THE LEBESGUE CONSTANTS FOR (γ, r) SUMMATION OF FOURIER SERIES

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The (γ, r) summation method, $0 < r < 1$, is the "circle method" employed by G. H. Hardy and J. E. Littlewood. It is also known as the Taylor method. Its Lebesgue constants, say $L(T_r, n)$, $n = 1, 2, \ldots$, were studied by K. Ishiguro [1] in the notation $L^*(n; 1-r)$. He noted that

$$(1) \qquad L(T_r, n) = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \frac{1}{\sin t} \left| \operatorname{Im} \left\{ \left(\frac{1-r}{1-re^{2it}} \right)^{n+1} e^{(2n+1)it} \right\} \right| dt ,$$

where $\operatorname{Im}\{z\}$ denotes the imaginary part of the complex number z, and proved that

$$(2) \qquad L(T_r, n) = \frac{2}{\pi^2} \log \frac{2n}{r} + \alpha + \underline{o}(1), \quad 0 < r < 1 .$$

Here

$$(3) \qquad \alpha = -\frac{2}{\pi^2} C + \frac{2}{\pi} \int_0^1 \frac{\sin t}{t} dt - \frac{2}{\pi} \int_1^\infty \frac{1}{t} \left\{ \frac{2}{\pi} - |\sin t| \right\} dt ,$$

where $C = .577\ldots$ is Euler's constant.

In this note, we provide an alternative derivation of (2), relating it to the computation of an asymptotic expression for

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the Lebesgue constants, $L(F, d_n)$, for the $[F, d_n]$ means of Fourier series [2]. This is done by representing (1), with error $\underline{o}(1)$, in terms of an expression [2, (5.2)] which yields also an asymptotic evaluation of $L(F, d_n)$ [2, (5.4)].

First, using the expansion for log (1-x) and exponentiating, we note

$$\frac{1-r}{1-re^{2it}} = \frac{1}{1+\frac{r}{1-r}(1-e^{2it})}$$

$$= \frac{1}{1-\frac{r}{1-r}(2it-2t^2)+\underline{O}(t^3)}$$

$$= 1+\frac{2r}{1-r}(it-t^2)+\frac{4r^2}{(1-r)^2}(it-t^2)^2+\underline{O}(t^3)$$

$$= \exp\left\{\frac{2r}{1-r}(it-t^2)+\frac{2r^2}{(1-r)^2}(it-t^2)^2\right\}+\underline{O}(t^3)$$

$$= \exp\left\{\frac{2ir}{1-r}t-\frac{2r}{(1-r)^2}t^2\right\}+\underline{O}(t^3).$$

And so, by the Lemma of [2, § 2],

$$(4) \qquad \left(\frac{1-r}{1-re^{2it}}\right)^{n+1} = \exp\left\{(n+1)\left[\frac{2ir}{1-r}t-\frac{2r}{(1-r)^2}t^2\right]\right\}+\underline{O}(nt^3).$$

As for $L(F, d_n)$, we need also an estimate particularly suited for t outside a neighbourhood of the origin. This is

$$(5) \qquad \left(\frac{1-r}{1-re^{2it}}\right)^{n+1} = \underline{O}(e^{-\delta nt^2}), \quad \text{for some } \delta > 0,\ 0 \le t \le \frac{1}{2}\pi.$$

Proof of (5): Now, $\cos 2t \le 1-\frac{1}{2}t^2$, $0 \le t \le \frac{1}{2}\pi$, and so

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$$\left|\frac{1-r}{1-re^{2it}}\right|^2 = \frac{(1-r)^2}{1 - 2r\cos 2t + r^2} \leq \frac{(1-r)^2}{(1-r)^2 + rt^2}$$

$$= \frac{1}{1 + r(1-r)^{-2}t^2},$$

and this is clearly $\leq e^{-2\delta t^2}$ for some $\delta > 0$, since $0 \leq t \leq \frac{1}{2}\pi$.

Analogously to what was done [2, § 3] in connection with $L(F, d_n)$, we decompose $L(T_r, n)$ and then introduce simplifying estimates. As there, we write, with $\xi = n^{-3/8}$

$$L(T_r, n) = \frac{2}{\pi}\int_0^{\xi} + \frac{2}{\pi}\int_{\xi}^{\frac{1}{2}\pi}$$

and infer, as in [2, § 3], that the integral from $n^{-3/8}$ to $\frac{1}{2}\pi$ is $\underline{o}(1)$, from (5). Likewise, the remainder term in (4) contributes only $\underline{o}(1)$ to the first integral. Moreover, $\sin t$ can be replaced by t with an error of $\underline{o}(1)$. Finally, the upper limit of integration, $\xi = n^{-3/8}$, can be replaced by $\frac{1}{2}\pi$ with an error of $\underline{o}(1)$.

This done, we have

$$(6) \qquad L(T_r, n) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \exp\{-S'_n t^2\} \left|\frac{\sin U'_n t}{t}\right| dt + \underline{o}(1),$$

where

$$(7) \qquad S'_n = \frac{2r(n+1)}{(1-r)^2} \text{ and } U'_n = \frac{2n+1+r}{1-r}.$$

Now, putting $S'_n = s_n$, $U'_n = u_n$, formula (6), with the error term $\underline{o}(1)$ disregarded, becomes $\lambda(n)$ defined by

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[2, (5.2)]. This identification is permissible since (7) clearly implies that the restrictions [2, (5.3)] placed on s_n and u_n are satisfied also by S'_n and U'_n.

This done, formula (5.4) of [2] for $\lambda(n)$ becomes Ishiguro's formula (2) above and the derivation is complete.

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