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DERIVATIONS WITH INVERTIBLE VALUES ON A LIE IDEAL

BY

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ABSTRACT. Let R be a ring which possesses a unit element, a Lie ideal $U \not\subset Z$, and a derivation d such that $d(U) \neq 0$ and $d(u)$ is 0 or invertible, for all $u \in U$. We prove that R must be either a division ring D or D_2, the 2×2 matrices over a division ring unless d is not inner, R is not semiprime, and either $2R$ or $3R$ is 0. We also examine for which division rings D, D_2 can possess such a derivation and study when this derivation must be inner.

In a recent paper [1], Bergen, Herstein and Lanski have related the structure of a ring R to the special behavior of one of its derivations. More precisely, they proved that if R is a ring with unit and $d \neq 0$ is a derivation of R such that for every $x \in R$, $d(x) = 0$ or $d(x)$ is invertible in R, then except for a special case which occurs when $2R = 0$, R must be a division ring D or the ring D_2 of 2×2 matrices over a division ring.

Here we shall examine what happens when R is a ring with unit, U is a non-central Lie ideal of R, and d is a derivation of R such that for every $u \in U$, $d(u) = 0$ or $d(u)$ is invertible in R. The results we will obtain have a similar flavor to those of [1]. In fact we shall prove the following:

THEOREM 1. *Let R be a ring with 1, $U \not\subset Z$ a Lie ideal of R, and d a derivation of R such that $d(U) \neq 0$ and $d(u) = 0$ or $d(u)$ is invertible, for every $u \in U$. Then R is either*

1. *a division ring D, or*
2. *D_2,*

unless $2R$ or $3R$ is zero, d is not inner, and R is not semiprime. In this case, $R = M + d(M)$, where M is the unique maximal ideal of R and $M^3 = 0$.

We then examine, for the case $R = D_2$, when d is inner and for which division rings D such a derivation exists. The result we obtain is

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THEOREM 2. *Suppose* $R = D_2$, *then*:

1. *if* D *is not commutative and* $2R \neq 0$, *every derivation of* R *such that* $d(u) = 0$ *or* $d(u)$ *is invertible, for all* u *in a non-central Lie ideal, must be inner.*

2. *there exists an inner derivation* d *such that* $d(U) \neq 0$ *and* $d(u) = 0$ *or* $d(u)$ *is invertible, for all* u *contained in a non-central Lie ideal* U, *if and only if* D *does not contain all quadratic extensions of* Z *or* D *is a field of characteristic* 2.

For $a, b \in R$ set $[a, b] = ab - ba$ and for subsets $U, V \subset R$ let $[U, V]$ be the additive subgroup generated by all $[u, v]$ for $u \in U$ and $v \in V$. We recall that a Lie ideal U of R is an additive subgroup of R such that $[U, R] \subset U$.

In all that follows, unless otherwise stated, R will be a ring with 1, $Z = Z(R)$ the center R, $U \not\subset Z$ a Lie ideal of R and d a derivation of R such that $d(U) \neq 0$ and $d(u) = 0$ or $d(u)$ is invertible, for all $u \in U$.

We begin with

LEMMA 1. $d([U, R]) \neq 0$.

PROOF. Suppose $d([U, R]) = 0$ and let $u \in U, r \in R$; then $0 = d([u, ur]) = d(u[u, r]) = d(u)[u, r]$. Therefore, $d(u) = 0$ or $[u, R] = 0$ thus, for all $u \in U$, either $d(u) = 0$ or $u \in Z$. It now follows that $d(U) = 0$ or $U \subset Z$, a contradiction.

We now show that R is d-simple, that is, has no non-zero, proper ideals invariant under d.

LEMMA 2. *If* $I \neq 0$ *is an ideal of* R *such that* $d(I) \subset I$, then $I = R$.

PROOF. Suppose $d([U, I]) \neq 0$; then $0 \neq d([U, I]) \subset d(U) \cap I$, therefore I contains invertible elements and so, $I = R$.

On the other hand, if $d([U, I]) = 0$ then for $u \in U$ and $i \in I$, we have $0 = d([u, ui]) = d(u[u, i]) = d(u)[u, i]$. As in the proof of Lemma 1, either $d(U) = 0$ or $[U, I] = 0$, thus $[U, I] = 0$. Hence $0 = [U, IR] = I[U, R]$.

