On the Numerical Solution of Integral Equations.

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1. *Introductory.*

The numerical solution of Integral equations with variable upper limits has been investigated by Professor Whittaker.* In this investigation the nucleus, supposed to be given numerically by a table of single entry, is replaced by an approximate expression consisting of a finite number of terms, each term involving an exponential or simply a power of the variable, and then the solution is found as an analytical expression from which its numerical values may be computed. The numerical solution of integral equations with fixed limits has been discussed by H. Bateman.† Methods for solving differential equations numerically have long been known‡ and extensive use of such methods has been made, specially for the calculation of "special perturbations" in Astronomy. The differential equations giving the forms of drops of fluid under the influence of capillary action have also been numerically solved by Bashforth and Adams.§ Methods for the numerical solution of differential equations from a somewhat different point of view have been investigated by Runge,‖ Heun,** Kutta,†† and Piaggio.‡‡ The aim of the present paper is to find a method for the numerical solution of integral equations on the lines of the methods for solving differential equations.

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* *Proc. Royal Soc.*, XCIV (A), 1918, pp. 367-383.

† *Proc. Royal Soc.*, C (A), 1921, pp. 441-449.

‡ See, for example, Bond, *Proc. American Academy*, IV., 1849, pp. 189–203, or Encke, *Astronomische Jahrbuch für* 1858.

§ Bashforth and Adams: *An attempt to test the Theories of Capillary Action*, Cambridge, 1883.

‖ *Mathematische Annalen*, XLVI., 1895, pp. 167–178.

** *Zeitschrift für Math. u. Phys.*, XLV., 1900, pp. 23–38.

†† *Zeitschrift für Math. u. Phys.*, XLVI., 1901, pp. 435–453.

‡‡ *Phil. Mag.*, XXXVI. (6th Ser.), 1919, pp. 596–600.

The advantages of the method given in Art. 2 below seem to be the following :—

(1). It is not necessary to replace either the nucleus or the function outside the integral by an approximate expression.

(2). Compared with other methods, this method is much less laborious.

(3). The computed values are accurate practically up to the last figure retained, although, if extreme accuracy is desired, it will be safer to retain one more figure.

(4). The method is applicable whether or not the nucleus $K(x, \xi)$ in equation (1) below is a function of $x - \xi$.

(5). The method is applicable to integral equations whose analytical solution cannot be found by the usual methods, for example to non-linear integral equations.

I wish to express my gratitude to Professor Whittaker for his kind help and encouragement.

2. *The integral equation of the second kind with a variable upper limit.*

Consider the integral equation

$$\phi(x) = f(x) + \int_a^x K(x, \xi)\, \phi(\xi)\, d\xi, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $f(x)$ is a continuous function of x in the range $a \leqq x \leqq b$, given either numerically or by an analytical expression and $K(x, \xi)$ is a real function of x and ξ, continuous in both the variables in the range $a \leqq \xi \leqq x \leqq b$, given either numerically or by an analytical expression, and $\phi(x)$ is the unknown function whose values are to be determined in the range $a \leqq x \leqq b$. We suppose further that the first few differential coefficients of $f(x)$ and also those of $K(x, \xi)$ with respect to x and ξ are continuous in the above ranges. We proceed to find accurately the values of $\phi(x)$ for $x = a$, $a + w$, $a + 2w$, ... , where w is taken to be so small that the intermediate values of $\phi(x)$ may be interpolated with certainty. It should also be so small that differences of

$$K(x, a+rw)\,\phi\,(a+rw)$$

above a certain order, say, to fix ideas, the fourth, are negligible.

