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# DIRECT SUMS OF INDECOMPOSABLE INJECTIVE MODULES

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This paper proves that every direct summand $N$ of a direct sum of indecomposable injective submodules of a module is the sum of a direct sum of indecomposable injective submodules and a sum of indecomposable injective submodules of $Z_2(N)$.

In this paper, *we shall always assume that every ring $R$ has an identity element.* By a *module*, we shall always mean a unitary left $R$-module.

LEMMA 1. *Let $M$ be an $R$-module and suppose that $\{M_i\}_{i\in I}$ and $\{N_i\}_{i\in I}$ are families of submodules of $M$ such that $\sum\limits_{i\in I}\oplus M_i = \sum\limits_{i\in I} N_i$ in $M$. Suppose further that $M_i \supseteq N_i$ for each $i \in I$. Then $M_i = N_i$ for each $i \in I$.*

PROOF: If $M_j \supset N_j$ for some $j \in I$, then $\sum\limits_{i\in I}\oplus M_i \supset \sum\limits_{i\in I}\oplus N_i = \sum\limits_{i\in I} N_i$, contradiction. ∎

Recall that the *singular submodule* $Z(M)$ of an $R$-module $M$ is defined by

$$Z(M) = \{m \in M \mid \operatorname{ann}_R(m) \text{ is essential in } R\}.$$

Recall further that the module $M$ is called *singular* if $M = Z(M)$, and *nonsingular* if $Z(M) = 0$.

It is fairly well known that for any prime $p$ in the ring $\mathbb{Z}$ of integers, the $\mathbb{Z}$-module $G = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$ has the property that not every submodule has a unique injective envelope in $G$. It is well-known that if there is an injective envelope of a submodule, within a given nonsingular module, then it is unique. (In fact, this follows from [**3**, Propositions 4.9, 3.28(b), 3.26] and [**2**, Lemma 2.1].) This can be further generalised as follows:

LEMMA 2. *If there is an injective envelope $E(N)$ of a submodule $N$, within a given $R$-module $M$, then for every injective envelope $E'(N)$ of $N$, within $M$,*

$$E'(N) + Z(M) = E(N) + Z(M).$$

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PROOF: It is well-known [**2**, Proposition 1.11] or [**3**, Proposition 4.9] that there exists an isomorphism $f$ from $E(N)$ onto $E'(N)$ such that $f|_N = id_N$. To show that $E(N)+Z(M) = E'(N)+Z(M)$ in $M$, it suffices to prove that for every $e \in E(N)\backslash N$, $\operatorname{ann}_R(e - f(e))$ is essential in $R$.

Let $\mathfrak{I}$ be a non-zero left ideal in $R$. Since $N$ is essential in $E(N)$, it follows that $E(N)/N$ is singular. This implies that $\operatorname{ann}_R(e+N)$ is essential in $R$, so that $\mathfrak{I} \cap \operatorname{ann}_R(e+N) \neq 0$. Take a non-zero element $i$ in $\mathfrak{I} \cap \operatorname{ann}_R(e+N)$. Then $ie \in N$. Since $f|_N = id_N$, it follows that $i$ belongs to $\mathfrak{I} \cap \operatorname{ann}_R(e-f(e))$, so that $\mathfrak{I} \cap \operatorname{ann}_R(e - f(e)) \neq 0$. Hence, $\operatorname{ann}_R(e - f(e))$ is essential in $R$, as required. ∎

If $M$ is a nonsingular $R$-module which is a sum of indecomposable injective submodules, then it is a direct sum of indecomposable injective submodules. The generalisation of this will be discussed.

PROPOSITION 3. *Let $M$ be an $R$-module which can be expressed in the form $M = \sum_{i\in I} M_i + Z(M)$, where the $M_i$'s are indecomposable injective submodules of $M$. Then there exists a subset $J$ of $I$ such that $M = \sum_{j\in J} \oplus M_j + Z(M)$.*

PROOF: Consider the family $\{M_i\}_{i\in I}$. By Zorn's lemma, the collection of subfamilies $\{M_k\}_{k\in K}$ $(K \subseteq I)$ of the family $\{M_i\}_{i\in I}$ such that the sum $\sum_{k\in K} M_k$ is direct, has a maximal member, say $\{M_j\}_{j\in J}$ $(J \subseteq I)$. Let $C = \sum_{j\in J} \oplus M_j + Z(M)$. In order to show the proposition, it is sufficient to prove that $M = C$.

