J. Austral. Math. Soc. Ser. B 30(1989), 313–325

EXACT SOLUTIONS OF NONLINEAR EVOLUTION EQUATIONS OF THE AKNS CLASS

W. L. CHAN[1] AND YU-KUN ZHENG[1]

(Received 1 August 1987; revised 29 April 1988)

Abstract

The problem of obtaining explicit and exact solutions of soliton equations of the AKNS class is considered. The technique developed relies on the construction of the wave functions which are solutions of the associated AKNS system; that is, a linear eigenvalue problem in the form of a system of first order partial differential equations. The method of characteristics is used and Bäcklund transformations are employed to generate new solutions from the old. Thus, families of new solutions for the KdV equation, the mKdV equation, the sine-Gordon equation and the nonlinear Schrödinger equation are obtained, avoiding the solution of some Riccati equations. Our results in the KdV case include those obtained recently by other investigators.

1. Introduction

The Bäcklund transformation (BT) is an important tool for constructing a new solution of a nonlinear evolution equation (NEE) from a known solution of that equation [5]. Earlier, Konno and Wadati [3] had derived some BTs for the NEEs of AKNS class [1]. These BTs explicitly express the new solutions in terms of the known solutions of the NEEs and the corresponding wave functions which are solutions of the associated AKNS system. Therefore the key step for obtaining new solutions by the BT is to obtain the wave functions. In this paper, we shall use some simple methods to find the wave functions and apply the BTs derived by Konno and Wadati to obtain families of new solutions.

The main objective of this article is not to derive BTs but rather to implement them in the construction of exact solutions. Indeed, our main contribution is the

[1]Department of Mathematics, The Chinese University of Hong Kong, Science Centre, Shantin, N. T., Hong Kong.

complete integration of the AKNS linear system when the field variable $u(x,t)$ is of the form $u(x-kt)$, i.e., a travelling wave.

The second generation of solutions, obtained from the seed solution u being a constant, is not all new. They include 1-soliton solutions. However, they are also travelling waves, so that, by our method, the wave functions associated with them can be found. Hence through the known BTs, a new third generation of solutions has been obtained. One distinct feature here is that the value of the parameter η is kept constant from generation to generation. This is unlike the case in which solutions are generated by the algebraic method [5] (theorem of permutability or nonlinear superposition) where the value of η must be kept changing.

It is known that many NEEs can be derived from the following AKNS system

$$\Phi_x = P\Phi, \qquad \Phi_t = Q\Phi, \tag{1}$$

where

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{2}$$

P and Q are two 2×2 null-trace matrices

$$P = \begin{pmatrix} \eta & q \\ r & -\eta \end{pmatrix}, \tag{3}$$

$$Q = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}. \tag{4}$$

Here η is a parameter, independent of x and t, q and r are functions of x and t. P and Q must satisfy the following integrability condition:

$$P_t - Q_x + PQ - QP = 0, \tag{5}$$

or in component form:

$$-A_x + qC - rB = 0, \tag{6}$$

$$q_t - B_x + 2\eta B - 2qA = 0, \tag{7}$$

$$r_t - C_x + 2rA - 2\eta C = 0. \tag{8}$$

By suitably choosing r, A, B and C in (6)–(8), we will obtain various NEEs which q must satisfy. Konno and Wadati introduced the function [3]

$$\Gamma = \varphi_1/\varphi_2, \tag{9}$$

and, for each of the NEE, derived a BT with the following form:

$$q' = q + f(\Gamma, \eta), \tag{10}$$

where q' is a new solution of the corresponding NEE. For use in the sequel, we list the NEEs and their corresponding BT in the following.

a) The Korteweg-de Vries (KdV) equation.

$$P = \begin{pmatrix} \eta & q \\ -1 & -\eta \end{pmatrix}, \tag{11}$$

$$Q = \begin{pmatrix} -4\eta^3 - 2\eta q - q_x & -4\eta^2 q - 2q^2 - 2\eta q_x - q_{xx} \\ 4\eta^2 + 2q & 4\eta^3 + 2\eta q + q_x \end{pmatrix}, \tag{12}$$

$$q_t + q_{xxx} + 6qq_x = 0, \tag{13}$$

$$q' = q - 2\Gamma_x. \tag{14}$$

b) The modified Korteweg-de Vries (mKdV) equation.