Now suppose that we have already calculated ϕ_0, ϕ_1, ϕ_2, ... ϕ_{n-1}, where ϕ_r denotes $\phi(a+rw)$, and we are next to calculate ϕ_n. Let the successive differences of these quantities be taken according to the following scheme:—

x	ϕ	$\Delta\phi$	$\Delta^2\phi$	$\Delta^3\phi$	$\Delta^4\phi$
...	...				
$a+(n-5)w$	ϕ_{n-5}				
		$\Delta\phi_{n-5}$			
$a+(n-4)w$	ϕ_{n-4}		$\Delta^2\phi_{n-5}$		
		$\Delta\phi_{n-4}$		$\Delta^3\phi_{n-5}$	
$a+(n-3)w$	ϕ_{n-3}		$\Delta^2\phi_{n-4}$		$\Delta^4\phi_{n-5}$
		$\Delta\phi_{n-3}$		$\Delta^3\phi_{n-4}$	
$a+(n-2)w$	ϕ_{n-2}		$\Delta^2\phi_{n-3}$		$(\Delta^4\phi_{n-4})$
		$\Delta\phi_{n-2}$			
$a+(n-1)w$	ϕ_{n-1}				

The general run of the differences $\Delta^4\phi$ will suggest a close guess to the value of $\Delta^4\phi_{n-4}$, say $(\Delta^4\phi_{n-4})$, which will lead to a provisional value of ϕ_n, say (ϕ_n). Let the true value of ϕ_n be $(\phi_n)+\eta$.

We must now evaluate the integral in (1) numerically. A suitable formula for this is

$$\begin{aligned}\frac{1}{w}\int_a^{a+rw} f(x)\,dx = {} & f_0+f_1+f_2+\dots+f_r-\tfrac{1}{2}(f_r+f_0)\\ & -\tfrac{1}{12}(\Delta f_{r-1}-\Delta f_0)-\tfrac{1}{24}(\Delta^2 f_{r-2}+\Delta^2 f_0)\\ & -\tfrac{19}{720}(\Delta^3 f_{r-3}-\Delta^3 f_0)-\tfrac{3}{160}(\Delta^4 f_{r-4}+\Delta^4 f_0)\\ & -\tfrac{863}{60480}(\Delta^5 f_{r-5}-\Delta^5 f_0)-\tfrac{275}{24192}(\Delta^6 f_{r-6}+\Delta^6 f_0)\\ & -\dots \qquad\qquad \text{(2)}\end{aligned}$$

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Neglecting the differences of order higher than the fourth, let the values of

$$\int_a^{a+nw} K(a+nw,\ \xi)\,\phi(\xi)\,d\xi,$$

computed by formula (2) with the value (ϕ_n) of ϕ_n, be (I_n). Find $f_n + (I_n) - (\phi_n)$ and denote it by ϵ.

The true value of the integral is

$$(I_n) + wK_{n,n}\,\eta\,\{1 - \tfrac{1}{2} - \tfrac{1}{12} - \tfrac{1}{24} - \tfrac{19}{720} - \tfrac{3}{160}\},$$

or

$$(I_n) + w\,K_{n,n}\,\eta\,\tfrac{95}{288},\text{*}$$

where $K_{n,n}$ stands for $K(a+nw,\ a+nw)$.

Substituting this in the integral equation (1), we find

$$(\phi_n) + \eta = f_n + (I_n) + \tfrac{95}{288}\,wK_{n,n}\,\eta, \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

or

$$\eta = \frac{\epsilon}{1 - \frac{95}{288}\,wK_{n,n}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Hence we have the following theorem:—

Theorem I. *If the values of the solution of the integral equation*

$$\phi(x) = f(x) + \int_a^x K(x,\ \xi)\,\phi(\xi)\,d\xi,$$

where $K(x, \xi)$ and $f(x)$ may be given numerically, have been computed for $x = a,\ a+w,\ a+2w,\ \ldots\ a+(n-1)\,w$, and are $\phi_0,\ \phi_1,\ \phi_2,\ \ldots\ \phi_{n-1}$, then its value for $x = a+nw$, viz., ϕ_n, is given by

$$\phi_n = f_n + (I_n) + cwK_{n,n}\,\frac{f_n + (I_n) - (\phi_n)}{1 - cwK_{n,n}},$$