By Lemma 1, there exist $u \in U$ and $r \in R$ such that $d([u, r]) \neq 0$. If $i \in I$ then $0 = d(i[u, r]) = id([u, r]) + d(i)[u, r]$. However, since $d(i) \in I$, we obtain $Id([u, r]) = 0$, a contradiction.

We proceed with

LEMMA 3. *If* $I \neq R$ *is an ideal of* R, *then* $I^3 = 0$.

PROOF. Since $d([U, I^2]) \subset d(U) \cap I$ and $I \neq R$, it follows that $d([U, I^2]) = 0$. Using the identical argument as in the proof of Lemma 2, $I^2[U, R] = 0$ and $0 = d(i[u, r]) = d(i)[u, r] + id([u, r])$, for $i \in I^2$, $u \in U$, and $r \in R$. However, if $i \in I^3$ then $d(i) \in I^2$, hence $I^3 d([U, R]) = 0$ and, by Lemma 1, $I^3 = 0$.

We continue with

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LEMMA 4. *If* $2R \neq 0$ *and* $3R \neq 0$ *then* R *is simple.*

PROOF. If R is not simple, let M be the sum of all proper ideals of R. By Lemma 3, every proper ideal of R has cube zero, hence M is nil and so, $M^3 = 0$. Obviously M is the unique maximal ideal of R and, by Lemma 2, $d(M) \not\subset M$.

Since $M + d(M)$ is an ideal of R properly containing M, $M + d(M) = R$. Consequently there exist $a, b \in M$ such that $1 = a + d(b)$. Now, $0 = d^3(b^3) = d^3(b^2b) = d^3(b^2)b + 3d^2(b^2)d(b) + 3d(b^2)d^2(b) + b^2d^3(b)$, hence $3d^2(b^2)d(b) \in M$. Since $d^2(b^2) = d^2(b)b + 2d(b)^2 + bd^2(b)$, we obtain $6d(b)^3 \in M$. If $2R \neq 0$ and $3R \neq 0$ then, by Lemma 2, $2R = 3R = R$, hence $6R = R$. However, $d(b) = 1 - a$ is invertible therefore $6 \in M$ and so, $M = R$, a contradiction. As a result, R is simple.

Combining Lemmas 2, 3, and 4 we immediately obtain

LEMMA 5. *If either* d *is inner,* R *is semiprime, or both* $2R$ *and* $3R$ *are nonzero then* R *is simple. In addition, if* R *is not simple then* $R = M + d(M)$ *where* M *is the unique maximal ideal of* R *and* $M^3 = 0$.

At this point, the proof of Theorem 1 reduces to showing that when R is simple either $R = D$ or D_2. By Theorem 1.5 of [3], if R is simple then either $U \supset [R, R]$ or R is of characteristic 2 and of dimension at most 4 over its center. In the latter case, there is nothing left to prove. However, in the first case it is relatively easy to see that $d([R, R]) \neq 0$ and $[R, R] \not\subset Z$. Therefore, throughout Lemmas 6, 7, 8, 9 we will assume that R is simple, $U = [R, R]$, and R is not of characteristic 2 with dimension $\leqq 4$ over its center.

LEMMA 6. *If* $0 \neq a \in R$ *is such that* $d(a) = 0$, *then* a *is invertible.*

PROOF. Suppose that $[a, d(R)] \neq 0$; then let $x \in R$ such that $[a, d(x)] \neq 0$. Since $d(a) = 0$, we obtain $d([a, x]) = [a, d(x)]$ and $d([a, ax]) = d(a[a, x]) = a[a, d(x)]$. Moreover $d([a, x]), d([a, ax]) \in d([R, R])$, therefore $[a, d(x)]$ is invertible, hence $a[a, d(x)]$ is non-zero and so, $a[a, d(x)]$ is also invertible, finally resulting in a invertible.