$$P = \begin{pmatrix} \eta & q \\ -q & -\eta \end{pmatrix}, \tag{15}$$

$$Q = \begin{pmatrix} -4\eta^3 - 2\eta q^2 & -4\eta^2 q - 2q^3 - 2\eta q_x - q_{xx} \\ 4\eta^2 q + 2q^3 - 2\eta q_x + q_{xx} & 4\eta^3 + 2\eta q^2 \end{pmatrix}, \tag{16}$$

$$q_t + q_{xxx} + 6q^2 q_x = 0, \tag{17}$$

$$q' = q - (2\Gamma_x)/(1 + \Gamma^2) = q - 2(\tan^{-1}\Gamma)_x. \tag{18}$$

c) The sine-Gordon (SG) equation.

$$P = \begin{pmatrix} \eta & -u_x/2 \\ u_x/2 & -\eta \end{pmatrix}, \tag{19}$$

$$Q = \frac{1}{4\eta} \begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix}, \tag{20}$$

$$u_{xt} = \sin u, \tag{21}$$

$$u' = u + 4\cot^{-1}\Gamma. \tag{22}$$

d) The nonlinear Schrödinger (NS) equation.

$$P = \begin{pmatrix} \eta & q \\ -q^* & -\eta \end{pmatrix}, \qquad (\eta\text{-real}), \tag{23}$$

$$Q = \begin{pmatrix} 2i\eta^2 + iqq^* & 2i\eta q + iq_x \\ -2i\eta q^* + iq_x^* & -2i\eta^2 - iqq^* \end{pmatrix}, \tag{24}$$

$$iq_t + q_{xx} + 2q^2 q^* = 0, \tag{25}$$

$$q' = -q - (4\eta\Gamma)/(1 + |\Gamma|^2). \tag{26}$$

Now we shall choose some known solutions of the above NEEs and substitute these solutions into the corresponding matrices P and Q. Next, we solve the equations (1) for φ_1 and φ_2. Then, by (9) and the corresponding BT we will obtain the new solutions of the NEEs.

2. The known solution is a constant q

a) The KdV equation.

Substitute q_0 into the matrices P and Q in (11) and (12), then by (1) we have

$$d\Phi = \Phi_x\, dx + \Phi_t\, dt = P\Phi\, d\rho, \tag{27}$$

where

$$P = \begin{pmatrix} \eta & q \\ -1 & -\eta \end{pmatrix}, \tag{28}$$

$$\rho = x - kt, \qquad k = 2q_0 + 4\eta^2. \tag{29}$$

The solution of equation (27) is

$$\Phi = (\exp \rho P)\Phi_0 = (I + \rho P + \rho^2 P^2/2! + \rho^3 P^3/3! + \cdots)\Phi_0, \tag{30}$$

where Φ_0 is a constant column vector. According to the sign of the quantity $\eta^2 - q_0$, the solution (30) may take the following three forms:

i) $\eta^2 - q_0 > 0$, $\alpha^2 = \eta^2 - q_0$

$$\Phi = \begin{pmatrix} \cosh \alpha\rho + (\eta/\alpha) \sinh \alpha\rho & (q_0/\alpha) \sinh \alpha\rho \\ -(1/\alpha) \sinh \alpha\rho & \cosh \alpha\rho - (\eta/\alpha) \sinh \alpha\rho \end{pmatrix} \Phi_0; \tag{31}$$

ii) $\eta^2 - q_0 < 0$, $\alpha^2 = q_0 - \eta^2$

$$\Phi = \begin{pmatrix} \cos \alpha\rho + (\eta/\alpha) \sin \alpha\rho & (q_0/\alpha) \sin \alpha\rho \\ -(1/\alpha) \sin \alpha\rho & \cos \alpha\rho - (\eta/\alpha) \sin \alpha\rho \end{pmatrix} \Phi_0; \tag{32}$$

iii) $\eta^2 - q_0 = 0$,

$$\Phi = \begin{pmatrix} 1 + \eta\rho & q_0\rho \\ -\rho & 1 - \eta\rho \end{pmatrix} \Phi_0. \tag{33}$$

