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The Co-annihilating-ideal Graphs of Commutative Rings

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Abstract. Let R be a commutative ring with identity. The co-annihilating-ideal graph of R, denoted by $\mathcal{A}_R$, is a graph whose vertex set is the set of all non-zero proper ideals of R and two distinct vertices I and J are adjacent whenever $\operatorname{Ann}(I) \cap \operatorname{Ann}(J) = \{0\}$. In this paper we initiate the study of the co-annihilating ideal graph of a commutative ring and we investigate its properties.

1 Introduction

Throughout this paper, R denotes a commutative ring with identity and with non-zero proper ideals. If X is either an element or a subset of R, then the *annihilator* of X is defined as $\operatorname{Ann}(X) = \{r \in R \mid rX = 0\}$. A *regular* element of R is a non-zero element that is not a zero divisor. We denote by $Z(R)$ and $\operatorname{Max}(R)$, the sets of zero divisors and maximal ideals of R, respectively. The ring R is said to be *reduced* if it has no non-zero nilpotent element. The intersection of all maximal ideals of R is called its Jacobson radical and is denoted by $J(R)$.

Let G be a simple graph with the vertex set $V(G)$ and edge set $E(G)$. For every vertex $v \in V(G)$, $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$. The *degree* of a vertex v is defined as $d_G(v) = |N_G(v)|$. The minimum degree of G is denoted by $\delta(G)$. A *universal vertex* is a vertex that is adjacent to all other vertices of G. The *distance* $d_G(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest uv-path in G. The greatest distance between any pair of vertices u and v in G is the *diameter* of G and denoted by $\operatorname{diam}(G)$. The complete graph is a graph in which any two distinct vertices are adjacent. If a graph G contains one vertex adjacent to all other vertices and with no extra edge, then G is called a *star graph*. The *girth* of a graph G, denoted by $g(G)$, is the length of its shortest cycle. The girth of a graph with no cycle is defined ∞. A *clique* in a graph G is a set of pairwise adjacent vertices and the number of vertices in a maximum clique of G, denoted by $\omega(G)$, is called the clique number of G. Let $\chi(G)$ denote the chromatic number of the graph G, that is, the minimal number of colors needed to color the vertices of G so that no two adjacent vertices have the same color. Obviously, $\chi(G) \geq \omega(G)$. We write P_n for a path

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of order n, C_n for a cycle of order n, and K_n for a complete graph of order n. For terminology and notation not defined here, the reader is referred to [20].

Applying the properties of graphs in the study of algebraic structures, has become an interesting research topic in the past two decades, leading to many fascinating results and questions. There are many papers that assign graphs to algebraic structures. We recall three graphs used to represent rings.

The zero divisor graph $\Gamma(R)$ [8,9]: The vertex set of this graph is $Z(R) \setminus \{0\}$ and two distinct vertices v_1 and v_2 are adjacent if and only if $v_1v_2 = 0$.

The annihilating-ideal graph $\mathbb{AG}(R)$ [3–6,13]: The vertex set of this graph is the set of non-zero ideals, whose annihilators are non-zero, and two distinct ideals I and J are adjacent if and only if $IJ = (0)$.

The comaximal ideal graph $\widetilde{\mathcal{G}}(R)$ [15,19]: The vertices are the ideals of R and two vertices v_1 and v_2 are adjacent if and only if v_1 and v_2 are comaximal. We denote the subgraph of the comaximal ideal graph induced by the set of non-zero proper ideals by $\mathcal{G}(R)$.

Here we propose another graph whose vertex set is all non-zero proper ideals of R. The co-annihilating-ideal graph of R, denoted by $\mathcal{A}_R$, is a graph whose vertex set is the set of all non-zero proper ideals of R and two distinct vertices I and J are adjacent whenever $\operatorname{Ann}(I) \cap \operatorname{Ann}(J) = (0)$. Clearly, for a commutative ring R if I is a non-zero proper ideal of R and I is a universal vertex in $\mathcal{A}_R$, then $\operatorname{Ann}(I) = (0)$ or I is a minimal ideal. In this paper we initiate the study of the co-annihilating-ideal graph of a commutative ring.

Also we show that two graphs $\mathcal{A}_R$ and $\mathcal{G}(R)$ are the same for Artinian rings.

Observation 1.1 *If R is an Artinian ring, then for any non-zero proper ideal I of R,* $\operatorname{Ann}_R(I) \neq (0)$.