Now suppose that $[a, d(R)] = 0$; then, by Theorem 1 of [5], $d^2 = 0$, $a^2 \in Z$, char $R = 2$, and d is an inner derivation induced by a central multiple of a. Furthermore, by Theorem 2 of [4], if $[d(R), d(R)] = 0$ then R has dimension $\leqq 4$, over its center. Therefore, without loss of generality, we may assume that $d^2 = 0$, $d(r) = [a, r]$ for all $r \in R$, and there exist $s, t \in R$ such that $[d(s), d(t)] \neq 0$. Consider $d([s, d(t)]) = [d(s), d(t)]$ and $d([s, ad(t)]) = [d(s), ad(t)] = a[d(s), d(t)]$. Since $[d(s), d(t)]$, $a[d(s), d(t)] \in d([R, R])$ we conclude, as in the previous paragraph, that a is invertible.

We continue with

LEMMA 7. *If $L \neq 0$ is a right ideal of R, then $R = L + d(L)$.*

PROOF. Since R is simple either R is a field or $[L, L] \neq 0$. If R is not a field, let $0 \neq x \in [L, L]$; by the previous lemma, we get that either x or $d(x)$ must be invertible. This implies that $L + d(L)$ is a right ideal which contains invertible elements, hence $L + d(L) = R$.

Since R is simple with 1, it is primitive. Therefore R has a faithful, irreducible right module V and R acts densely on V, viewing V as a vector space over the division ring D where D is the commuting ring of R on V.

We now prove the technical, but very useful

LEMMA 8. *Let V be a faithful, irreducible, right R-module. If $0 \neq v \in V$ and $0 \neq a \in R$ are such that $va = 0$, then $vd(a) \neq 0$.*

PROOF. By Lemma 7 we get $aR + d(aR) = R$. Therefore, $V = vR = v(aR + ad(R) + d(a)R) = vd(a)R$, thus $vd(a) \neq 0$.

We now narrow in on the structure of R.

LEMMA 9. *$R = D$ or $R = D_2$.*

PROOF. It suffices to show that $\dim_D V = 1$ or 2. Suppose $\dim_D V \geqq 3$; then there exist linearly independent $v_1, v_2, v_3 \in V$ and an $r \in R$ such that $v_1 r = 0$, $v_2 r = 0$, and $v_3 r = v_3$. Let $T = \{r \in R | v_1 r = v_2 r = 0\}$; since $r \neq 0 \in T$, T is a non-zero right ideal of R, hence, by Lemma 7, $R = T + d(T)$.

Now, let $x, y \in R$ such that $v_1 x = v_1$, $v_2 x = v_2$, $v_1 y = 0$, and $v_2 y = v_2$. In addition, since $R = T + d(T)$, let $a, b \in T$ such that $x = a + d(b)$. As a result, $v_1 = v_1 x = v_1(a + d(b)) = v_1 d(b)$ and $v_2 = v_2 x = v_2(a + d(b)) = v_2 d(b)$. Hence $v_1 d(by) = v_1(bd(y) + d(b)y) = v_1 d(b)y = v_1 y = 0$ which, by Lemma 8, implies $by = 0$. However, in this case $0 = v_2 d(by) = v_2(bd(y) + d(b)y) = v_2 d(b)y = v_2 y = v_2$, a contradiction, thereby proving the lemma.

By combining Lemmas 5 and 9 we obtain our first main result, which we mentioned at the outset of this paper.

THEOREM 1. *Let R be a ring with 1, $U \not\subset Z$ a Lie ideal of R, and d a derivation of R such that $d(U) \neq 0$ and $d(u) = 0$ or $d(u)$ is invertible, for every $u \in U$. Then R is either*

1. *a division ring D, or*
2. *D_2*

unless $2R$ or $3R$ is zero, d is not inner, and R is not semiprime. In this case, $R = M + d(M)$, where M is the unique maximal ideal of R and $M^3 = 0$.

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In [1] it is shown that the only example of a ring $R \neq D$ or D_2 with a derivation $d \neq 0$ such that $d(x) = 0$ or is invertible, for all $x \in R$, is $D[x]/(x^2)$ where char $D = 2$, $d(D) = 0$, and $d(x) = 1 + ax$, for some a in the center of D. Therefore, with the hypothesis of Theorem 1, when $2R = 0$ there exists an example where $R \neq D$ or D_2. However, when $3R = 0$ we have neither been able to either prove that $R = D$ or D_2 nor been able to produce a counterexample. On the other hand, it does follow from Theorem 1 that if $R \neq D$ or D_2, with $3R = 0$, then R and d are rather special.