Now, we choose $\Phi_0 = (1, 0)^{\mathsf{T}}$ in (31)–(33) and use (9) and the BT (14); we obtain the new solutions of the KdV equation (13) corresponding to the known constant KdV solution q_0 as follows:

i) $\eta^2 - q_0 > 0$,

$$q = q_0 - 2\alpha^2 \operatorname{csch}^2(\alpha\rho), \qquad \alpha^2 = \eta^2 - q_0; \tag{34}$$

ii) $\eta^2 - q_0 < 0$,

$$q = q_0 - 2\alpha^2 \csc^2(\alpha\rho), \qquad \alpha^2 = q_0 - \eta^2; \tag{35}$$

iii) $\eta^2 - q_0 = 0$,

$$q = q_0 - (2/\rho^2). \tag{36}$$

These solutions had been obtained in [2] by a different method involving the solution of some Riccati equations. If we choose $\Phi_0 = (0, 1)^{\mathsf{T}}$ in (31)–(33), we will obtain another group of solutions. Obviously all of these solutions are travelling waves with velocity $k = 4\eta^2 + 2q_0$.

b) The mKdV equation.

Substitute q_0 into the matrices P and Q in (15) and (16); then by (1) we have

$$d\Phi = \Phi_x\,dx + \Phi_t\,dt = P\Phi\,d\rho, \tag{37}$$

where

$$P = \begin{pmatrix} \eta & q_0 \\ -q_0 & -\eta \end{pmatrix}, \tag{38}$$

$$\rho = x - kt, \qquad k = 2(2\eta^2 + q_0^2). \tag{39}$$

The solution of equation (37) is

$$\Phi = (\exp\rho P)\Phi_0 = (I + \rho P + \rho^2 P^2/2! + \rho^3 P^3/3! + \cdots)\Phi_0. \tag{40}$$

Similar to the case of KdV equation, by (40), taking $\Phi_0 = (1,0)^{\mathsf{T}}$, and by (9) and (18), we obtain three new solutions of the mKdV equation corresponding to the known constant solution q_0 as follows:

i) $\eta^2 - q_0^2 > 0$, $\alpha^2 = \eta^2 - q_0^2$,

$$\Phi = \begin{pmatrix} \cosh\alpha\rho + (\eta/\alpha)\sinh\alpha\rho & (q_0/\alpha)\sinh\alpha\rho \\ -(q_0/\alpha)\sinh\alpha\rho & \cosh\alpha\rho - (\eta/\alpha)\sinh\alpha\rho \end{pmatrix}\Phi_0, \tag{41}$$

$$q = q_0 + 2\{\tan^{-1}(1/q_0)[\alpha\coth\alpha\rho + \eta]\}_x; \tag{42}$$

ii) $\eta^2 - q_0^2 < 0$, $\alpha^2 = q_0^2 - \eta^2$,

$$\Phi = \begin{pmatrix} \cos\alpha\rho + (\eta/\alpha)\sin\alpha\rho & (q_0/\alpha)\sin\alpha\rho \\ -(q_0/\alpha)\sin\alpha\rho & \cos\alpha\rho - (\eta/\alpha)\sin\alpha\rho \end{pmatrix}\Phi_0, \tag{43}$$

$$q = q_0 + 2\{\tan^{-1}(1/q_0)[\alpha\cot\alpha\rho + \eta]\}_x; \tag{44}$$

iii) $\eta^2 - q_0^2 = 0$, $\rho = x - 6q_0^2 t$,

$$\Phi = \begin{pmatrix} 1 - q_0\rho & q_0\rho \\ -q_0\rho & 1 - q_0\rho \end{pmatrix}\Phi_0, \tag{45}$$

$$q = q_0 + 2[\tan^{-1}((1/q_0\rho) + 1)]_x. \tag{46}$$

c) The SG equation.

