INTEGRALS OF E-FUNCTIONS EXPRESSED IN TERMS OF E-FUNCTIONS

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PART I

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§ 1. *Introductory.* In § 2 the following two formulae will be established.

If, when $p \geqq q+1$, $R(\alpha_r + k) > 0$, $r = 1, 2, \ldots, p$ and $|\operatorname{amp} z| < \pi$,

$$\int_0^\infty e^{-\lambda} \lambda^{k-1} E(p ; \alpha_r : q ; \rho_s : \lambda z)\, d\lambda$$
$$= \frac{\pi}{\sin k\pi} \left\{ \begin{array}{l} E(p ; \alpha_r : 1-k, \rho_1, \ldots, \rho_q : e^{\pm i\pi} z) \\ -z^{-k} E(p ; \alpha_r + k : 1+k, \rho_1 + k, \ldots, \rho_q + k : e^{\pm i\pi} z) \end{array} \right\} \ldots\ldots\ldots (1)$$

If $p \leqq q$, the result holds if the integral is convergent.

If when $p \geqq q+1$, $l \geqq m+1$, $R(\alpha_r + k) > 0$, $r = 1, 2, \ldots, p$, $R(\beta_t - k) > 0$, $t = 1, 2, \ldots, l$, and $|\operatorname{amp} z| < \pi$,

$$\int_0^\infty \lambda^{k-1} E(p ; \alpha_r : q ; \rho_s : \lambda) E(l ; \beta_t : m ; \sigma_u : z/\lambda)\, d\lambda$$
$$= \frac{\pi}{\sin k\pi} \left\{ \begin{array}{l} z^k E\left(\begin{array}{l} \alpha_1, \ldots, \alpha_p, \beta_1 - k, \ldots, \beta_l - k \quad : e^{\pm i\pi} z \\ 1-k, \rho_1, \ldots, \rho_q, \sigma_1 - k, \ldots, \sigma_m - k \end{array} \right) \\ -E\left(\begin{array}{l} \alpha_1 + k, \ldots, \alpha_p + k, \beta_1, \ldots, \beta_l \quad : e^{\pm i\pi} z \\ 1+k, \rho_1 + k, \ldots, \rho_q + k, \sigma_1, \ldots, \sigma_m \end{array} \right) \end{array} \right\} \ldots\ldots\ldots (2)$$

For other values of p, q, l, m the result holds if the integral converges.

In § 3 some particular cases will be considered.

In Part II a further integral formula is given, and from it is deduced the discontinuous Integral of Weber and Schafheitlin (*cf.* Watson's *Bessel Functions*, p. 401.).

The following formulae will be required in the proof.

If $R(\alpha_{p+1}) > 0$,

$$\int_0^\infty e^{-\mu} \mu^{\alpha_{p+1}-1} E(p ; \alpha_r : q ; \rho_s : z/\mu)\, d\mu = E(p+1 ; \alpha_r : q ; \rho_s : z) \ldots\ldots\ldots (3)$$

$$\frac{1}{2\pi i} \int e^{\zeta} \zeta^{-\rho_{q+1}} E(p ; \alpha_r : q ; \rho_s : \zeta z)\, d\zeta = E(p ; \alpha_r : q+1 ; \rho_s : z), \ldots\ldots\ldots (4)$$

where the contour starts at $-\infty$ on the ξ-axis, passes positively round the origin and returns to $-\infty$ on the ξ-axis, the initial value of amp ζ being $-\pi$.

If $R(\beta) > 0$,

$$\Gamma(\alpha) \int_0^\infty e^{-\lambda} \lambda^{\beta-1} (1 + \lambda/z)^{-\alpha}\, d\lambda = \sum_{\alpha, \beta} \Gamma(\beta - \alpha) \Gamma(\alpha) z^\alpha F(\alpha ; \alpha - \beta + 1 ; z). \ldots\ldots\ldots (5)$$

§ 2. *Proofs of the Formulae.* In formula (5) replace α by α_1, β by $k + \alpha_1$, and z by $1/z$, and it can be written

$$\Gamma(\alpha_1) \int_0^\infty e^{-\lambda} \lambda^{k-1} \left(1 + \frac{1}{\lambda z}\right)^{-\alpha_1} d\lambda$$
$$= \Gamma(k) \Gamma(\alpha_1) F(\alpha_1 ; 1-k ; 1/z) + \Gamma(-k) \Gamma(\alpha_1 + k) z^{-k} F(\alpha_1 + k ; 1+k ; 1/z),$$

where $R(\alpha_1 + k) > 0$.

