X by a Fréchet differentiable convex function.*
(2) *There exists a Fréchet differentiable convex function ϕ defined on X such that $\phi \geqslant f$ on X.*

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PROOF: Clearly (1) $\Longrightarrow$ (2). The proof (2) $\Longrightarrow$ (1) is divided into a few steps.

PROPOSITION 2. *Let X be a WCG Banach space, ϕ be a Fréchet differentiable convex function defined on X and let $Y := \overline{span}^{\|\cdot\|}\{\phi'(x) : x \in X\}$. Then there exists a projectional resolution of the identity (PRI) $\{P_\alpha : \omega \leqslant \alpha \leqslant \mu\}$ on X such that*

(i) $P_\mu^* = I$, $\|P_\alpha^*\| = 1$ *for all* α.
(ii) $P_\alpha^* P_\beta^* = P_\beta^* P_\alpha^* = P_{min(\alpha,\beta)}^*$.
(iii) $P_\alpha^* Y \subset Y$ *for all* α.
(iv) $\operatorname{dens}(P_\alpha^* Y) \leqslant \alpha$ *for all* $\alpha \leqslant \mu$.
(v) $P_\alpha^* Y = \overline{span}^{\|\cdot\|} \bigcup_{\beta<\alpha} P_{\beta+1}^* Y$ *for all* $\alpha \leqslant \mu$.

PROOF: Using standard techniques for constructing projectional resolutions of the identity (see for example [**3**, Chapter VI]), we only need to show $P_\alpha^* Y \subset Y$ and $P_\alpha^* Y = \overline{span}^{\|\cdot\|} \bigcup\limits_{\beta<\alpha} P_{\beta+1}^* Y$ for all α. The proof of this is contained in Lemmas 3 to 5.

LEMMA 3. *In notation as above, we can construct a PRI $\{P_\alpha : \omega \leqslant \alpha \leqslant \mu\}$ so that $P_\alpha^* \phi'(y) = \phi'(y)$ for all $y \in P_\alpha X$.*

PROOF: See Lemma 5 in [**5**]. ∎

LEMMA 4. *With notation as above, $P_\alpha^* Y = \overline{span}^{\|\cdot\|}\{\phi'(x) : x \in P_\alpha X\}$.*

PROOF: To see $P_\alpha^* Y \supset \overline{span}^{\|\cdot\|}\{\phi'(x) : x \in P_\alpha X\}$, we let $x \in P_\alpha X$, and show that $\phi'(x) \in P_\alpha^* Y$. Since ϕ is C^1-smooth, given $\varepsilon > 0$, there exists an $x_\beta \in P_{\beta+1} X$ for some $\beta < \alpha$, such that $\|\phi'(x) - \phi'(x_\beta)\| < \varepsilon$. By Lemma 3, $\phi'(x_\beta) = P_{\beta+1}^* \phi'(x_\beta)$. Therefore $\phi'(x_\beta) \in P_{\beta+1}^* Y \subset P_\alpha^* Y$. As $P_\alpha^* Y$ is closed, $\phi'(x) \in P_\alpha^* Y$. For the converse inclusion, we follow the idea in [4]. Let $\phi'(x) \in Y$. Clearly $g(\cdot) = \phi(\cdot) - \phi'(x)(\cdot)$ is a continuous bounded below function on X. Hence its restriction $g_{\restriction P_\alpha X}$ is also continuous and bounded below. By Ekeland's variational principle, given $\varepsilon > 0$, there exists $x_\alpha \in P_\alpha X$ such that for every $w \in B_{P_\alpha X}$, $t > 0$, we have $g(x_\alpha + tw) \geqslant g(x_\alpha) - \varepsilon t$, thus, $\phi'(x)(w) \leqslant (\phi(x_\alpha + tw) - \phi(x_\alpha))/t - \varepsilon$. Hence, by taking limits, we have $\phi'(x)(w) - \phi'(x_\alpha)(w) \leqslant \varepsilon$. Therefore $\sup\{|\phi'(x)(v) - \phi'(x_\alpha)(v)| : v \in B_{P_\alpha X}\} \leqslant \varepsilon$. Given any $h \in B_X$, we have $(h, P_\alpha^*\phi'(x) - \phi'(x_\alpha)) = (h, P_\alpha^*\phi'(x) - P_\alpha^*\phi'(x_\alpha)) = (P_\alpha h, \phi'(x) - \phi'(x_\alpha)) \leqslant \varepsilon$. Therefore $\|P_\alpha^*\phi'(x) - \phi'(x_\alpha)\| \leqslant \varepsilon$. Finally, since Y is the closed linear span of the derivatives of ϕ and P_α is bounded, the assertion follows. ∎