Let the constant solution of (21) be

$$q_n = n\pi, \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{47}$$

Substitute (47) into the matrices P and Q in (19) and (20), then by (1) we have

$$d\Phi = \Phi_x\,dx + \Phi_t\,dt = P\Phi\,d\rho_n, \tag{48}$$

where

$$P = \begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix}, \tag{49}$$

$$\rho_n = x - (-1)^{n-1}t/(4\eta^2). \tag{50}$$

The solution of (48) is

$$\Phi_n = \begin{pmatrix} \cosh \eta\rho_n + \sinh \eta\rho_n & 0 \\ 0 & \cosh \eta\rho_n - \sinh \eta\rho_n \end{pmatrix} \Phi_0. \tag{51}$$

Choose $\Phi_0 = (1, 1)^{\mathsf{T}}$ in (51), then we have

$$\Phi_n = \begin{pmatrix} \exp(\eta\rho_n) \\ \exp(-\eta\rho_n) \end{pmatrix}. \tag{52}$$

Substitute (52) into (9), then by (22), we obtain the new solutions of the SG equation (21)

$$q_n = n\pi + 4\cot^{-1}(\exp 2\eta\rho_n), \qquad n = 0, 1, 2, \cdots. \tag{53}$$

d) The NS equation.

Note that $q_0 = 0$ is the only constant solution of the NS equation. Substitute it into the matrices P and Q in (23) and (24), then by (1) we have

$$d\Phi = \Phi_x\, dx + \Phi_t\, dt = P\Phi\, d\rho, \tag{54}$$

where

$$P = \begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix}, \tag{55}$$

$$\rho = x + 2i\eta t. \tag{56}$$

The solution of (54) is

$$\Phi = \begin{pmatrix} \exp(\eta\rho) & 0 \\ 0 & \exp(-\eta\rho) \end{pmatrix} \Phi_0. \tag{57}$$

Choosing $\Phi_0 = (1, 1)^{\mathsf{T}}$ in (57), then, by (9) and (26), we obtain the new solution of the NS equation which reads:

$$q = -2\eta \exp(4i\eta^2 t)\,\mathrm{sech}(2\eta x). \tag{58}$$

3. The known solution $q = q(x, t)$ is a simple function

In this case we cannot solve the system (1)–(4) for the vector Φ as a whole, but we can solve its components φ_1 and φ_2 separately. From (1)–(4), after inserting the known solution $q(x, t)$ of the NEE into the corresponding matrices P and Q, we will have the following system of partial differential equations for the unknowns φ_1 and φ_2:

$$\varphi_{1x} = \eta\varphi_1 + q\varphi_2, \tag{59}$$

$$\varphi_{2x} = r\varphi_1 - \eta\varphi_2, \tag{60}$$

$$\varphi_{1t} = A\varphi_1 + B\varphi_2, \tag{61}$$

$$\varphi_{2t} = C\varphi_1 - A\varphi_2. \tag{62}$$

These equations are compatible under the conditions of the assumed values of matrices P and Q connected with the considered NEEs. Solve φ_1 from (60) giving

$$\varphi_1 = (1/r)(\varphi_{2x} + \eta\varphi_2). \tag{63}$$

Substituting this φ_1 into (62) and together with (8) we get

$$C\varphi_{2x} - r\varphi_{2t} = (1/2)(C_x - r_t)\varphi_2. \tag{64}$$

This is a linear first order partial differential equation with φ_2 as its unknown function; it can be solved by the method of characteristics. After φ_2 has been obtained from (64), and substituting it into (63), we will obtain φ_1. Thus we have obtained two general solutions φ_1 and φ_2 which contain an arbitrary function f. This arbitrary function can be determined by demanding that the two solutions φ_1 and φ_2 satisfy either (59) or (61), which will yield a second order linear ordinary differential equation with the function f as its unknown. If we can solve for the function f, we will eventually obtain the two particular solutions φ_1 and φ_2. Finally, by applying (9) and the BT corresponding to the NEE we shall obtain a new solution of the NEE.

Example 1. The KdV equation.