Assuming that $|\operatorname{amp} z|<\pi$, this can be written

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(\alpha_1::\lambda z)\,d\lambda$$
$$=\frac{\pi}{\sin k\pi}\{E(\alpha_1:1-k:e^{\pm i\pi}z)-z^{-k}E(\alpha_1+k:1+k:e^{\pm i\pi}z)\},$$

where $R(\alpha_1+k)>0$.

This is a particular case of formula (1). On replacing z by z/μ and applying formula (3) repeatedly; and then replacing z by ζz and applying formula (4) repeatedly, the general case is obtained.

Next, in (1) with $p\geqq q+1$ replace λ by λ/z, where for the moment z is taken real and positive, and the formula can be written

$$\int_0^\infty \lambda^{k-1}E(p\,;\alpha_r:q\,;\rho_s:\lambda)E(\;::z/\lambda)\,d\lambda$$
$$=\frac{\pi}{\sin k\pi}\left\{\begin{matrix} z^kE(p\,;\alpha_r:1-k,\,\rho_1,\,\ldots,\,\rho_q:e^{\pm i\pi}z) \\ -E(p\,;\alpha_r+k:1+k,\,\rho_1+k,\,\ldots,\,\rho_q+k:e^{\pm i\pi}z)\end{matrix}\right\},$$

where $R(\alpha_r+k)>0$, $r=1, 2, \ldots, p$, and we can take $|\operatorname{amp} z|<\pi$. This is a special case of formula (2), and the general case can be deduced in the same manner as was that of formula (1).

§ 3. *Special Cases.* In (1) take $p=0$, $q=1$, with $\rho_1=n+1$; then, since

$$E(\;:n+1:z)=z^{\frac{1}{2}n}J_n(2/\surd z), \qquad \text{(6)}$$

if $R(k+\frac{1}{2}n)>-\frac{3}{4}$,

$$z^{\frac{1}{2}n}\int_0^\infty e^{-\lambda}\lambda^{k+\frac{1}{2}n-1}J_n\{2/\surd(\lambda z)\}\,d\lambda=\frac{\Gamma(k)}{\Gamma(n+1)}F\left(\;;1-k,\,n+1\,;\frac{1}{z}\right)$$
$$+\frac{\Gamma(-k)}{\Gamma(n+k+1)}z^{-k}F\left(\;;1+k,\,n+k+1\,;\frac{1}{z}\right) \qquad \text{(7)}$$

Next, in (2) take $p=l=0$, $q=m=1$, put $\rho_1=m+1$, $\sigma_1=n+1$, replace z by $16/z^2$, λ by $4\lambda^2$ and k by $\frac{1}{2}(\rho-m+n)$; then, from (6), if $R(m-\rho)>-\frac{3}{2}$, $R(\rho+n)>-\frac{3}{2}$,

$$\int_0^\infty \lambda^{\rho-1}J_m(1/\lambda)J_n(\lambda z)\,d\lambda$$
$$=\frac{\Gamma(\frac{1}{2}\rho-\frac{1}{2}m+\frac{1}{2}n)z^{m-\rho}}{2^{2m-\rho+1}\Gamma(\frac{1}{2}m+\frac{1}{2}n-\frac{1}{2}\rho+1)\Gamma(m+1)}F\left(\;;m+1,\,\tfrac{1}{2}m-\tfrac{1}{2}n-\tfrac{1}{2}\rho+1,\,\tfrac{1}{2}m+\tfrac{1}{2}n-\tfrac{1}{2}\rho+1\,;\frac{z^2}{16}\right)$$
$$+\frac{\Gamma(\frac{1}{2}m-\frac{1}{2}n-\frac{1}{2}\rho)z^n}{2^{2n+\rho+1}\Gamma(\frac{1}{2}m+\frac{1}{2}n+\frac{1}{2}\rho+1)\Gamma(n+1)}F\left(\;;n+1,\,\tfrac{1}{2}n-\tfrac{1}{2}m+\tfrac{1}{2}\rho+1,\,\tfrac{1}{2}m+\tfrac{1}{2}n+\tfrac{1}{2}\rho+1\,;\frac{z^2}{16}\right). \qquad \text{(8)}$$