We now try to characterize those division rings D for which $R = D_2$ has a derivation $d \neq 0$ all of whose values are zero or invertible on a non-central Lie ideal. In addition, we shall examine when such a d must be inner. To do this, we will refer to several calculations which were done in Lemma 8 of [1] and will be omitted here for brevity.

LEMMA 10. *If $R = D_2$, where $2R \neq 0$ and D is non-commutative, then d is inner.*

PROOF. If d is a derivation of D_2, then d has the form:

$$d\begin{pmatrix} a & b \\ c & e \end{pmatrix} = \begin{pmatrix} f(a) - b\beta - \alpha e & f(b) + a\alpha + b\gamma - \alpha e \\ f(c) + \beta a - e\beta - \gamma c & f(e) + e\gamma - \gamma e + \beta b + c\alpha \end{pmatrix}$$

for all $a, b, c, e \in D$; where $\alpha, \beta, \gamma \in D$ and f is a derivation of D. Furthermore, it is shown in Lemma 7 of [1] that d is inner on D_2 if and only if f is inner on D. Therefore it will be enough to show that f is inner.

Let

$$T = \left\{ a \in D \,\middle|\, \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in [R, R] \right\};$$

since D is non-commutative, T is a non-central subset of D invariant under all automorphisms of D. By a result of Brauer-Cartan-Hua [2], the subdivision ring $\bar{T}$ of D generated by T is all of D. As noted in the discussion before Lemma 5, we may assume that $U \supset [R, R]$.

Suppose $\alpha = 0$; if $a \in T$ then

$$d\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} f(a) & 0 \\ \beta a & 0 \end{pmatrix}$$

is zero or invertible. Therefore $f(a) = 0$, hence $0 = f(T) = f(\bar{T}) = f(D)$, implying that f is inner. As a result, we may now assume that $\alpha \neq 0$. It now follows from the calculations in Lemma 8 of [1], that there is a $\tau \in D$ such that $f(a) = \tau a - a\tau$, for all $a \in D$ satisfying

$$\begin{pmatrix} a & 0 \\ \alpha^{-1} f(a) & \alpha^{-1} a\alpha \end{pmatrix} \in [R, R].$$

However, if $a \in T$ then $\alpha a \alpha^{-1} \in T$, therefore

$$\begin{pmatrix} a & 0 \\ \alpha^{-1} f(a) & \alpha^{-1} a \alpha \end{pmatrix}$$

$$= \begin{pmatrix} a + \alpha^{-1} a \alpha & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha^{-1} f(a) & 0 \end{pmatrix} + \begin{pmatrix} -\alpha^{-1} a \alpha & 0 \\ 0 & \alpha^{-1} a \alpha \end{pmatrix}$$

$$= \begin{pmatrix} a + \alpha^{-1} a \alpha & 0 \\ 0 & 0 \end{pmatrix} + \left[\begin{pmatrix} 0 & 0 \\ \alpha^{-1} f(a) & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] + \left[\begin{pmatrix} 0 & 0 \\ \alpha^{-1} a \alpha & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right]$$

$$\in [R, R].$$

Thus $f(a) = [\tau, a]$ for all $a \in T$, hence $f(a) = [\tau, a]$ for al $a \in \bar{T} = D$, thereby proving that f is inner on D.

At this point, we should note that the assumption $2R \neq 0$ in Lemma 10 cannot be dropped, as an example is given in [1] of a division ring D of characteristic 2 such that $R = D_2$ has a derivation $d \neq 0$ all of whose values on R are 0 or invertible, yet d is not inner. We have not, however, been able to determine whether the assumption in Lemma 10, that D be non-commutative, is necessary.