Let

$$q = x/(6t). \tag{65}$$

By direct calculation one can check that (65) is a solution of the KdV equation (13). Inserting (65) into (64), together with (11) and (12), gives

$$\left(4\eta^2 + \frac{x}{3t}\right)\varphi_{2x} + \varphi_{2t} = \frac{1}{6t}\varphi_2. \tag{66}$$

Equation (66) has the following system of ordinary differential equations as its characteristic equations,

$$dx/dt = (x/3t) + 4\eta^2, \tag{67}$$

$$d\varphi_2/dt = \varphi_2/(6t). \tag{68}$$

Solving these two equations gives the general solution of the unknown φ_2 in equation (66), which reads

$$\varphi_2 = t^{1/6}f(\xi), \qquad \xi = xt^{-1/3} - 6\eta^2 t^{2/3}, \tag{69}$$

where f is an arbitrary differentiable function. Substituting (65) and (69) into (63) gives the general solution of φ_1 which reads

$$\varphi_1 = -t^{-1/6}\,dt/d\xi - \eta t^{1/6} f(\xi). \tag{70}$$

To determine the function $f(\xi)$, we substitute (65), (69) and (70) into (59), and find that $f(\xi)$ must satisfy the following Airy equation [4]:

$$d^2f/d\xi^2 + (\xi/6)f = 0. \tag{71}$$

Therefore we obtain the function $f(\xi)$ as follows:

$$f = C_1 A + C_2 B, \tag{72}$$

where A and B are two Airy functions,

$$A = A(\xi), \tag{73}$$

$$B = B(\xi), \tag{74}$$

and C_1 and C_2 are two arbitrary constants.

After f has been determined, (69), (70) and (9) lead to

$$\Gamma = -t^{-1/3} \frac{d}{d\xi}(\ln f) - \eta; \tag{75}$$

then substituting this Γ and (65) into the BT (14), we arrive at the new solution q' of the KdV equation (13) corresponding to the known solution (65):

$$q' = \frac{x}{6t} + 2t^{-2/3} \frac{d^2}{d\xi^2}(\ln f). \tag{76}$$

Example 2. The NS equation.

We take (58) as the known solution of the NS equation. Referring to the matrices (23) and (24), the PDE (64) now reads

$$(iq_x^* - 2i\eta q^*)\varphi_{2x} + q^*\varphi_{2t} = -(i\eta q_x^* + i|q|^2 q^*)\varphi_2. \tag{77}$$

Substituting (58) into (77), after simplifying we have

$$2\eta(1 + \tanh 2\eta x)\varphi_{2x} + i\varphi_{2t} = 2\eta^2(2\operatorname{sech}^2 2\eta x - \tanh 2\eta x)\varphi_2. \tag{78}$$

Solving (78) we obtain the general solution for φ_2:

$$\varphi_2 = \exp(\eta x + (1/8)\varsigma - 2i\eta^2 t)\operatorname{sech}(2\eta x) f(\varsigma), \tag{79}$$

where f is an arbitrary differentiable function of ς and

$$\varsigma = 4\eta x + \sinh 4\eta x - \cosh 4\eta x + 16i\eta^2 t. \tag{80}$$

Substituting (79) into (63) we obtain the general solution for φ_1:

$$\varphi_1 = \exp(\eta x + (1/8)\varsigma + 2i\eta^2 t)[(1 - \tanh 2\eta x + \rho) f + 16\rho f'_\varsigma], \tag{81}$$

where

$$\rho = 1/(16\eta)\varsigma_x. \tag{82}$$

To determine the function $f(\varsigma)$, (79) and (81) are substituted into (61) and using (24) we arrive at a second order ordinary differential equation

$$64 f''_{\varsigma\varsigma} + 16 f'_\varsigma + f = 0. \tag{83}$$

Solving this equation gives

$$f(\varsigma) = (c_1\varsigma + c_2)\exp(-(1/8)\varsigma), \tag{84}$$

where C_1 and C_2 are two constants. Now, by (9), (79) and (81) we have

$$\Gamma = \exp(4i\eta^2 t)[1 - \tanh 2\eta x + \rho + 16\rho(\ln f)_\varsigma] \cosh 2\eta x. \tag{85}$$

Substituting (84) into (85) we get

$$\Gamma = \exp(4i\eta^2 t)(1 - \tanh 2\eta x - \rho + (16\rho C_1)/(C_1\varsigma + C_2)) \cosh 2\eta x. \tag{86}$$