This formula was given by Hanumanta Rao [*Mess. of Maths.*, XLVII. (1918), pp. 134–137].

Again, in (2) take $p=2$, $q=0$, $\alpha_1=\frac{1}{2}+n$, $\alpha_2=\frac{1}{2}-n$ and replace λ and z by 2λ and $2z$; then, from the formula

$$\cos n\pi\,E(\tfrac{1}{2}+n,\,\tfrac{1}{2}-n::2z)=\surd(2\pi z)e^zK_n(z) \qquad \text{(9)}$$

it follows that, if $R(k\pm n)>-\frac{1}{2}$, $R(\beta_t-k)>0$, $t=1, 2, \ldots l$, $l\geqq m+1$,

$$\int_0^\infty \lambda^{k-\frac{1}{2}}e^\lambda K_n(\lambda)E(l\,;\beta_t:m\,;\sigma_u:z/\lambda)\,d\lambda$$
$$=\sqrt{\left(\frac{\pi}{2}\right)}\frac{\cos n\pi}{\sin k\pi}\left\{\begin{matrix} z^kE\begin{pmatrix}\frac{1}{2}+n,\,\frac{1}{2}-n,\,\beta_1-k,\,\ldots,\,\beta_l-k:2e^{\pm i\pi}z\\ 1-k,\,\sigma_1-k,\,\ldots,\,\sigma_m-k\end{pmatrix} \\ -2^{-k}E\begin{pmatrix}\frac{1}{2}+n+k,\,\frac{1}{2}-n+k,\,\beta_1,\,\ldots,\,\beta_l:2e^{\pm i\pi}z\\ 1+k,\,\sigma_1,\,\ldots,\,\sigma_m\end{pmatrix}\end{matrix}\right\}. \qquad \text{(10)}$$

In particular, from (6), if $l=0$, $m=1$ and σ_1 is replaced by $m+1$;

$$\int_0^\infty \lambda^{k-\frac{1}{2}m-\frac{1}{2}} e^{\lambda} K_n(\lambda) J_m\left(2\sqrt{\frac{\lambda}{z}}\right) d\lambda$$
$$=\sqrt{\left(\frac{\pi}{2}\right)}\frac{\cos n\pi}{\sin k\pi} z^{-\frac{1}{2}m}\left\{\begin{matrix} z^k E(\frac{1}{2}+n, \frac{1}{2}-n : 1-k, m+1-k : 2e^{\pm i\pi}z) \\ -2^{-k} E(\frac{1}{2}+n+k, \frac{1}{2}-n+k : 1+k, m+1 : 2e^{\pm i\pi}z)\end{matrix}\right\}, \quad \ldots\ldots(11)$$

where $R(\pm n-\frac{1}{2})<R(k)<R\left(\frac{1}{2}m+\frac{3}{4}\right)$.

Next, in (10) put $l=1$, $m=0$, $\beta_1=\beta$ and get

$$\int_0^\infty \lambda^{k-\frac{1}{2}} (\lambda+z)^{-\beta} e^{\lambda} K_n(\lambda)\, d\lambda$$
$$=\sqrt{\left(\frac{\pi}{2}\right)}\frac{\cos n\pi}{\sin k\pi}\frac{z^{-\beta}}{\Gamma(\beta)}\left[\begin{matrix} z^k E(\frac{1}{2}+n, \frac{1}{2}-n, \beta-k : 1-k : 2e^{\pm i\pi}z) \\ -2^{-k} E(\frac{1}{2}+n+k, \frac{1}{2}-n+k, \beta : 1+k : 2e^{\pm i\pi}z)\end{matrix}\right], \quad \ldots\ldots(12)$$

where $R(\pm n-\frac{1}{2})<R(k)<R(\beta)$ and $|\operatorname{amp} z|<\pi$.