where $\quad f_n$ *denotes* $f(a+nw)$,

$K_{r,s}$ *denotes* $K(a+rw,\ a+sw)$,

(ϕ_n) *is any assumed value of* ϕ_n,

$$\begin{aligned}(I_n) = w\,[u_0 + u_1 + u_2 + \ldots + u_{n-1} + (u_n) - \tfrac{1}{2}\,\{(u_n) + u_0\} \\ - \tfrac{1}{12}\,\{(\Delta u_{n-1}) - \Delta u_0\} - \tfrac{1}{24}\{(\Delta^2 u_{n-2}) + \Delta^2 u_0\} \\ - \ldots \textit{ up to the term involving the } r^{\text{th}} \textit{ differences}],\end{aligned}$$

$$u_p = K_{n,p}\,\phi_p,\ (u_n) = K_{n,n}(\phi_n),\ (\Delta u_{n-1}) = (u_n) - u_{n-1},\ \ldots$$

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* It may be useful to note that if only the differences up to the 2nd, 3rd, 4th, ... order are included in the evaluation of (I_n), this coefficient becomes $\frac{3}{8}$, $\frac{251}{720}$, $\frac{95}{288}$, $\frac{19087}{60480}$, $\frac{5257}{17280}$, etc., respectively. See Bashforth and Adams, *loc. cit.*, p. 20. In this work are given tables which greatly facilitate the calculation of

$$\tfrac{19}{720}\,(\Delta^3 f_{r-3} - \Delta^3 f_0),\ \tfrac{3}{160}\,(\Delta^4 f_{r-4} + \Delta^4 f_0),\ \ldots\ \tfrac{8183}{1036800}\,(\Delta^8 f_{r-8} + \Delta^8 f_0).$$

and c is a numerical constant whose value is ·3299, ·3156 or ·3042, *according as the order r of the highest order difference used in the evaluation of* (I_n) *is* 4, 5 *or* 6.

It is easy to see that η need not be a small quantity in order that ϕ_n may be found correctly. In fact we may put $(\phi_n)=0$ and then find η, that is, ϕ_n, from (4). The procedure outlined above, however, saves a great deal of unnecessary labour, for with (ϕ_n) put equal to zero, the differences will all be large numbers, the 4th difference in this case being $-(\phi_{n-1}+\Delta\phi_{n-2}+\Delta^2\phi_{n-3}+\Delta^3\phi_{n-4})$. There is another point which should be mentioned here. We need not, if we prefer it, form a table of differences of ϕ to find an approximate value of ϕ_n. For, for the numerical evaluation of the integral $\int_a^{a+nw} K(a+nw,\xi)\,\phi(\xi)\,d\xi$, we shall have to form a table of the differences of the integrand, which will enable us to find an approximate value of $K_{n,n}\,\phi_n$. The table of differences of ϕ, however, serves as a useful check against accidental errors being made in the work, and, moreover, such a table is useful for interpolating intermediate values of ϕ.

Equation (3) shows that the true value of (ϕ_n), viz. $(\phi_n)+\eta$, differs from $f_n+(I_n)$ merely by $\frac{95}{288}wK_{n,n}\eta$, *i.e.* an error in the assumed value of ϕ_n gives rise to a much smaller error in the value of ϕ_n calculated from the integral equation (1) by using this assumed value for the evaluation of the integral occurring in it. In some cases $\frac{95}{288}wK_{n,n}\eta$ may be negligible and then we shall have simply

$$\phi_n=f_n+(I_n),$$

but in many cases at least $\frac{1}{288}wK_{n,n}\epsilon$ and $w^2K_{n,n}^2\epsilon$ will be negligible, and then we can write

$$\phi_n=f_n+(I_n)+\tfrac{1}{3}wK_{n,n}\epsilon. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

It is interesting to notice that η produces no error in the calculated value of ϕ_n if $K_{n,n}$ is zero. If, on the other hand, $K_{n,n}$ is very large, we must take w sufficiently small to secure that the divisor $1-\frac{95}{288}wK_{n,n}$ shall not unduly magnify the error of the omitted decimals in (I_n).