Finally substituting (58) and (86) into (26), we obtain the new solution of the NS equation (25)

$$\begin{aligned} q' = {} & 2\eta \exp(4i\eta^2 t) \operatorname{sech} 2\eta x \\ & \cdot \{1 - 2(1 - \tanh 2\eta x - \rho + (16C_1\rho)/(C_1\varsigma + C_2)) \\ & /[\operatorname{sech}^2 2\eta x + (1 - \tanh 2\eta x - \rho + (16C_1\rho)/(C_1\varsigma + C_2)) \\ & (1 - \tanh 2\eta x - \rho + (16C_1^*\rho)/(C_1^*\varsigma^* + C_2^*)]\}. \end{aligned} \tag{87}$$

4. The known solution is a travelling wave

$$q = q(\rho), \ \rho = x - kt, \ (k \text{ a constant}). \tag{88}$$

Such a solution does exist for many NEEs as we have seen in Section 2. In this case the AKNS system (1)–(4) has a general solution. Let us consider the more general case. Suppose that the components q and r of the matrix P are functions of ρ:

$$q = q(\rho), \qquad r = r(\rho); \tag{89}$$

then the components A, B and C of the matrix Q determined by equations (6)–(8) are also functions of ρ:

$$A = A(\rho), \qquad B = B(\rho), \qquad C = C(\rho). \tag{90}$$

Under these assumptions, we have the following result, which is crucial in our subsequent exact solution.

PROPOSITION 1. *The matrix*

$$M = (kP + Q)^2 \tag{91}$$

and the quantity

$$\beta = (A + k\eta)^2 + (B + kq)(C + kr) \tag{92}$$

are constants with respect to ρ (or x and t).

PROOF. Substituting the matrices (4) and (5) into (91) we have

$$\begin{aligned} M &= \left[k\begin{pmatrix} \eta & q \\ r-\eta & \end{pmatrix} + \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\right]^2 = \begin{pmatrix} A+k\eta & B+kq \\ C+kr & -A-k\eta \end{pmatrix}^2 \\ &= [(A+k\eta)^2 + (B+kq)(C+kr)]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \beta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{93}$$

To prove the assertions of the proposition we only need to show that the derivative of the matrix M in (91) with respect to ρ is zero; this can be done by the following direct calculation.

Using (88)–(90) and (5) we get

$$\begin{aligned} dM/d\rho &= (k\,dP/d\rho + dQ/d\rho)(kP+Q) + (kP+Q)(k\,dP/d\rho + dQ/d\rho) \\ &= (-P_t + Q_x)(kP+Q) + (kP+Q)(-P_t+Q_x) \\ &= (PQ-QP)(kP+Q) + (kP+Q)(PQ-QP) \\ &= kPQP + PQ^2 - kQP^2 - QPQ + kP^2Q - kPQP + QPQ - Q^2P \\ &= P(A^2+BC) - kQ(\eta^2+qr) + k(\eta^2+qr)Q - (A^2+BC)P \\ &= 0. \end{aligned} \tag{94}$$

We now solve the system (59)–(62) by applying the method of characteristics as in Section 3. The PDE (64) possesses the following characteristic equations:

$$\frac{dt}{-r} = \frac{dx}{C} = \frac{d\varphi_2}{\frac{1}{2}(C_x - r_t)\varphi_2}. \tag{95}$$

Using (88)–(90), we have

$$C_x - r_t = \frac{dC}{d\rho} + k\,\frac{dr}{d\rho} = \frac{d}{d\rho}(C+kr). \tag{96}$$

Substituting (96) into (95) gives

$$\frac{dt}{-r} = \frac{d\rho}{C+kr} = \frac{d\varphi_2}{\frac{1}{2}(C+kr)'_\rho\varphi_2}. \tag{97}$$