Proof By the structure theorem of Artinian rings [10, Theorem 8.7, p. 90], there exist Artinian local rings $R_1, \dots, R_n$ such that $R = R_1 \times \cdots \times R_n$. Then $I = I_1 \times \cdots \times I_n$, where I_i is an ideal of R_i for $1 \leq i \leq n$. Let $\mathfrak{m}_i$ be the maximal ideal of R_i for each i. Then $\mathfrak{m}_i = \operatorname{Ann}_{R_i}(x_i)$ for some $x_i \in R_i$. Hence $\operatorname{Ann}_{R_i}(\mathfrak{m}_i) \neq (0)$ for each i. Since I is a proper ideal, $I_i \subseteq \mathfrak{m}_i$ for some i and so $(0) \neq \operatorname{Ann}_{R_i}(\mathfrak{m}_i) \subseteq \operatorname{Ann}_{R_i}(I_i)$. Since $\operatorname{Ann}_R(I) = \operatorname{Ann}_{R_1}(I_1) \times \cdots \times \operatorname{Ann}_{R_n}(I_i)$, the result follows. ∎

The next result is an immediate consequence of Observation 1.1.

Corollary 1.2 *If R is an Artinian ring, then $\mathcal{A}_R = \mathcal{G}(R)$.*

Observation 1.3 *Let R be a commutative ring with non-zero identity. If R has a non-unit regular element x, then (x^n) for each positive integer n is adjacent to all vertices of $\mathcal{A}_R$. In particular, R has an infinite clique. Thus if R is an integral domain that is not a field, then $\chi(\mathcal{A}_R) = \omega(\mathcal{A}_R) = \infty$.*

2 Basic Properties of Co-annihilating-ideal Graphs

In this section, we classify all rings whose co-annihilating-ideal graphs are empty, complete, and connected.

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Theorem 2.1 *Let R be a commutative ring with non-zero identity. Then $\mathcal{A}_R$ is an empty graph if and only if $(R, \mathfrak{m})$ is a local ring and* $\operatorname{Ann}(\mathfrak{m}) \neq (0)$.

Proof Let $\mathcal{A}_R$ be an empty graph. Assume to the contrary that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are two distinct maximal ideals of R. Then $\mathfrak{m}_1 + \mathfrak{m}_2 = R$, implying that $\operatorname{Ann}(\mathfrak{m}_1) \cap \operatorname{Ann}(\mathfrak{m}_2) = (0)$. It follows that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are adjacent in $\mathcal{A}_R$, a contradiction. Hence R is a local ring. Now let $\mathfrak{m}$ be the unique maximal ideal of R. We show that $\operatorname{Ann}(\mathfrak{m}) \neq (0)$. Suppose to the contrary that $\operatorname{Ann}(\mathfrak{m}) = (0)$. Since $\mathcal{A}_R$ is an empty graph, we deduce that R has exactly three ideals (0), $\mathfrak{m}$, R. Since $\mathfrak{m}^2 \in \{(0), \mathfrak{m}, R\}$, we should have $\mathfrak{m}^2 = (0)$ or $\mathfrak{m}^2 = \mathfrak{m}$. It follows from $\operatorname{Ann}(\mathfrak{m}) = (0)$ that $\mathfrak{m}^2 = \mathfrak{m}$. Now applying Nakayama's Lemma, we obtain a contradiction.

Conversely, let $(R, \mathfrak{m})$ be a local ring and $\operatorname{Ann}(\mathfrak{m}) \neq \{0\}$. Then for every two distinct non-zero proper ideals I and J, $\operatorname{Ann}(\mathfrak{m}) \subseteq \operatorname{Ann}(I) \cap \operatorname{Ann}(J)$ and hence I and J are not adjacent. Thus $\mathcal{A}_R$ is an empty graph and the proof is complete. ∎

The next result is an immediate consequence of Observation 1.1 and Theorem 2.1.

Corollary 2.2 *If $(R, \mathfrak{m})$ is an Artinian local ring, then $\mathcal{A}_R$ is an empty graph.*

Theorem 2.3 *Let R be a commutative ring with non-zero identity. Then $\mathcal{A}_R$ is a complete graph if and only if one of the following holds.*

(i) *R has exactly one non-zero proper ideal.*
(ii) *R is an integral domain.*
(iii) *R is a direct product of two fields.*

Proof If R has exactly one non-zero proper ideal or R is an integral domain, then clearly $\mathcal{A}_R$ is a complete graph. If $R = F_1 \times F_2$, where F_1 and F_2 are fields, then R has exactly two non-zero proper ideals, $(0) \times F_2$ and $F_1 \times (0)$. Obviously, $\operatorname{Ann}(F_1 \times (0)) \cap \operatorname{Ann}((0) \times F_2) = \{0\}$ and so $(0) \times F_2$ and $F_1 \times (0)$ are adjacent. Thus $\mathcal{A}_R$ is a complete graph.