LEMMA 5. $P_\alpha^* Y = \overline{span}^{\|\cdot\|} \bigcup\limits_{\beta<\alpha} P_{\beta+1}^* Y$ *for every* $\alpha \leqslant \mu$.

PROOF: Clearly $P_\alpha^* Y \supset \overline{span}^{\|\cdot\|} \bigcup\limits_{\beta<\alpha} P_{\beta+1}^* Y$. The converse inclusion follows from Lemma 4 and the continuity of ϕ'. ∎

PROOF OF THEOREM 1: Since $f \leqslant \phi$, using Ekeland's variational principle as in Lemma 4 we show that $\operatorname{dom} f^* \subset Y$. Using Lemma 5, and the classical Troyanski's

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construction (see for example [**3**, Chapter VII]) we obtain a dual norm $\|\cdot\|^*$ in X^* such that its restriction on Y is locally uniformly rotund (LUR). Define a sequence of functions $\{h_n\}$ on X^* by $h_n(x^*) = f^*(x^*) + \|x^*\|^{*2} / (4n^4)$. Clearly, $\operatorname{dom} h_n = \operatorname{dom} f^*$. Define $g_n := f \square n^4 \|\cdot\|^2$, where $\square$ denotes the infimal convolution. Note that g_n is convex and continuous on X and $g_n^* = h_n$ for all n. Given $n \in \mathbb{N}$, $x \in X$ and $y \in \partial g_n(x)$, note that h_n is rotund at y with respect to x in the sense of [**1**], that is, for every $\varepsilon > 0$, there exist $\delta > 0$ such that $\{v : h_n(y+v) - h_n(y) - (x,v) \leqslant \delta\} \subset \varepsilon B_{X^*}$ (see, [**10**]). By [**1**, Proposition 4], g_n is Fréchet differentiable at x with the derivative y. One can also show that $\lim g_n = f$ uniformly on X (see for example [**8**, Lemma 2.4]). ∎

Since the function f can be quite "flat" in Theorem 1, there is a difficulty in applying the techniques of Smooth Variational Principles (see, [**3**, Chapter I]) in this situation. However, under more restrictive assumptions we can use the Stegall-Fabian variational principle and obtain our variational result by duality. We shall say that $x \in X$ is a point of Lipschitz smoothness of a convex function f if $f(x+h) + f(x-h) - 2f(x) = O\left(\|h\|^2\right)$.

Lemma 6. *Let f be a convex continuous function on a Banach Space X and g be its dual function. Suppose there exists a constant C such that for any $x \in X$, $y \in \partial f(x)$, and for any $\varepsilon > 0$, we have*

$$\{v : g(y+v) - g(y) - (x,v) \leqslant C\varepsilon^2\} \subset \varepsilon B_{X^*}.$$

Then f is Fréchet differentiable and f' is Lipschitz on X.

Proof: By taking polars, we have $\varepsilon^{-1} B_X \subset \{v : g(y+v) - g(y) - (x,v) \leqslant C\varepsilon^2\}^0$. According to Proposition 3 of [**1**], $\{v : g(y+v) - g(y) - (x,v) \leqslant C\varepsilon^2\}^0 \subset C^{-1}\varepsilon^{-2}\{u : f(x+u) - f(x) - (y,u) \leqslant C\varepsilon^2\}$. Therefore, $\varepsilon C B_X \subset \{u : f(x+u) - f(x) - (y,u) \leqslant C\varepsilon^2\}$, that is, for any $u \in \varepsilon C B_X$, $f(x+u) + f(x-u) - 2f(x) \leqslant 2/C(\varepsilon C)^2$. Thus f' exists at x and we have that f' is Lipschitz on X (see, for example [**3**, Lemma V.3.5]). ∎