These equations yield the following system of ordinary differential equation:

$$d(\ln\varphi_2)/d\rho = (C+kr)'_\rho/(2(C+kr)), \tag{98}$$

$$d\rho/dt = -(C+kr)/r. \tag{99}$$

Integrating equation (98) leads to

$$\varphi_2 = k_2(C+kr)^{1/2}, \tag{100}$$

where k_2 is an integration constant. Integrating (99) we get

$$-t + k_1 = \int \frac{r}{C+kr}\,d\rho, \tag{101}$$

where k_1 is another integration constant. Denote

$$\sigma(\rho) = \int \frac{r}{C + kr}\, d\rho. \tag{102}$$

Substituting (102) into (101), we have

$$\sigma(\rho) + t = k_1. \tag{103}$$

From (100) and (103), we obtain the general solution of the equation (64):

$$\varphi_2 = (C + kr)^{1/2} f(\varsigma), \tag{104}$$

where

$$\varsigma = \sigma(\rho) + t, \tag{105}$$

and $f(\varsigma)$ is a differentiable function of ς. Substituting (104) into (63) gives the general solution of φ_1:

$$\varphi_1 = (C + kr)^{-1/2}[f_\varsigma + (A + k\eta)f]. \tag{106}$$

To determine the function f, (104) and (106) are substituted into (59) and we find that f must satisfy the following second order ordinary differential equation

$$f''_{\varsigma\varsigma} - \beta f = 0, \tag{107}$$

where β is a constant defined in (92). According to the sign of β, (107) will have the following three different solutions:

$$f = c_1\varsigma + c_2, \qquad \text{when } \beta = 0, \tag{108}$$

$$f = c_1 \sinh\omega(\varsigma + c_2), \qquad \text{when } \beta > 0,\ \omega^2 = \beta, \tag{109}$$

$$f = c_1 \sin\omega(\varsigma + c_2), \qquad \text{when } \beta < 0,\ \omega^2 = -\beta, \tag{110}$$

where c_1 and c_2 are integration constants. Substituting these solutions into (106) and (104) respectively, we obtain the corresponding different solutions of the system (1)–(4):

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}(C + kr)^{-1/2}[(A + k\eta)(c_1\varsigma + c_2) + c_1]\\ (C + kr)^{1/2}(c_1\varsigma + c_2)\end{pmatrix}, \qquad \text{when } \beta = 0, \tag{111}$$

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}c_1(C + kr)^{-1/2}[(A + k\eta)\sinh\omega(\varsigma + c_2) + \omega\cosh\omega(\varsigma + c_2)]\\ c_1(C + kr)^{1/2}\sinh\omega(\varsigma + c_2)\end{pmatrix}, \qquad \text{when } \beta > 0, \tag{112}$$

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}c_1(C + kr)^{-1/2}[(A + k\eta)\sin\omega(\varsigma + c_2) + \omega\cos\omega(\varsigma + c_2)]\\ c_1(C + kr)^{1/2}\sin\omega(\varsigma + c_2)\end{pmatrix} \qquad \text{when } \beta < 0. \tag{113}$$

These results (111)–(113) are valid for any NEE contained in the AKNS system (1)–(4), provided that they meet the assumptions (89) and (90).

We now apply the results obtained here and the known travelling wave solutions of the NEEs obtained in Section 2 to construct some new solutions of the corresponding NEEs by means of the BTs. We will only consider the KdV equation and the SG equation. In these cases, the constant β defined by (92) is zero and therefore the corresponding solution of the AKNS system (1)–(4) is (111). By substituting (111) into (9) we get the common expression of Γ of these NEEs.

$$\Gamma = (C + kr)^{-1}(A + k\eta + 1/(\varsigma + c_0)), \qquad c_0 = C_1/C_2. \tag{114}$$

In the following, we omit some tedious calculations but only list the main results of the KdV equation and SG equation.

(a) The KdV equation.