3. *Initial values of the solution.*

It remains now to see how a few initial values of ϕ are to be calculated in order to start the solution. Obviously $\phi_0 = f_0$. To find the other values of ϕ, we choose a value of w, say w_1, where w_1 is so small that w_1^3 is negligible. Suppose that

$$\phi(a+w_1) = f(a+w_1) + \eta_1 .$$

Then, applying the Trapezoidal rule for the evaluation of integrals to (1), we obtain a linear equation to find η_1, which gives

$$\eta_1 = \frac{\tfrac{1}{2}w_1\{K(a+w_1,\, a)f(a) + K(a+w_1,\, a+w_1)f(a+w_1)\}}{1 - \tfrac{1}{2}w_1 K(a+w_1,\, a+w_1)}. \quad \ldots\ldots\ldots(6)$$

Having thus calculated $\phi(a+w_1)$, we assume that

$$\phi(a+2w_1) = \phi(a+w_1) + \{\phi(a+w_1) - \phi(a)\} + \eta_2 .$$

Applying this time Simpson's rule for evaluating the integral in (1), we again get a linear equation which gives η_2 and thus $\phi(a+2w_1)$ is found. If now $(2w_1)^5$ is negligible, we next calculate $\phi(a+4w_1)$ instead of $\phi(a+3w_1)$, Simpson's rule being again applied. If however $(2w_1)^5$ is not negligible, we must proceed more slowly; $\phi(a+3w_1)$ must also be calculated, and this time we may employ the Three-Eighths rule, or formula (2).* In this way we calculate $\phi(a+rw_1)$ for increasing values of r, always using the longest practicable interval between the successive ordinates to be summed and the best method of approximating to the integral as far as the materials in hand permit. In this way we shall soon have a sufficient number of known values of ϕ to employ the method given in Art. 2.

4. *An illustrative example.*

As an example of the method, let us solve the integral equation

$$\phi(x) = \frac{1}{1+x} - \frac{2}{2+x}\log_e(1+x) + \int_0^x \frac{1}{1+x-\xi}\,\phi(\xi)\,d\xi, \quad \ldots\ldots\ldots\ldots(7)$$

whose analytical solution can be seen to be $\phi(x) = \dfrac{1}{1+x}$.

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* It must be borne in mind that the generalised Simpson's rule is less exact than formula (2) when we have calculated $\phi(a)$, $\phi(a+w)$, $\phi(a+2w)$ and $\phi(a+3w)$.

We notice that the successive derivatives of the nucleus rapidly increase, and the equation, therefore, is not one which will show the method of the present paper at its best.

Supposing that we wish to retain seven places of decimals, we choose w_1 to be ·005. Application of (6) gives us the correct value of $\phi(\cdot 005)$. By four successive applications of Simpson's rule we find $\phi(\cdot 01)$, $\phi(\cdot 02)$, $\phi(\cdot 04)$ and $\phi(\cdot 08)$. By applying the Three-Eighths rule we find $\phi(\cdot 12)$. With the help of (2), we now find $\phi(\cdot 16)$, $\phi(\cdot 20)$, $\phi(\cdot 24)$, ... $\phi(\cdot 40)$. Since we have now a sufficient number of values of ϕ to obtain differences up to the fifth order when the interval w is ·08, and since actual computation shows that the fourth and fifth differences are fairly small (omission of the fifth differences is found to affect the value of $\phi(\cdot 48)$ by less than half a unit in the eighth decimal place), we increase w to ·08 and thus calculate $\phi(\cdot 48)$ by using the values of $\phi(\xi)$ for $\xi = 0$, ·08, ·16, ·24, ·32 and ·40 only.