Finally, in (10) take $l=2$, $m=0$, put $\beta_1=\frac{1}{2}+m$, $\beta_2=\frac{1}{2}-m$ and replace z by $2z$; then, from (9),

$$\int_0^\infty \lambda^{k-1} e^{\lambda+z/\lambda} K_n(\lambda) K_m(z/\lambda)\, d\lambda$$
$$=\frac{\cos m\pi \cos n\pi}{2z^{\frac{1}{2}}\sin k\pi}\left\{\begin{matrix} (2z)^k E(\frac{1}{2}+n, \frac{1}{2}-n, \frac{1}{2}+m-k, \frac{1}{2}-m-k : 1-k : 4e^{\pm i\pi}z) \\ -2^{-k} E(\frac{1}{2}+n+k, \frac{1}{2}-n+k, \frac{1}{2}+m, \frac{1}{2}-m : 1+k ; 4e^{\pm i\pi}z)\end{matrix}\right\}, \quad \ldots(13)$$

where $R(\pm n-\frac{1}{2})<R(k)<R(\frac{1}{2}\pm m)$ and $|\operatorname{amp} z|<\pi$.

Numerous other special cases can be derived from (1) and (2).

Part II

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§ 4. *A third Integral.* The formula to be proved is

$$\int_0^\infty \lambda^{-\alpha_{p+1}-1} E(p ; \alpha_r : q ; \rho_s : \lambda) E(l ; \beta_t : m ; \sigma_u : z\lambda)\, d\lambda$$
$$=\pi^{p-q} z^{\alpha_{p+1}} \sum_{r=1}^{p+1} \prod_{t=1}^{q} \sin(\rho_t-\alpha_r)\pi \left\{\prod_{s=1}^{p+1}{}' \sin(\alpha_s-\alpha_r)\pi\right\}^{-1} z^{-\alpha_r}$$
$$\times E\left\{\begin{matrix}\alpha_r, \alpha_r+\beta_1-\alpha_{p+1}, \ldots, \alpha_r+\beta_l-\alpha_{p+1}, \alpha_r-\rho_1+1, \ldots, \alpha_r-\rho_q+1 : e^{\pm i\pi(p-q)}z \\ \alpha_r-\alpha_1+1, \ldots * \ldots, \alpha_r-\alpha_{p+1}+1, \alpha_r+\sigma_1-\alpha_{p+1}, \ldots, \alpha_r+\sigma_m-\alpha_{p+1}\end{matrix}\right\}, \quad \ldots(14)$$

where $p\geqq q+1$, $l\geqq m+1$, $R(\alpha_{p+1})>0$, $R(\alpha_r+\beta_t-\alpha_{p+1})>0$, $r=1, 2, \ldots, p$, $t=1, 2, \ldots, l$, and $|\operatorname{amp} z|<\pi$. For other values of p, q, l, m the formula is valid if the integral is convergent.

The following formulae will be required in the proof.

If $p\geqq q+1$,

$$E(p ; \alpha_r : q ; \rho_s : z)=\sum_{r=1}^{p}\prod_{s=1}^{p}{}' \Gamma(\alpha_s-\alpha_r)\left\{\prod_{t=1}^{q}\Gamma(\rho_t-\alpha_r)\right\}^{-1}$$
$$\times\Gamma(\alpha_r)z^{\alpha_r} F\left\{\begin{matrix}\alpha_r, \alpha_r-\rho_1+1, \ldots, \alpha_r-\rho_q+1 ; (-1)^{p-q}z \\ \alpha_r-\alpha_1+1, \ldots * \ldots, \alpha_r-\alpha_p+1\end{matrix}\right\}. \quad \ldots\ldots\ldots(15)$$

If $p\leqq q+1$, $z\neq 0$,

$$E(p ; \alpha_r : q ; \rho_s : z)=\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_p)}{\Gamma(\rho_1)\ldots\Gamma(\rho_q)} F\left(p ; \alpha_r : q ; \rho_s : -\frac{1}{z}\right), \quad \ldots\ldots\ldots\ldots(16)$$

where, if $p=q+1$, $|z|>1$.