We will now characterize those D such that $R = D_2$ possesses an inner derivation d such that $d(U) \neq 0$ and $d(u) = 0$ or is invertible, for all u in a Lie ideal $U \not\subset Z$. The condition "D does not contain all quadratic extensions of Z" will come up. By this we mean that there exist γ, δ in the center of D such that the polynomial $t^2 + \gamma t + \delta$ has no root in D. Note that the following lemma places no restriction on either the characteristic or the non-commutativity of D.

LEMMA 11. *$R = D_2$ has an inner derivation d such that $d(U) \neq 0$ and $d(u)$ is 0 or invertible, for all u in a Lie ideal $U \not\subset Z$, if and only if D does not contain all quadratic extensions of Z or D is a field of characteristic* 2.

PROOF. It is shown in Lemma 9 of [1] that if D does not contain all quadratic extensions of Z, then there exists an inner derivation $d \neq 0$ such that $d(x) = 0$ or is invertible for all $x \in R$. In addition, if D is a field of characteristic 2, then

$$U = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \middle| a, b \in D \right\}$$

is a non-central Lie ideal of R and it is easy to see that the inner derivation d induced by $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has the properties that $d(U) \neq 0$ and $d(U) \subset Z(R)$.

Conversely, suppose that D is not a field of characteristic 2 and that $d \neq 0$ is inner such that $d(U) \neq 0$ and $d(u) = 0$ or is invertible, for all u in a Lie ideal $U \not\subset Z$. Therefore, by Lemma 6 and the discussion preceeding it, we may assume that $U = [R, R]$ and that every element in the kernel of d is 0 or invertible. Using essentially the same argument as in Lemma 9 of [1], we may also assume that d is induced by an element of the form $\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$.

We claim that α and β lie in the center of D. Clearly when D is commutative there is nothing to prove. If D is non-commutative, let T be as in Lemma 10; then $\bar{T}$, the subdivision ring of D generated by T, is all of D. Suppose $a \in T$; then

$$d\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \alpha a - a\alpha & \beta a - a\beta \end{pmatrix}$$

is a non-invertible element of $d([R, R])$, hence is zero. Therefore $\alpha a = a\alpha$ and $\beta a = a\beta$ for all $a \in T$, hence also for all $a \in \bar{T} = D$, thereby proving the claim. Furthermore, since $d\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix} = 0$, $\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$ must be invertible thus $\alpha \neq 0$.

Suppose $\beta = 0$; if $x \in D$ then

$$d\begin{pmatrix} x & 1 \\ \alpha & x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}\begin{pmatrix} x & 1 \\ \alpha & x \end{pmatrix} - \begin{pmatrix} x & 1 \\ \alpha & x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $\begin{pmatrix} x & 1 \\ \alpha & x \end{pmatrix}$ is not zero, it must be invertible, hence its determinant $x^2 - \alpha \neq 0$. As a result the quadratic polynomial $t^2 - \alpha$ has coefficients in Z, but no roots in D, thus D does not contain all quadratic extensions of Z. Finally, suppose $\beta \neq 0$ and for $x \in D$, consider

$$d\begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}\begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \alpha & \beta \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \alpha x & -\beta x \\ \beta & \alpha x - 1 \end{pmatrix} \in d([R, R]).$$

Since $\begin{pmatrix} 1-\alpha x & -\beta x \\ \beta & \alpha x - 1 \end{pmatrix}$ is not zero, it also is invertible, hence its determinant $-(\alpha x - 1)^2 + \beta^2 x = -\alpha^2 x^2 + (2\alpha + \beta^2)x - 1 \neq 0$. Therefore the polynomial

$$t^2 - \frac{1}{\alpha^2}(2\alpha + \beta^2)t + \frac{1}{\alpha^2}$$

has no roots in D, thereby concluding the proof.

We now conclude this paper by combining Lemmas 10 and 11 to obtain

THEOREM 2. *Suppose* $R = D_2$; *then*:

1. *if D is not commutative and* $2R \neq 0$, *every derivation d such that* $d(u) = 0$ *is invertible, for all u in a non-central Lie ideal, must be inner.*

2. *there exists an inner derivation d such that* $d(U) \neq 0$ *and* $d(u) = 0$ *or $d(u)$ is invertible, for all u contained in a non-central Lie ideal U, if and only if D does not contain all quadratic extensions of Z or D is a field of characteristic* 2.

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