Suppose that $M \neq C$. Then there exists $i \in I$ such that $M_i$ is not contained in $C$. By the maximality of $\{M_j\}_{j\in J}$, we have $M_i \cap \sum_{j\in J} \oplus M_j \neq 0$. We can now pick out a finite collection $\{M_{j_1}, M_{j_2}, \dots, M_{j_r}\}$ of members of $\{M_j\}_{j\in J}$ such that

$$M_i \cap (M_{j_1} \oplus \cdots \oplus M_{j_r}) \neq 0.$$

Since $M_{j_1} \oplus \cdots \oplus M_{j_r}$ is injective, $M_i \cap (M_{j_1} \oplus \cdots \oplus M_{j_r})$ has an injective envelope which is a submodule of $M_{j_1} \oplus \cdots \oplus M_{j_r}$ (see [**5**, Proposition 2.22]). Further, $M_i$ is an injective envelope for $M_i \cap (M_{j_1} \oplus \cdots \oplus M_{j_r})$. Therefore, according to Lemma 2,

$$M_i \subseteq M_i + Z(M) \subseteq M_{j_1} \oplus \cdots \oplus M_{j_r} + Z(M) \subseteq C.$$

This is a contradiction. Hence, $M = C$. ∎

Let $A$ be any indecomposable injective $R$-module and let $B$ be any non-zero submodule of $A$. Then $A$ is an injective envelope for $B$. In particular, $B$ is essential in $A$. Hence, the residue class $R$-module $A/B$ is singular. This shows that the homomorphic image of an indecomposable injective $R$-module $A$ is either an indecomposable injective $R$-module (which is isomorphic to $A$) or a singular $R$-module.

Every homomorphic image $N$ of a sum of indecomposable injective submodules of an $R$-module is the sum of a direct sum of indecomposable injective submodules which are not singular and $Z(N)$. More precisely, we proceed to the following:

**THEOREM 4.** *Let $M$ and $N$ be $R$-modules and let $\{M_i\}_{i\in I}$ be a family of indecomposable injective submodules of $M$. Then for every epimorphism $f : \sum_{i\in I} M_i \to N$, there is a subset $J$ of $I$ such that*

$$N = \left(\sum_{j\in J} \oplus f(M_j)\right) + Z(N)$$

*in which each $f(M_j)$ is indecomposable injective but not singular.*

PROOF: Let $\{M_i\}_{i\in I}$ be a family of submodules of an $R$-module $M$. Let $N_i = f(M_i)$ for each $i \in I$. Then $N = \sum_{i\in I} N_i$. Hence,

$$N = \left(\sum_{i\in I} N_i\right) + Z(N).$$

Now, assume that each $M_i$ is indecomposable injective. Then, since $N_i$ is the homomorphic image of $M_i$, it follows from the argument immediately following the proof of Proposition 3 that each $N_i$ is either indecomposable injective or singular. It may be assumed that every singular submodule of $N$ is contained in $Z(N)$. Thus, we may assume that each $N_i$ is indecomposable injective but not singular.

Therefore, by Proposition 3, there is a subset $J$ of $I$ such that

$$N = \left(\sum_{j\in J} \oplus N_j\right) + Z(N);$$

and each $N_j$ is indecomposable injective but not singular, as required. ∎

Theorem 4 provides us with the natural generalisation of a theorem of Harada [**4**, (8.2.7)].

If $R$ is a nonsingular ring, then the factor $R$-module $M/Z(M)$ is nonsingular (see [**3**, Proposition 3.29].) However, we cannot say in general that the factor $R$-module $M/Z(M)$ is nonsingular.

Let $R$ be a ring and let $M$ be an $R$-module. Since $Z\big(M/Z(M)\big)$ is a submodule of the factor $R$-module $M/Z(M)$, it follows from the one-to-one correspondence theorem for modules that there exists a unique submodule $G$ of $M$ containing $Z(M)$ such that

$$G/Z(M) = Z\big(M/Z(M)\big).$$

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Then $G$ is called the *Goldie torsion submodule* (or *second singular submodule*) of $M$ and is denoted by $Z_2(M)$ (see [7]).

We should mention three well known facts. The first is that

$$\frac{M/Z(M)}{Z_2(M)/Z(M)} \cong M/Z_2(M).$$

The second is that $Z_2(M)/Z(M)$ is the singular submodule of the factor $R$-module $M/Z(M)$. The third is that the factor $R$-module $M/Z_2(M)$ is nonsingular. Clearly, $Z_2\big(Z_2(M)\big) = Z_2(M)$.