Conversely, let $\mathcal{A}_R$ be a complete graph. If $\operatorname{Ann}(I) = (0)$ for each non-zero proper ideal I of R, then R is an integral domain as desired. Hence, assume there is a non-zero ideal I such that $\operatorname{Ann}(I) \neq (0)$. Obviously, I is a minimal ideal of R.

First let $I^2 = (0)$. Then $I \subseteq \operatorname{Ann}(I) \cap \operatorname{Ann}(\operatorname{Ann}(I))$. If $I \neq \operatorname{Ann}(I)$, then I and $\operatorname{Ann}(I)$ are not adjacent, a contradiction. Henceforth, we assume $I = \operatorname{Ann}(I)$. Since I is a minimal ideal, $I = (a)$ for some $a \in I \setminus (0)$ and so $R/\operatorname{Ann}(I) \cong I$. It follows that $R/\operatorname{Ann}(I)$ is a simple R-module that implies $I = \operatorname{Ann}(I)$ is a maximal ideal. If R has a maximal ideal $\mathfrak{m}$ different from I, then $I^2 \subset \mathfrak{m}$ and hence $I \subseteq \mathfrak{m}$ which is a contradiction because I is a maximal ideal. Thus (R, I) is a local ring. Since I is minimal, we conclude that I is the only non-zero proper ideal of R and so (i) holds.

Now let $I^2 \neq (0)$. By Brauer's Lemma [17, Lemma 10.22, p. 172], there exists an idempotent element $a \in R$ such that $I = Ra$ and $R = Ra \times R(1-a)$. Now we show that both of Ra and $R(1-a)$ are fields. Clearly, Ra and $R(1-a)$ are commutative rings with identity a and $1-a$, respectively. If $R(1-a)$ has a non-zero proper ideal J, then $(0) \times J$ is a non-zero proper ideal of R such that $\operatorname{Ann}((0) \times R(1-a)) \cap \operatorname{Ann}((0) \times J) \neq (0)$ and hence $(0) \times R(1-a)$ and $(0) \times J$ are not adjacent in $\mathcal{A}_R$, a contradiction. Thus $R(1-a)$

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has no non-zero proper ideal implying that $R(1-a)$ is a field. Also, obviously $I = Ra$ has no non-zero proper ideal and hence I is a field. This completes the proof. ∎

Observation 2.4 *Let R be a commutative ring with non-zero identity. If $\operatorname{Max}(R)$ is the set consisting of all maximal ideals of R, then the subgraph $\mathcal{A}_R[\operatorname{Max}(R)]$ induced by $\operatorname{Max}(R)$ is a clique in $\mathcal{A}_R$.*

Proof If R has exactly one maximal ideal, then the result is immediate. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be two arbitrary distinct maximal ideals of R. Then $\mathfrak{m}_1+\mathfrak{m}_2 = R$ and so $\operatorname{Ann}_R(\mathfrak{m}_1)\cap \operatorname{Ann}_R(\mathfrak{m}_2) = \{0\}$. It follows that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are adjacent in $\mathcal{A}_R$ and the proof is complete. ∎

Theorem 2.5 *Let R be a commutative ring with non-zero identity. Then $\mathcal{A}_R$ is connected if and only if one of the following holds.*

(i) *There exists a non-zero proper ideal I for which $\operatorname{Ann}(I) = (0)$.*
(ii) $J(R) = (0)$.
(iii) *$J(R)$ is the unique non-zero ideal of R.*

Proof If R has a non-zero proper ideal I such that $\operatorname{Ann}(I) = (0)$, then for any non-zero proper ideal J of R, we have $\operatorname{Ann}(I) \cap \operatorname{Ann}(J) = (0)$ implying that I and J are adjacent in $\mathcal{A}_R$. Thus I is adjacent to all vertices of $\mathcal{A}_R$ and hence $\mathcal{A}_R$ is connected. Now let $J(R) = (0)$. Assume that I is a non-zero proper ideal of R. Since $I \nsubseteq J(R)$, there exists a maximal ideal $\mathfrak{m}$ such that $I \nsubseteq \mathfrak{m}$. It follows that $I + \mathfrak{m} = R$ and hence $\operatorname{Ann}(I) \cap \operatorname{Ann}(\mathfrak{m}) = (0)$. Thus I and $\mathfrak{m}$ are adjacent in $\mathcal{A}_R$. In fact, every non-zero proper ideal is adjacent to some maximal ideal in $\mathcal{A}_R$. Now the result follows from Observation 2.4. Finally, if $J(R)$ is the unique non-zero proper ideal of R, then $\mathcal{A}_R$ has exactly one vertex and so it is connected.