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If $p \geqq q+1$, it follows from (15) and (16) that

$$E(p\,;\alpha_r:q\,;\rho_s:z)=\pi^{p-q-1}\sum_{r=1}^{p}\prod_{t=1}^{q}\sin(\rho_t-\alpha_r)\pi\left\{\prod_{s=1}^{p}{}'\sin(\alpha_s-\alpha_r)\pi\right\}^{-1}z^{\alpha_r}$$
$$\times E\left\{\begin{matrix}\alpha_r,\,\alpha_r-\rho_1+1,\,\ldots,\,\alpha_r-\rho_q+1 \\ \alpha_r-\alpha_1+1,\,\ldots *\ldots,\,\alpha_r-\alpha_p+1\end{matrix} : \frac{e^{\pm i\pi(p-q-1)}}{z}\right\}. \quad (17)$$

§ 5. *Proof of the formula.* In (3) put $\mu=\lambda z$ and then replace λ by $1/\lambda$ and z by $1/z$, so obtaining

$$z^{-\alpha_{p+1}}\int_0^\infty \lambda^{-\alpha_{p+1}-1}E(p\,;\alpha_r:q\,;\rho_s:\lambda)\,E(::\lambda z)\,d\lambda=E(p+1\,;\alpha_r:q\,;\rho_s:1/z),$$

where $R(\alpha_{p+1})>0$. Hence, on applying formula (17), formula (14) with $l=0$, $m=0$ is obtained.

The general case is deduced in the same manner as the general case of formula (1).

§ 6. *The discontinuous Integral of Weber and Schafheitlin.* If

$$I\equiv\int_0^\infty J_m(\lambda x)\,J_n(\lambda)\,\lambda^{-k}\,d\lambda,$$

where x is real and positive and $-1<R(k)<R(m+n+1)$, then

$$I=\begin{cases}\dfrac{\Gamma\left(\frac{m+n-k+1}{2}\right)2^{-k}}{\Gamma\left(\frac{n-m+k+1}{2}\right)\Gamma(m+1)}\,x^m F\left(\begin{matrix}\frac{m+n-k+1}{2},\ \frac{m-n-k+1}{2}\,;x^2\\ m+1\end{matrix}\right),\ 0<x<1,\\[2ex] \dfrac{\Gamma\left(\frac{m+n-k+1}{2}\right)2^{-k}}{\Gamma\left(\frac{m-n+k+1}{2}\right)\Gamma(n+1)}\,x^{k-n-1}F\left(\begin{matrix}\frac{m+n-k+1}{2},\ \frac{n-m-k+1}{2}\,;\frac{1}{x^2}\\ n+1\end{matrix}\right),\ x>1.\end{cases}\quad \ldots(18)$$

To prove this take $p=0$, $q=1$, $l=0$, $m=1$ in (14) and replace a_{p+1} by d, ρ_1 by $n+1$ and σ_1 by $m+1$; then, from (6),

$$\pi x^{\frac{1}{2}m}\int_0^\infty \lambda^{-d-1+\frac{1}{2}m+\frac{1}{2}n}J_m\{2/\surd(\lambda x)\}\,J_n(2/\surd\lambda)\,d\lambda$$
$$=-\sin(n-d)\pi\,E(d,\,d-n:m+1:\,-x)$$
$$=\pi\frac{\Gamma(d)}{\Gamma(n-d+1)\Gamma(m+1)}F\left(\begin{matrix}d,\,d-n\,;\frac{1}{x}\\ m+1\end{matrix}\right),$$

provided that $x>1$.

Now replace λ by $4/\lambda^2$, x by $1/x^2$ and d by $\frac{1}{2}(m+n-k+1)$, and so obtain the first case of (18).

To obtain the second case interchange m and n, replace λ by λ/x and then replace x by $1/x$.

Many other special cases can be derived from (14).

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