**Theorem 5.** *Let $M$ and $N$ be $R$-modules and let $\{M_i\}_{i\in I}$ be a family of indecomposable injective submodules of $M$. Then for every epimorphism $f : \sum_{i\in I} M_i \to N$, there are subsets $K, J$ of $I$ with $K \subseteq J$ such that*

$$N = \Big(\sum_{k\in K} \oplus f(M_k)\Big) \oplus Z_2(N)$$

*where each $f(M_k)$ is indecomposable injective and nonsingular;*

$$Z_2(N) = \Big(\sum_{j\in J\smallsetminus K} \oplus f(M_j)\Big) + Z(N)$$

*where each $f(M_j)$ is indecomposable injective, not singular but $Z_2\big(f(M_j)\big) \neq 0$.*

Proof: By Theorem 4, there is a subset $J$ of $I$ such that

$$N = \Big(\sum_{j\in J} \oplus N_j\Big) + Z(N) \tag{1}$$

where $N_j = f(M_j)$ and $N_j$ is indecomposable injective but not singular. Then since $Z_2(N)$ contains $Z(N)$, it follows that

$$N = N + Z_2(N) = \Big(\sum_{j\in J} \oplus N_j\Big) + Z(N) + Z_2(N) = \Big(\cdot\sum_{j\in J} \oplus N_j\Big) + Z_2(N).$$

Let $K$ be the set of all $j \in J$ with $Z_2(N_j) = 0$. Then

$$N = \Big(\sum_{k\in K} \oplus N_k\Big) \oplus \Big(\sum_{j\in J\smallsetminus K} \oplus N_j\Big) + Z_2(N). \tag{2}$$

Let $j$ be any element of $J\backslash K$. Since $N_j$ is indecomposable injective and $Z_2(N_j) \neq 0$, $N_j/Z_2(N_j)$ is both singular and nonsingular. This implies that $N_j = Z_2(N_j)$. Hence,

$$\sum_{j\in J\smallsetminus K} \oplus N_j = \sum_{j\in J\smallsetminus K} \oplus Z_2(N_j) \subseteq Z_2(N).$$

Further, by making a direct computation, we can see that

$$\left(\sum_{k\in K}\oplus N_k\right)\cap Z_2(N)\subseteq\sum_{k\in K}\oplus Z_2(N_k)=0.$$

Therefore, it follows from (2) that

$$N=\left(\sum_{k\in K}\oplus N_k\right)\oplus Z_2(N). \tag{3}$$

Here, notice that each $N_k$ is nonsingular, because $N/Z_2(N)$ is nonsingular so that $\sum_{k\in K}\oplus N_k$ is nonsingular.

From (1) and (3), we have

$$\left(\sum_{k\in K}\oplus N_k\right)\oplus Z_2(N)=\left(\sum_{k\in K}\oplus N_k\right)\oplus\left(\sum_{j\in J\smallsetminus K}\oplus N_j\right)+Z(N).$$

Therefore, by Lemma 1 or by the modularity, we get

$$Z_2(N)=\left(\sum_{j\in J\smallsetminus K}\oplus N_j\right)+Z(N).$$

The remainder of the proof is obvious. □

COROLLARY 6. *Let $M$ be an $R$-module which is a direct sum of indecomposable injective submodules. Then for every direct summand $N$ of $M$, $N/Z_2(N)$ is a direct sum of indecomposable injective submodules.* □

Theorem 5 says that if $M$ is a module which is a direct sum of indecomposable injective submodules, then every direct summand $N$ of $M$ is a direct sum of a direct sum of indecomposable injective submodules and $Z_2(N)$. This implies that $Z_2(N)$ is a direct summand of a direct sum of indecomposable injective modules.

We turn our attention to the decomposition of $Z_2(N)$ into indecomposable injective submodules when $Z_2(N)\neq 0$. More generally, we proceed to the following:

LEMMA 7. *Let $X$, $Y$, $Z$ be submodules of an $R$-module such that $X\oplus Y=X\oplus Z$. Then there exists an isomorphism $\varphi$ from $Y$ onto $Z$ such that for every submodule $B$ of $Y$ and for every submodule $C$ of $Z$, $\varphi(B)\cap C=(X\oplus B)\cap C$.*

PROOF: Let $\pi: X\oplus Z\to Z$ be the canonical projection. Then for any submodule $B'$ of $X\oplus Z$, $\pi(B')=(X+B')\cap Z$. If $X\oplus Y=X\oplus Z$, then the composite map $\varphi: Y\xrightarrow{inc} X\oplus Y=X\oplus Z\xrightarrow{\pi} Z$ is an isomorphism. Now, let $B$ be any submodule of $Y$ and let $C$ be any submodule of $Z$. Then

$$\varphi(B)\cap C=\pi(B)\cap C=(X\oplus B)\cap Z\cap C=(X\oplus B)\cap C,$$

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as required. $\square$

Compare the following theorem with Azumaya's Decomposition Theorem [**1**, Theorem 12.6].