To illustrate the process of computation, suppose that values of $\phi(\xi)$ for $\xi = 0$, ·08, ·16, ... 2·00, have already been computed and that we want to compute $\phi(2\cdot 08)$. We form a table of the function $K(2\cdot 08, \xi)\,\phi(\xi)$ for $\xi = 0$, ·08, ·16, ... 2·00, and form the successive differences for six values of this function at the beginning and six or seven at the end. The latter part of the table is reproduced below.

ξ	$K(2\cdot 08, \xi)\phi(\xi)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
...	...					
1·52	·2543753					
		55000				
1·60	·2598753		11492			
		66492		1611		
1·68	·2665245		13103		469	
		79595		2080		163
1·76	·2744840		15183		632	
		94778		2712		217
1·84	·2839618		17895		849	
		112673		3561		(300)
1·92	·2952291		21456		(1149)	
		134129		(4710)		
2·00	·3086420		(26166)			
		(160295)				
2·08	(·3246715)					

From the fact that Δ^5 is increasing we put down 300 as an approximate value of the next fifth difference. This gives 1149 as the value of the next Δ^4 and so on, leading to ·3246715 as an approximate value of $K(2{\cdot}08, 2{\cdot}08)\phi(2{\cdot}08)$. Since $K(2{\cdot}08, 2{\cdot}08) = 1$, this is also the approximate value of $\phi(2{\cdot}08)$. By an application of the formula (2) we now find (I) to be ·5514360, and since $f(2{\cdot}08) = -{\cdot}2267608$, we find ϵ to be 37. Then (5) gives the value of $\phi(2{\cdot}08)$ as ·3246753, which is correct to the last place.

5. *Comparison with other methods.*

The heaviest part of the work consists in the various multiplications in order to tabulate $K(x, \xi)\,\phi(\xi)$, x remaining fixed for one integration, but varying from one integration to another. The work can be much shortened by using a larger value of the interval w. Thus in the above example we could have used $w = {\cdot}16$ instead of ·08 as we did, only in this case it would have been necessary to retain differences up to the ninth order. However, with a machine like the "Millionaire" for performing the multiplications, and an adding and listing machine (like a typewriter with the adding mechanism attached or a Burroughs adding and listing machine) to print the results of the multiplications and to add them automatically, combined with Bashforth's table giving the values of $\frac{19}{720}(\Delta^3 f_{r-3} - \Delta^3 f_0)$, ..., computations are quickly made.

The method given above appears to be a tedious and slow one, but this is partly due to the fact that we are working with seven decimal figures. If we desire, say, only two-place accuracy, we can find values of $\phi(\xi)$ from $\xi = 0$ to, say, $\xi = 6$ very quickly. Thus with $w = {\cdot}25$ and an application of the Trapezoidal rule, we find $\phi({\cdot}25)$, then two successive applications of Simpson's rule give $\phi({\cdot}5)$ and $\phi(1{\cdot}0)$ and ten more steps give the remaining values of ϕ, at intervals of half a unit, all correct to 2 decimal places.

This compares very favourably indeed with all the other methods. For, in order to use a method which requires an approximate representation of the nucleus by a polynomial, the first thing to do is to find this polynomial, but to represent the nucleus $\dfrac{1}{1+x}$ for $0 \leqq x \leqq 6$ correctly even only to 2 decimals, we shall have to use a polynomial of something like the sixth degree, a cubic being seen to give an error of 8 units in the second decimal place for some values of x. This will necessitate the solution of an algebraic equation of the sixth degree with coefficients involving two or three figures each and finally either a dozen numerical integrations or the representation of $\frac{1}{1+x} - \frac{2}{2+x} \log_e(1+x)$, supposed to be given numerically, by some approximate expression and the tabulation of the function, consisting of at least six terms, obtained as the result of the integration. Similarly, in view of the work required to represent a function, given numerically, by exponentials, a method using such a representation will be equally laborious. The solution in terms of iterated functions will be still more troublesome to compute. For seven-place accuracy these methods will naturally be far more tedious.