(i)

$$q = q_0 - 2\alpha^2 \operatorname{csch}^2(\alpha\rho), \tag{115}$$

$$\varsigma = (\sinh 2\alpha\rho - 2\alpha\rho + 16\alpha^3 t)/(16\alpha^3), \tag{116}$$

$$\Gamma = -\alpha \coth \alpha\rho + 4\alpha \sinh^2(\alpha\rho)/[\sinh 2\alpha\rho - 2\alpha\rho + 16\alpha^3(t + c_0)], \tag{117}$$

$$q' = q_0 - 8\alpha^2 \left\{[2\alpha\rho - 16\alpha^3(t + c_0)] \sinh 2\alpha\rho + 2(1 - \cosh \alpha\rho)\right\} \\ /[\sinh 2\alpha\rho - 2\alpha\rho + 16\alpha^3(t + c_0)]^2. \tag{118}$$

(ii)

$$q = q_0 - 2\alpha^2 \csc^2(\alpha\rho), \tag{119}$$

$$\varsigma = (2\alpha\rho - \sin 2\alpha\rho + 16\alpha^3 t)/(16\alpha^3), \tag{120}$$

$$\Gamma = -\alpha \cot \alpha\rho + 4\alpha \sin(\alpha\rho)/[2\alpha\rho - \sin 2\alpha\rho + 16\alpha^3(t + c_0)], \tag{121}$$

$$q' = q_0 + 8\alpha^2 \left\{[2\alpha\rho + 16\alpha^3(t + c_0)] \sin 2\alpha\rho - 2(1 - \cos 2\alpha\rho)\right\} \\ /[2\alpha\rho - \sin 2\alpha\rho + 16\alpha^3(t + c_0)]^2. \tag{122}$$

(iii)

$$q = q_0 - 2/\rho^2, \tag{123}$$

$$\varsigma = \rho^3/12 + t, \tag{124}$$

$$\Gamma = -1/\rho + 3\rho^2/[\rho^3 + 12(t + c_0)], \tag{125}$$

$$q' = q_0 - 6\rho[\rho^3 - 24(t + c_0)]/[\rho^3 + 12(t + c_0)]^2. \tag{126}$$

(b) The SG equation.

$$q_n = n\pi + 4\cot^{-1}[\exp(2\eta\rho_n)], \qquad \rho_n = x - k_n t, \\ k_n = (-1)^{n+1}/4\eta^2,\ n = 0, \pm 1, \pm 2, \cdots, \tag{127}$$

$$\varsigma_n = t + \frac{1}{2}\eta(-1)^{n+s1}(4\eta\rho_n + \sinh 4\eta\rho_n - \cosh 4\eta\rho_n), \tag{128}$$

$$\Gamma_n = [(-1)^n \eta(1 + \cosh 2\eta\rho_n)]/(\varsigma + c_n) - 1, \tag{129}$$

$$q'_n = n\pi + 4\cot^{-1}[\exp(2\eta\rho_n)] + 4\cot^{-1}\Gamma_n. \tag{130}$$

5. Concluding remarks

In this paper, we have considered the construction of exact solutions to the NEEs of the AKNS class. It has been shown that the implementation of certain Bäcklund transformations for a class of nonlinear partial differential equation requires the solution of the underlying linear differential equation whose coefficients depend on the initial known solution $q(x,t)$ of the nonlinear equation. We obtain the solution (wave function) of the underlying linear equations by the method of characteristics. This solution has usually been given only for specific input solutions $q_0(x,t)$, but here our method produces some new explicit solutions $q_1(x,t)$ from a wide class of input solution, including any travelling wave solution $q = q_0(x - kt)$. Employing Bäcklund transformations involving explicitly the wave function, new solutions are generated. Some of our results, when specialised to the case of KdV equation, include those obtained by Fung et al. [2]. Our approach here is new and enables us to construct new second and third generations of solutions of these NEEs. One other feature here is that the parameter η of the AKNS system is kept unchanged from generation to generation.

References

[1] M. J. Ablowitz and H. Segur, *Solitons and the inverse scattering transform*, SIAM, Philadelphia, 1981).

[2] C. Au and P. C. W. Fung, "Vacuum states of the KdV equation," *Phys. Rev. B* **25** (1982), 6460.

[3] K. Konno and M. Wadati, "Simple derivation of Bäcklund transformation from Riccati form of inverse method," *Progr. Theoret. Phys.* **53** (1975), 1652–1656.

[4] F. W. Oliver, *Asymptotics and special functions*, (Academic Press, New York, 1974).

[5] C. Rogers and W. E. Shadwick, *Bäcklund transformation and their applications* (Academic Press, New York, 1982).