**THEOREM 8.** *Let $M$ be an $R$-module which is a direct sum of indecomposable injective submodules $M_i$, $i \in I$. Then every non-zero direct summand of $M$ has an indecomposable injective direct summand isomorphic to one of the $M_i$.*

PROOF: Let $\{M_i\}_{i\in I}$ be a family of indecomposable injective submodules of an $R$-module $M$ such that $M = \sum_{i\in I} \oplus M_i$. Let $N$ be a direct summand of $M$. Then there is a submodule $N'$ of $M$ such that $M = N \oplus N'$. If $N$ is non-zero, then there exists a finite subset $J$ of $I$ such that $\left(\sum_{j\in J} \oplus M_j\right) \cap N \neq 0$. Consider the family $\mathcal{F}$ of all finite subsets $F$ of $I$ such that $\left(\sum_{f\in F} \oplus M_f\right) \cap N \neq 0$, and consider the set $S = \{|F| \mid F \in \mathcal{F}\}$, where $|F|$ denotes the number of elements of $F$. Then $S$ is a non-empty subset of the set $\mathbb{N}$ of natural numbers. By the well-ordering property of integers, $S$ has the least element $l$. Then $l = |F_*|$ for some finite subset $F_*$ of $I$ with $\left(\sum_{f\in F_*} \oplus M_f\right) \cap N \neq 0$. Write $F_* = \{i_1, \dots, i_l\}$.

Assume that $l = 1$. Then $(M_{i_1} \cap N) \cap (M_{i_1} \cap N') = 0$, $M_{i_1} \cap N \neq 0$, and $M_{i_1}$ is indecomposable injective. So, $M_{i_1} \cap N' = 0$. Hence, by [**6**, Lemma 2.10], $M_{i_1} \oplus N'$ is a direct summand of $M$. In fact, there exists a submodule $N_1$ of $N$ such that $M = M_{i_1} \oplus N' \oplus N_1$, that is, $N' \oplus M_{i_1} \oplus N_1 = N' \oplus N$. By Lemma 7, there exists an isomorphism $\varphi$ from $M_{i_1} \oplus N_1$ onto $N$. Hence $\varphi(M_{i_1}) \oplus \varphi(N_1) = \varphi(M_{i_1} \oplus N_1) = N$ so that $N$ has an indecomposable injective direct summand $\varphi(M_{i_1})$ which is isomorphic to $M_{i_l}$.

Assume now that $l > 1$. Then by the minimality of $l$, we must have $(M_{i_1} \oplus \cdots \oplus M_{i_{l-1}}) \cap N = 0$. According to [**6**, Lemma 2.10], $M_{i_1} \oplus \cdots \oplus M_{i_{l-1}} \oplus N$ is a direct summand of $M$. In fact, there exists a submodule $N_1'$ of $N'$ such that $M = M_{i_1} \oplus \cdots \oplus M_{i_{l-1}} \oplus N \oplus N_1'$. Write $M$ as follows:

$$M = M_{i_1} \oplus \cdots \oplus M_{i_{l-1}} \oplus \sum_{i\in I\setminus\{i_1,\dots,i_{l-1}\}} \oplus M_i.$$

By Lemma 7, there exists an isomorphism $\varphi$ from $\sum_{i\in I\setminus\{i_1,\dots,i_{l-1}\}} \oplus M_i$ onto $N \oplus N_1'$ such that $\varphi(M_{i_l}) \cap N = (M_{i_1} \oplus \cdots \oplus M_{i_{l-1}} \oplus M_{i_l}) \cap N$. Note that

$$\left(\varphi(M_{i_l}) \cap N\right) \cap \left(\varphi(M_{i_l}) \cap N_1'\right) = 0.$$

According to the construction of $l$,

$$\varphi(M_{i_l}) \cap N = (M_{i_1} \oplus \cdots \oplus M_{i_l}) \cap N = \Big(\sum_{f \in F_*} \oplus M_f\Big) \cap N \neq 0.$$

$M_{i_l} \cong \varphi(M_{i_l})$ implies that $\varphi(M_{i_l})$ is indecomposable injective. Hence, as in the case $l = 1$, we can prove that $N$ has an indecomposable injective direct summand which is isomorphic to $\varphi(M_{i_l})$ and hence to $M_{i_l}$. ∎

REMARK. The word "isomorphic" in Theorem 8 cannot be replaced by the word "equal". An example of this is given in the following example.