Conversely, let $\mathcal{A}_R$ be connected. If $J(R) = (0)$, then we are done. Let $J(R) \neq (0)$. If $|V(\mathcal{A}_R)| = 1$, then $J(R)$ is the unique non-zero proper ideal of R and (iii) holds. Let $\mathcal{A}_R$ have at least two vertices. We show that R has a non-zero proper ideal I such that $\operatorname{Ann}(I) = (0)$. Assume to the contrary that for each non-zero proper ideal I, $\operatorname{Ann}(I) \neq (0)$. Let J be a non-zero proper ideal different from $J(R)$. Clearly $(0) \neq J(R) + J \subsetneq R$. By assumption $\operatorname{Ann}(J(R)) \cap \operatorname{Ann}(J) = \operatorname{Ann}(J(R) + J) \neq (0)$, and so $J(R)$ and J are not adjacent. It follows that $J(R)$ is an isolated vertex in $\mathcal{A}_R$, which is a contradiction. This completes the proof. ∎

3 When Is $\mathcal{A}_R$ a Tree?

In this section, we characterize all rings whose co-annihilating-ideal graphs are bipartite graphs with no isolated vertex.

Theorem 3.1 *Let R be a commutative ring. Then $\mathcal{A}_R$ is a star if and only if one of the following holds.*

(i) *$(R,\mathfrak{m})$ is a local ring, $\mathfrak{m}^2 = \mathfrak{m}$, $\operatorname{Ann}(\mathfrak{m}) = (0)$ and for each pair of distinct non-zero proper ideals I and J, different from $\mathfrak{m}$, $\operatorname{Ann}(I) \cap \operatorname{Ann}(J) \neq (0)$.*
(ii) *$(R,\mathfrak{m})$ is a local ring, $\mathfrak{m}^2 = (0)$, and R has exactly three ideals.*

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(iii) *$R \cong F_1 \times F_2$, where F_1 and F_2 are fields.*

Proof One side is clear. Suppose that $\mathcal{A}_R$ is a star. Since $\mathcal{A}_R$ is triangle-free, we have $|\operatorname{Max}(R)| \leq 2$. Two cases can be considered:

Case 1: $\operatorname{Max}(R) = \{\mathfrak{m}\}$. First suppose that $\operatorname{Ann}(\mathfrak{m}) = (0)$. It follows that $\mathfrak{m}^2 \neq (0)$, which implies $\operatorname{Ann}(\mathfrak{m}^2) = (0)$. We claim that $\mathfrak{m}^2 = \mathfrak{m}$. Assume to the contrary that $\mathfrak{m}^2 \neq \mathfrak{m}$. Then $\mathfrak{m}$ and $\mathfrak{m}^2$ are adjacent to all other vertices. Since $\mathcal{A}_R$ is a star, we conclude that R has at most two non-zero proper ideals, $\mathfrak{m}$ and $\mathfrak{m}^2$. We have $\mathfrak{m}^3 \in \{(0), \mathfrak{m}, \mathfrak{m}^2\}$. If $\mathfrak{m}^3 = (0)$, then $\mathfrak{m}^2 = (0)$, a contradiction. Thus $\mathfrak{m}^3 = \mathfrak{m}^2$ or $\mathfrak{m}^3 = \mathfrak{m}$. So by Nakayama's Lemma, $\mathfrak{m}^2 = (0)$, a contradiction and the claim is proved. Since $\mathcal{A}_R$ is a star, we deduce that for each pair of distinct non-zero proper ideals I and J, different from $\mathfrak{m}$, $\operatorname{Ann}(I) \cap \operatorname{Ann}(J) \neq (0)$.

Now, assume that $\operatorname{Ann}(\mathfrak{m}) \neq (0)$. Thus $\mathfrak{m}$ is an isolated vertex in $\mathcal{A}_R$. Since $\mathcal{A}_R$ is a star, R has exactly three ideals, (0), $\mathfrak{m}$, R. Clearly, $\mathfrak{m}$ is a principal ideal. If $\mathfrak{m}^2 = \mathfrak{m}$, then by Nakayama's Lemma, $\mathfrak{m} = (0)$, a contradiction. So $\mathfrak{m}^2 = (0)$.