6. *Solution by a power-series.*

The method of solving integral equations indicated by the following theorem will sometimes be found useful.

Theorem II. *The solution of the integral equation*

$$\phi(x) = f(x) + \int_0^x K(x-\xi)\,\phi(\xi)\,d\xi, \qquad \dots\dots\dots\dots\dots\dots(8)$$

where the nucleus $K(x)$ and the function $f(x)$ are both supposed to be expansible in the Taylor's series *

$$K(x) = K_0 + K_1 x + K_2 \frac{x^2}{2\,!} + K_3 \frac{x^3}{3\,!} + \dots$$

and
$$f(x) = f_0 + f_1 x + f_2 \frac{x^2}{2\,!} + f_3 \frac{x^3}{3\,!} + \dots, \qquad \dots\dots\dots\dots(9)$$

is given by

$$\phi(x) = \phi_0 + \phi_1 x + \phi_2 \frac{x^2}{2\,!} + \phi_3 \frac{x^3}{3\,!} + \dots, \qquad \dots\dots\dots\dots(10)$$

* Of course $f_1, f_2, \dots, \phi_1, \phi_2, \dots$ do not now have the same meaning as in Art. 2.

where ϕ_0, ϕ_1, ϕ_2, ... *are to be calculated successively from the equations*

$$\left.\begin{aligned}
&\phi_0 = f_0,\\
&\phi_1 = f_1 + K_0\,\phi_0,\\
&\phi_2 = f_2 + K_0\,\phi_1 + K_1\,\phi_0,\\
&\quad\ldots\qquad\ldots\qquad\ldots\\
&\phi_r = f_r + K_0\,\phi_{r-1} + K_1\phi_{r-2} + \ldots + K_{r-1}\,\phi_0.
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(11)$$

Equations (11) are obtained by differentiating (8) repeatedly and then putting $x=0$.

It is easy to see that, under the supposition made above that $f(x)$ is expansible in a Taylor's series, this result is equivalent to the following theorem given by Professor Whittaker * :—

The solution of the integral equation

$$\phi(x) + \int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x),$$

where the nucleus $\kappa(x)$ is supposed to be expansible in a Taylor's series

$$\kappa(x) = \kappa_0 + \kappa_1\,x + \kappa_2\frac{x^2}{2\,!} + \kappa_3\,\frac{x^3}{3\,!} + \ldots,$$

is
$$\phi(x) = f(x) - \int_0^x K(x-s)\,f(s)\,ds, \quad\ldots\ldots\ldots\ldots\ldots(12)$$

where
$$K(x) = \kappa_0 - \begin{vmatrix}\kappa_0 & 1\\ \kappa_1 & \kappa_0\end{vmatrix} x + \begin{vmatrix}\kappa_0 & 1 & 0\\ \kappa_1 & \kappa_0 & 1\\ \kappa_2 & \kappa_1 & \kappa_0\end{vmatrix}\frac{x^2}{2\,!}$$
$$- \begin{vmatrix}\kappa_0 & 1 & 0 & 0\\ \kappa_1 & \kappa_0 & 1 & 0\\ \kappa_2 & \kappa_1 & \kappa_0 & 1\\ \kappa_3 & \kappa_2 & \kappa_1 & \kappa_0\end{vmatrix}\frac{x_3}{3\,!} + \ldots(13)$$

For, solving equations (11), we get

$$\phi_r = f_r + a_0 f_{r-1} + a_1 f_{r-2} + \ldots + a_{r-1} f_0, \quad\ldots\ldots\ldots\ldots(14)$$

where, for brevity, a_r has been written for

$$\begin{vmatrix}K_0 & -1 & 0 & \ldots & 0\\ K_1 & K_0 & -1 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ K_r & K_{r-1} & K_{r-2} & \ldots & K_0\end{vmatrix}$$