EXAMPLE 9. Let $R = \mathbb{Z}$ be the ring of integers and let $\mathbb{N}$ be the set of non-negative integers. Let $V$ be a vector space over the field $\mathbb{Z}/2\mathbb{Z}$ with a countable basis $v_i$, $i \in \mathbb{N}$. Let $M$ be the injective envelope of the $\mathbb{Z}$-module $V$. Then $M = \sum_{i \in \mathbb{N}} \oplus M_i$, where each $M_i$ is an indecomposable injective $\mathbb{Z}$-module whose socle is the submodule with two elements $\mathbb{Z}v_i$. Let $N$ be an injective envelope in $M$ of the submodule generated by the countable set $v_i - v_{i+1}$, $i \in \mathbb{N}$, and $N'$ be an injective envelope in $M$ of the submodule generated by $v_0$. Clearly $M = N \oplus N'$. Note that $N = Z_2(N) \neq 0$. Then

$$K = \{i \in \mathbb{N} \mid Z_2(M_i) \neq 0\} = \mathbb{N}.$$

Now $N = Z_2(N) \neq 0$ has an indecomposable injective direct summand $N_1$ which is isomorphic to $M_{k_1}$ for some $k_1 \in \mathbb{N}$. For instance, take as $N_1$ the injective envelope in $M$ of the submodule generated by $v_0 - v_1$. This $N_1$ is isomorphic to $M_{k_1}$ for any $k_1 \in \mathbb{N}$. Let $N_1'$ be a submodule of $N = Z_2(N)$ such that

$$N = Z_2(N) = N_1 \oplus N_1'.$$

For instance, take as $N_1'$ the injective envelope in $M$ of the submodule generated by the countable set $v_i - v_{i+1}$, $i \geq 1$. From this we cannot deduce that

$$N = Z_2(N) = M_{k_1} \oplus N_1'.$$

Otherwise, the elements $v_{k_1}$ in the socle of $M_{k_1}$ would belong to $N$. But this would imply that all the elements $v_i$ are in $N$, that is, $N = M$, contradiction.

To investigate the Goldie torsion submodule of an indecomposable injective module, we need to introduce the notion of a module with $(C_{11})$.

Let $M$ be an $R$-module and let $N$ be a submodule of $M$. By Zorn's lemma, the collection of submodules $L$ of $M$ such that $N \cap L = 0$ has a maximal member. A *complement of* $N$ *in* $M$ is a submodule $K$ of $M$ maximal with respect to the property

$N \cap K = 0$. A submodule $K$ of $M$ is called a *complement in* $M$ if there exists a submodule $N$ of $M$ such that $K$ is a complement of $N$ in $M$. A module $M$ is a *$CS$-module or satisfies* $(C_1)$ if every complement in $M$ is a direct summand of $M$. A module $M$ is said to *satisfy* $(C_{11})$ if every submodule of $M$ has a complement in $M$ which is a direct summand of $M$.

If an $R$-module $M$ satisfies $(C_1)$, then it satisfies $(C_{11})$. For example, every injective $R$-module satisfies $(C_1)$ and hence it satisfies $(C_{11})$. In fact, let $M$ be an injective $R$-module and let $K$ be any complement in $M$. Then there is a submodule $N$ of $M$ such that $K$ is a complement of $N$ in $M$. Since $M$ is injective and $N \oplus K$ is a submodule of $M$, $N \oplus K$ has an injective envelope $E(N \oplus K)$ in $M$. Note that $E(N \oplus K) = E(N) \oplus E(K)$. According to [**5**, Proposition 2.15], $E(N) \oplus E(K)$ is a direct summand of $M$. Let $P$ be a submodule of $M$ such that

$$M = E(N) \oplus E(K) \oplus P.$$

Then $N \cap \big(E(K) \oplus P\big) = 0$ and $K \subseteq E(K) \oplus P$. By the maximality of $K$, $E(K) \oplus P = K$. Hence $E(K) \subseteq K$ and $P \subseteq K$, so $K = E(K)$ and $P = P \cap K = 0$. Thus, $M = E(N) \oplus K$. This shows that $K$ is a direct summand of $M$. Therefore, $M$ satisfies $(C_1)$ and hence it satisfies $(C_{11})$.