Theorem 7. *Let f be a Lipschitz convex function on a Banach space X and $Y = \overline{span}^{\|\cdot\|}\{\partial f(x) : x \in X\}$. Suppose that Y admits an equivalent norm with modulus of convexity of power type 2. Then f can be majorised by a convex function ψ that has a Lipschitz derivative and $\psi(x) = f(x)$ for some $x \in X$. In particular, f has points of Lipschitz smoothness.*

Proof: Let $\|\cdot\|$ be an equivalent norm on X^* such that its restriction on Y has modulus of convexity of power type 2 (see, for example [**3**, Lemma II.8.1]). We note that Y is w^*-closed. Indeed, since Y is reflexive, B_Y is compact in the weak topology of X^* and thus B_Y is w^*-compact in X^*. By the Banach-Dieudonné theorem, Y is w^*-closed. Assume that $f(0) = 0$, and thus we have $f^* \geqslant 0$ on X^*. Let

$$h(x^*) = \begin{cases} \frac{1}{2}\|x^*\|^2 - \frac{1}{2}m^2 \text{ if } x^* \in Y \\ \infty \text{ otherwise,} \end{cases}$$

where $m = Lip(f)$. Since Y is w^*-closed, h is w^*-lower semicontinuous and $h = (h^*_{\restriction X})^*$. We show that h satisfies the condition on the function g given in Lemma 6. Indeed, by the modulus of rotundity of $\|\cdot\|$, there exists $L > 0$ such that for any $y_1, y_2 \in Y$, we have

$$(*) \qquad \frac{1}{2}\{\|y_1\|^2 + \|y_2\|^2\} - \left\|\frac{y_1 + y_2}{2}\right\|^2 \geqslant L\|y_1 - y_2\|^2$$

(see, for example [**2**, Lemma 5.I.4]). Assume that for every $k \in \mathbb{N}$ there exist $\varepsilon_k > 0$, $x_k \in X$, $y_k \in \partial h^*_{\restriction X}(x_k)$ and $v_k \in X^*, \|v_k\| > \varepsilon_k$, such that $h(y_k + v_k) - h(y_k) - v_k(x_k) \leqslant {\varepsilon_k}^2/k$. Then $\|y_k + v_k\|^2/2 - \|y_k\|^2/2 - (x_k, v_k) \leqslant {\varepsilon_k}^2/k$ for all k. From the definition of a subdifferential, we have $-(x_k, v_k) \geqslant \|y_k\|^2 - \|y_k + v_k/2\|^2$. Therefore, $\left(\|y_k\|^2 + \|y_k + v_k\|^2\right)/2 - \|y_k + v_k/2\|^2 \leqslant {\varepsilon_k}^2/k \leqslant \|v_k\|^2/k$, which contradicts (*). Now, for each $x^* \in \operatorname{dom} f^* \subset mB_{X^*}$, we have $h(x^*) \leqslant 0 \leqslant f(x^*)$. Therefore $f^* - h$ is a lower semicontinuous convex function on $\operatorname{dom} f^*$ that is bounded below. Note that $f^* - h \geqslant \|\cdot\| - m$. By the Stegall-Fabian result (see, for example [**9**, Corollary 5.22]), there exists $\widehat{x} \in Y^*$ such that $f^* - h - \widehat{x}$ attains its minimum in $\operatorname{dom} f^*$, that is, there is a $x^* \in \operatorname{dom} f^*$ such that $f^*(x^*) - h(x^*) - \widehat{x}(x^*) = \alpha \leqslant f^*(y^*) - h(y^*) - \widehat{x}(y^*)$ for all $y^* \in \operatorname{dom} f^*$. Therefore we have $h(\cdot) + \widehat{x}(\cdot) + \alpha \leqslant f^*(\cdot)$ on $\operatorname{dom} f^*$ and the equality holds at x^*. Since Y is reflexive, there exists $x \in X$ such that $y^*(x) = \widehat{x}(y^*)$ for each $y^* \in Y$. Let $k : X^* \to \mathbb{R}$ be a function defined by $k(\cdot) = h(\cdot) + x(\cdot) + \alpha$. Then k is a convex function such that $k \leqslant f^*$ and $k(x^*) = f^*(x^*)$. Put $l = k_{\restriction Y}$. The function l is continuous and convex on Y. Let $\widehat{y} \in \partial l(x^*) \subset Y^*$. As Y is reflexive, there exists $y \in X$ such that $\widehat{y}(y^*) = y^*(y)$ for each $y^* \in Y$. We claim that $y \in \partial k(x^*)$. Indeed, let $z^* \in X^*$. If $z^* \in Y$, $y(z^* - x^*) = \widehat{y}(z^* - x^*) \leqslant k(z^*) - k(x^*)$. If $z^* \notin Y$, then $y(z^* - x^*) < k(z^*) - k(x^*) = \infty$. Hence $y \in \partial k(x^*)$. Since $k(x^*) = f^*(x^*)$, we have $y \in \partial f^*(x^*)$. Thus $k^*(y) + k(x^*) = (x^*, y) = f^*(x^*) + f(y)$. Therefore $f(y) = k^*(y)$. Since $f^* \geqslant k$, we have $k^* \geqslant f$. Put $\psi = k^*_{\restriction X}$. The function ψ has a Lipschitz derivative and is our required function. Indeed, $k^* = (h(\cdot) + x(\cdot) + \alpha)^* = (h + x)^* - \alpha = h^*(\cdot)\Box\delta_x(\cdot) - \alpha = h^*(\cdot - x) - \alpha$ (where δ_x is the indicator function of the singleton $\{x\}$) and h^* has the desired differentiability by Lemma 6. Finally, since $f(y) = k^*(y) = \psi(y)$ and $f \leqslant \psi$ on X, we have $f(y + v) + f(y - v) - 2f(y) \leqslant \psi(y + v) + \psi(y - v) - 2\psi(y) \leqslant C\|v\|^2$, for some constant C. Therefore the function f is Fréchet differentiable at y and f' is Lipschitz at y. ∎