* *Loc. cit.*

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Substituting the values of $K(x-s)$ and $f(s)$ from (13) and (9) in (12), multiplying out and integrating term by term, and remembering that

$$\int_0^x (x-\xi)^p \xi^q \, d\xi = \frac{p! \, q!}{(p+q+1)!} x^{p+q+1},$$

we find, after making the necessary changes of notation,

$$\phi(x) = f(x) + a_0 f_0 x + (a_0 f_1 + a_1 f_0) \frac{x^2}{2!} + \dots$$
$$+ (a_0 f_{r-1} + a_1 f_{r-2} + \dots + a_{r-1} f_0) \frac{x^r}{r!} + \dots,$$

which is the same as (10) by virtue of (14).

The solution in the form given in Theorem II. will generally be found more convenient, because no integration has to be performed and because it does not necessitate the evaluation of determinants.

The application of Theorem II. to equation (7) gives, when $0 \leqq x < 1$,

$$\phi(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$$

We can generalise Theorem II. as follows:—

Theorem III. *The solution of the integral equation*

$$\phi(x) = f(x) + \int_0^x K(x, \xi) \, \phi(\xi) \, d\xi,$$

where the nucleus $K(x, \xi)$ is supposed to be expansible in a Taylor's series

$$K(x, \xi) = K_{00} + (K_{10} x + K_{01} \xi)$$
$$+ \frac{1}{2!} (K_{20} x^2 + 2K_{11} x\xi + K_{02} \xi^2)$$
$$+ \frac{1}{3!} (K_{30} x^3 + 3K_{21} x^2 \xi + 3K_{12} x \xi^2 + K_{03} \xi^3) + \dots,$$

and $f(x)$ is supposed to be expansible in a Taylor's series,

$$f(x) = f_0 + f_1 x + f_2 \frac{x^2}{2!} + f_3 \frac{x^3}{3!} + \dots,$$

is given by

$$\phi(x) = \phi_0 + \phi_1 x + \phi_2 \frac{x^2}{2!} + \phi_3 \frac{x^3}{3!} + \dots,$$

where $\phi_0 = f_0$,

$$\phi_1 = f_1 + K_{00}\,\phi_0,$$

$$\phi_2 = f_2 + (2K_{10} + K_{01})\,\phi_0 + K_{00}\,\phi_1,$$

$$\phi_3 = f_3 + (3K_{20} + 3K_{11} + K_{02})\,\phi_0 + (3K_{10} + 2K_{01})\,\phi_1 + K_{00}\,\phi_2,$$

$$\cdots \qquad \cdots \qquad \cdots$$

and generally

$$\phi_r = f_r + \Sigma \frac{r\,!}{(r-p-q-1)\,!\,p\,!\,q\,!}\,\frac{1}{(p+q+1)!}\,K_{r-p-q-1,\,p}\,\phi_q,$$

in which the summation Σ *is to be extended over all positive integral values of* p *and* q, *including* 0, *such that*

$$r - p - q - 1 \geqq 0.$$

The proof of this theorem is similar to that of Theorem II.

7. *Non-linear integral equations and equations with an infinite nucleus.*

The method of Art. 2 is applicable, not only to the linear integral equation (1), but also to the more general integral equation

$$\phi(x) = F\left(x,\ \int_{f(x)}^{g(x)} K\{x,\ \xi,\ \phi(\xi)\}\,d\xi\right),$$

where $\phi(x)$ is the unknown function, whose values are to be determined for values of $x \geqq a$, and $F(x, y)$, $f(x)$, $g(x)$ and $K(x, y, z)$ are known functions, such that $a \leqq f(x) \leqq x$, $a \leqq g(x) \leqq x$, provided that certain conditions regarding the continuity of the functions F, K, f and g and certain of their differential coefficients are satisfied, and we are justified in assuming that a continuous solution exists. The procedure will be obvious from the following theorem regarding a simpler type of the non-linear integral equation, for which it is known that a solution exists under suitable conditions.*

* See Vergerio, *Annali di Matematica*, XXXI., 1922, pp. 81–119.