**PROPOSITION 10.** *Let $M$ be an $R$-module. If $M$ is indecomposable injective, then either $Z_2(M) = 0$ or $Z_2(M) = M$.*

PROOF: Assume that $M$ is indecomposable injective. Then since $M$ is injective, $M$ satisfies $(C_{11})$. $Z_2(M)$ is a direct summand of $M$ (see [**6**, Theorem 2.7].) Since $M$ is indecomposable, either $Z_2(M) = 0$ or $Z_2(M) = M$. ∎

**LEMMA 11.** *Let $M$ be a direct sum of indecomposable injective submodules. Then every direct summand of $M$ is the sum of all its indecomposable injective submodules.*

PROOF: Let $M = \sum_{i \in I} \oplus M_i$ be a direct sum of indecomposable injective submodules $M_i$. Let $N$ be any direct summand of $M$. Let $S$ be the sum of all indecomposable injective submodules of $N$. The goal is to show that $S = N$.

Suppose on the contrary that $S \neq N$. Then we can take an element $x \in N \backslash S$. There is a finite set $\{i_1, \dots, i_m\} \subseteq I$ with $x \in M_{i_1} \oplus \cdots \oplus M_{i_m}$. Set $A = M_{i_1} \oplus \cdots \oplus M_{i_m}$. By [**1**, Lemma 26.4] there are direct summands $P$ of $N$ and $P'$ of $N'$ such that

$$M = A \oplus P \oplus P'.$$

Let

$$H = N \cap (A \oplus P').$$

Then $x \in N \cap A \subseteq H$, and by modularity

$$N = N \cap M = N \cap \big((A \oplus P') \oplus P\big) = H \oplus P.$$

Since $P'$ is a direct summand of $N'$, there is a submodule $P''$ of $N'$ with $N' = P' \oplus P''$. Thus,

$$\begin{aligned} A &\cong M/(P \oplus P') \\ &= (N \oplus N')/(P \oplus P') \\ &= (H \oplus P \oplus P' \oplus P'')/(P \oplus P') \\ &\cong H \oplus P''. \end{aligned}$$

But, by [**1**, Corollary 12.7] the decomposition $A = M_{i_1} \oplus \cdots \oplus M_{i_m}$ complements direct summands. Hence, in particular, $H$ is isomorphic to $\bigoplus_{k \in F} M_k$ for some finite subset $F \subseteq I$, so that $H$ is a direct sum of indecomposable injective submodules of $N$. Therefore, we get $x \in H \subseteq S$, contradiction. ∎

Recently, study of direct sum decompositions into indecomposable injective modules has been done in [**8**].

**Theorem 12.** *Let $M$ be an $R$-module which is a direct sum of indecomposable injective submodules. Then every direct summand $N$ of $M$ is the sum of a direct sum of indecomposable injective submodules and a sum of indecomposable injective submodules of $Z_2(N)$.*

Proof: Let $\{M_i\}_{i \in I}$ be a family of indecomposable injective submodules of an $R$-module $M$ such that $M = \sum_{i \in I} \oplus M_i$. Let $N$ be a direct summand of $M$. Let $\pi : M \to N$ be the canonical projection. Then by Theorem 5, there is a subset $J$ of $I$ such that

$$N = \left(\sum_{j \in J} \oplus \pi(M_j)\right) \oplus Z_2(N) \tag{1}$$

where each $\pi(M_j)$ is indecomposable injective. Let $N'$ be a submodule of $\sum_{i \in I} \oplus M_i$ such that $\sum_{i \in I} \oplus M_i = N \oplus N'$. Then

$$\sum_{i \in I} \oplus Z_2(M_i) = Z_2(N) \oplus Z_2(N').$$

Let

$$K = \{\, i \in I \mid Z_2(M_i) \neq 0\}.$$

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Then by Proposition 10,

$$\sum_{k \in K} \oplus M_k = Z_2(N) \oplus Z_2(N').$$

Hence, by Lemma 11, $Z_2(N)$ is the sum of all its indecomposable injective submodules. By (1), $N$ is a direct sum of a direct sum of indecomposable injective submodules and the sum of all indecomposable injective submodules of $Z_2(N)$. Therefore, by Zorn's lemma, $N$ is the sum of a direct sum of indecomposable injective submodules and a sum of indecomposable injective submodules of $Z_2(N)$. $\square$

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