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# ON THE MODULII OF ANALYTIC FUNCTIONS

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The problem to be considered in this note, in its most concrete form, is the determination of all quartets $f_1, f_2, g_1, g_2$ of functions analytic on some domain and satisfying

$$(*) \qquad |f_1(z)|^p + |f_2(z)|^p = |g_1(z)|^p + |g_2(z)|^p,$$

where $p > 0$. When $p = 2$ the question can be reformulated in terms of finding a necessary and sufficient condition for (two-dimensional) Hilbert space valued analytic functions to have equal pointwise norms, and the answer (Theorem 1) justifies this point of view. If $p \neq 2$, the problem is solved by reducing to the case $p = 2$, and the reformulation in terms of the norm equality of $l^p$ valued analytic functions gives no clue to the answer.

The following theorem is essentially due to Nevanlinna and Polya [2] for finite-dimensional $H$, as pointed out to the author by David Drasin and Harley Flanders. It is of greatest interest in the formulation given; i.e., in terms of Hilbert space valued mappings, but does not appear to be widely known. The following proof was suggested by Henry Helson.

THEOREM 1. *Let $H$ be a Hilbert space and let $F(z)$, $G(z)$ be analytic in some region $\Omega$ with values in $H$ and such that*

$$\|F(z)\| = \|G(z)\| \quad \text{for all } z \in \Omega.$$

*Then there is an isometry* U *defined on the range of $F$ such that*

$$G(z) = \mathrm{U}\, F(z) \quad \text{for all } z \in \Omega.$$

**Proof.** Assume $0 \in \Omega$ and let $F(z)$, $G(z)$ have power series expansions

$$F(z) = \sum_{n=0}^{\infty} \phi_n z^n, \qquad G(z) = \sum_{n=0}^{\infty} \psi_n z^n,$$

where the coefficients $\phi_n, \psi_n \in H$. Then the hypothesis is that

$$\left\| \sum_{n=0}^{\infty} \phi_n r^n e^{ni\theta} \right\| = \left\| \sum_{n=0}^{\infty} \psi_n r^n e^{ni\theta} \right\|,$$

or

$$\sum_{n,m} (\phi_n, \phi_m) r^{n+m} e^{(n-m)i\theta} = \sum_{n,m} (\psi_n, \psi_m) r^{n+m} e^{(n-m)i\theta},$$

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where $(\cdot, \cdot)$ denotes the inner product of $H$. A series of the form $\sum_{n,m\geq 0} a_{n,m}\, r^{n+m}\, e^{(n-m)i\theta}$ uniquely determine its coefficients, since if

$$\sum_{n,m\geq 0} a_{n,m} r^{(n+m)}\, e^{(n-m)i\theta} = \sum_{p=-\infty}^{\infty} \sum_{m=0}^{\infty} a_{m+p,m} r^{2m+p}\, e^{pi\theta} = 0,$$

then $\sum_{m=0}^{\infty} a_{m+p,m} r^{2m+p} = 0$ because of the uniqueness of the coefficients in a Fourier series, and then for each $p$, $a_{m+p,m} = 0$ for all $m$ by the uniqueness of power series coefficients. Thus $(\phi_n, \phi_m) = (\psi_n, \psi_m)$ for all $n$, $m$ and this guarantees that the linear mapping defined (unambiguously) on the span of the $\phi$'s by $\mathrm{U}(\sum a_n\phi_n) = \sum a_n\psi_n$ is an isometry. The result, proved for a neighborhood of 0, then extends to the connected open set $\Omega$.

If $H$ is two-dimensional, and

$$F(z) = f_1(z)e_1 + f_2(z)e_2, \qquad G(z) = g_1(z)e_1 + g_2(z)e_2,$$

where $e_1$, $e_2$ is an orthonormal basis, then the condition $\|F(z)\| = \|G(z)\|$ reduces to $(*)$ for $p=2$. If we define the $p$ norm of $F(z)$ to be $\|F(z)\|_p = [|f_1(z)|^p + |f_2(z)|^p]^{1/p}$, then a sufficient condition for $\|F(z)\|_p = \|G(z)\|_p$ is, of course, that $F(z) = \mathrm{U}\, G(z)$, where U is an isometry in this metric. If this condition were necessary then the only $g_1$, $g_2$ satisfying $(*)$ for $p \neq 2$ would be obtained by multiplying $f_1$, $f_2$ by constants of modulus 1 and permuting them. (The scarcity of isometries of finite-dimensional Banach spaces with an $l^p$ metric has surely been observed before, though the author could find no reference. When $p=1$, the vertices $(0, \ldots, 0, e^{i\theta}, 0, \ldots, 0)$ are extreme points of the unit ball and are therefore sent into one another by an isometry. If $p \neq 1$, these points must also be sent into one another (except, of course, when $p=2$), but here their distinguishing property among points on the unit sphere is maximal or minimal curvature (depending on whether $p<2$ or $p>2$), in a sense which needs to be made precise. The maximum $\delta$ for a fixed $\epsilon$ in the definition of "localized uniform rotundity" given in [1, p. 113] is one way of doing this. The details are somewhat complicated.)

Corollary 2. *Let $F$, $G$ be analytic on $\Omega$ with values in a finite-dimensional Hilbert space $H$, and suppose for some $p>0$*

$$\|F(z)\|_p = \|G(z)\|_p \quad \textit{for all } z \in \Omega.$$

*Then there is an isometry* U *of $H$ such that*

$$G(z)^{p/2} = \mathrm{U}\, F(z)^{p/2} \quad \text{for } z \in \Omega$$

*for some* $(p/2)$th *power of $F$, $G$.*

The proof is immediate. We comment that finite dimensionality is needed to assure the existence of a disk in which none of the coordinate functions of $F$ or $G$ vanish. If $p=1$, the above condition reduces to

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$$g_1(z) = a^2 f_1(z) + b^2 f_2(z) + 2ab\sqrt{f_1(z) f_2(z)}$$
$$g_2(z) = c^2 f_1(z) + d^2 f_2(z) + 2cd\sqrt{f_1(z) f_2(z)}$$

when $H$ is two-dimensional, and where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unitary. Most of the interest of the corollary is in the existence and construction of such nontrivial quartets satisfying (*).

COROLLARY 3. *If $f$, $g$, $h$ are analytic in some region in which*

$$|f(z)|^p = |g(z)|^p + |h(z)|^p$$

*then $g$, $h$ are linearly dependent. In particular, the sum of the modulii of two independent analytic functions is* NOT *the modulus of an analytic function.*

The above can be easily generalized to larger sets of functions, but except when $p=2$ or the dimension of $H$ is 2, the hypothesis cannot be stated very gracefully in terms of linear dependence or independence of the original functions.

## REFERENCES

1. M. Day, *Normed linear spaces*, Academic Press, New York, 1962.
2. R. Nevanlinna and G. Polya, Jahresbericht Deutsche Mathematikes Vereinigung **43** (1934), 6–7.

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