Similarly, using Troyanski's result that reflexive spaces admit equivalent LUR norms (see, for example [**3**, Chapter VII]], we can show the following result.

COROLLARY 8. *Let f be a Lipschitz convex function on a Banach space X and $Y = \overline{span}^{\|\cdot\|}\{\partial f(x) : x \in X\}$. If Y is reflexive, then f can be majorised on X by a convex function ϕ that is Fréchet differentiable and $\phi(x) = f(x)$ for some $x \in X$.*

Under the assumptions in Theorem 7, the techniques in Theorem 1 may be applied to obtain approximation by functions with Lipschitz derivatives.

THEOREM 9. *Let X, Y and f be as in Theorem 7. Then f can be uniformly approximated on X by convex functions that have a Lipschitz derivative.*

PROOF: As in the proof of Theorem 7, let $\|\cdot\|$ be an equivalent norm of X^* such that its restriction on Y is LUR. Let $h = \|\cdot\|^2/2$ and $g := h + f^*$ on X^*. The function g is w^*-lower semicontinuous on X^*. Let k be a convex function on X such that $k^* = g$. We claim that there exists a constant C such that for any $\varepsilon > 0$, $x \in X$ and $y \in \partial k(x)$, we have $\{v : g(v+y) - g(y) - (x,v) \leqslant C\varepsilon\} \subset \varepsilon B_{X^*}$. Since $g(u) = \infty$ whenever $u \notin Y$, we only need to consider points in Y. Let $v \in Y$, then $(g(y) + g(y+v))/2 - g((2y+v)/2) \geqslant (h(y) + h(y+v))/2 - h((2y+v)/2)$ for any $y \in Y$. Using (*), we have $(g(y) + g(y+v))/2 - g((2y+v)/2) \geqslant L\|v\|^2$ for any $v \in Y$ and for any $y \in Y$. Following the same idea as in the proof of Theorem 7, we can complete the proof of the claim. By Lemma 6, k is Fréchet differentiable and k' is Lipschitz. For each $n \in \mathbb{N}$ define $g_n := f^* + h/(2n^4)$ and k_n such that $k_n^* = g$. By the above argument, the function k_n is Fréchet differentiable and k_n' is Lipschitz for each $n \in \mathbb{N}$. By [**8**, Lemma 2.1], $\lim g_n = f$ uniformly on X. ∎

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