Theorem IV. *If the values of the solution of the integral equation*

$$\phi(x) = f(x) + \int_a^x K(x, \xi) F\{\phi(\xi)\} d\xi$$

have been computed for $x = a,\ a + w,\ \dots\ a + (n-1)w$, *and are* $\phi_0, \phi_1, \phi_2, \dots \phi_{n-1}$, *then its value for* $x = a + nw$, *viz.* ϕ_n, *is given by*

$$\phi_n = f_n + (I_n) + cwkK_{n,n}\ \frac{f_n + (I_n) - (\phi_n)}{1 - cwkK_{n,n}},$$

where the symbols have the same meaning as in Theorem I., except that now

(ϕ_n) *is a value of* ϕ_n, *extrapolated from a table of differences of* ϕ, *so near to* ϕ_n *that squares and higher powers of* $\phi_n - (\phi_n)$ *may be neglected,*

$u_p = K_{n,p} F\{\phi_p\}$, $(u_n) = K_{n,n} F\{(\phi_n)\}$,

and $k =$ *the value of* $\dfrac{dF(x)}{dx}$ *at* $x = (\phi_n)$.

The proof is similar to that of Theorem I. The initial values may be calculated almost exactly as before, the necessary changes being obvious, or they may be derived from the power series obtained by the method of Art. 6.

An interesting application can be made to the solution of the integral equation with an infinite nucleus of the type

$$\phi(x) = f(x) + \int_0^x \frac{G(x, \xi)}{(x-\xi)^{p/q}} \phi(\xi)\, d\xi \quad (0 < p < q), \quad \dots\dots\dots(15)$$

where $G(x, \xi)$ satisfies the same conditions as $K(x, \xi)$ did in equation (1) and p and q are integers. Making a change of variables, (15) can be written as

$$\phi(x) = f(x) + q\int_0^{x^{1/q}} G(x, x - \xi^q)\phi(x - \xi^q)\, \xi^{q-p-1}\, d\xi,$$

which can be solved numerically like equation (1).

8. *An analogue of the formula of Kutta.*

The methods investigated by Runge and others for the solution of differential equations give corresponding methods for the solution of integral equations, but these methods are not so good as the method of Art. 2. A single example will suffice. Kutta's symmetrical formula, correct to the 4th order in w, gives the following result.

Theorem V. *If the values of the solution of the integral equation* (1) *have been computed for $x = a$, $a + w$, $a + 2w$, ... $a + (n-1)w$, then its value for $x = a + nw$ may be computed from*

$$\phi_n = f_n + \sum_{r=0}^{n-1} \frac{\Delta'_r + 3\Delta''_r + 3\Delta'''_r + \Delta''''_r}{8}$$

or $$\phi_n = f_n + \int_a^{a+(n-1)w} K(a+nw, \xi)\, \phi(\xi)\, d\xi + \frac{\Delta'_{n-1} + 3\Delta''_{n-1} + 3\Delta'''_{n-1} + \Delta''''_{n-1}}{8},$$

where $$\Delta'_r = K_{n,r}\, \phi_r\, w,$$

$$\Delta''_r = K_{n,r+1/3} \left\{ \phi_r + \frac{\Delta'_r}{3} \right\} w,$$

$$\Delta'''_r = K_{n,r+2/3} \left\{ \phi_r + \Delta''_r - \frac{\Delta_r}{3} \right\} w,$$

$$\Delta''''_r = K_{n,r+1} \left\{ \phi_r + \Delta'''_r - \Delta''_r + \Delta'_r \right\} w,$$

and the other symbols have the same meaning as in Theorem I.

The second form of the formula is more convenient than the first, but it cannot be satisfactorily employed for small values of n.

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