Case 2: $\operatorname{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. If $\operatorname{Ann}(\mathfrak{m}_1) = (0)$, then $\mathfrak{m}_1$ is adjacent to all other vertices of $\mathcal{A}_R$. Let $x \in m_1 \setminus m_2$. Since $(x) + \mathfrak{m}_2 = R$, we conclude that $\operatorname{Ann}(x) \cap \operatorname{Ann}(\mathfrak{m}_2) = \{0\}$. Hence (x) and $\mathfrak{m}_2$ are adjacent. If $\mathfrak{m}_1 \neq (x)$, then we obtain a contradiction. Therefore $\mathfrak{m}_1 = (x)$. So x is a non-unit regular element and by Observation 1.3 there is an infinite clique, a contradiction.

Thus we can suppose that $\operatorname{Ann}(\mathfrak{m}_1) \neq (0)$ and $\operatorname{Ann}(\mathfrak{m}_2) \neq (0)$. If $\mathfrak{m}_1 \cap \mathfrak{m}_2 \neq (0)$, then noting that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are adjacent and $\mathcal{A}_R$ is a star, we get a contradiction. Hence $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$. Now by the Chinese Remainder Theorem [10, Theorem 1.10, p. 7] we have $R \cong R/\mathfrak{m}_1 \times R/\mathfrak{m}_2$, as desired. This completes the proof. ∎

Theorem 3.2 *Let R be a commutative ring and $g(\mathcal{A}_R) \geq 5$. Then R is a local ring or $R \cong F_1 \times F_2$, where F_1 and F_2 are fields.*

Proof Since $\mathcal{A}_R$ is triangle-free, $|\operatorname{Max}(R)| \leq 2$. If R is a local ring, then we are done. Thus assume that $\operatorname{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. Since $\mathfrak{m}_1 + \mathfrak{m}_2 = R$, there are $a \in \mathfrak{m}_1$ and $b \in \mathfrak{m}_2$ such that $a + b = 1$. If $\mathfrak{m}_1 \neq (a)$ and $\mathfrak{m}_2 \neq (b)$, then $(a), (b), \mathfrak{m}_1, \mathfrak{m}_2$ form a 4-cycle, a contradiction. Without loss of generality assume that $\mathfrak{m}_1 = (a)$. We claim that $J(R) = (0)$. By contradiction suppose that $0 \neq x \in J(R)$. If $\operatorname{Ann}(\mathfrak{m}_1) = (0)$, then a is a non-unit regular element and recalling Observation 1.3, we get a contradiction. Thus assume that $\operatorname{Ann}(\mathfrak{m}_1) \neq (0)$. Since $\mathcal{A}_R$ is connected, (x) is adjacent to some ideal, say L. If $\operatorname{Ann}(\mathfrak{m}_2) = (0)$, then $\mathfrak{m}_2$ is adjacent to all other vertices and so $\mathcal{A}_R$ should be a star, and recalling the previous theorem we are done. So assume that $\operatorname{Ann}(\mathfrak{m}_2) \neq (0)$. Note that $L \subseteq \mathfrak{m}_1$ or $L \subseteq \mathfrak{m}_2$. If $L \subseteq \mathfrak{m}_2$, then x should be adjacent to $\mathfrak{m}_2$. But since $(x) \subseteq \mathfrak{m}_2$, we get a contradiction. On the other hand, since $\operatorname{Ann}(\mathfrak{m}_1) \neq (0)$, we deduce that L is not contained in $\mathfrak{m}_1$. Therefore we conclude that $J(R) = (0)$ and by the Chinese Remainder Theorem, R is a direct product of two fields and the proof is complete. ∎

Corollary 3.3 *If R is an Artinian ring, then $g(\mathcal{A}_R) = 3, 4$, or ∞.*

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Theorem 3.4 *Let R be a commutative ring. Then $\mathcal{A}_R$ is a tree if and only if $\mathcal{A}_R$ is a star.*

Proof Let $\mathcal{A}_R$ be a tree. By Theorem 3.2, R is a local ring or $R \cong F_1 \times F_2$, where F_1 and F_2 are fields. If $R \cong F_1 \times F_2$, where F_1 and F_2 are fields, then by Theorem 3.1 we are done. Let R be a local ring with the unique maximal ideal $\mathfrak{m}$. First let $\mathrm{Ann}(\mathfrak{m}) \neq (0)$. Then $\mathcal{A}_R$ is an empty graph by Theorem 2.1. Since $\mathcal{A}_R$ is a tree, we deduce that $\mathcal{A}_R$ is trivial. Thus R has exactly three ideals (0), $\mathfrak{m}$, R. Since $\mathfrak{m}^2 \in \{(0), \mathfrak{m}, R\}$, it follows from Nakayama's Lemma that $\mathfrak{m}^2 = (0)$ and hence $\mathcal{A}_R$ is a star. Now let $\mathrm{Ann}(\mathfrak{m}) = (0)$. Then clearly $\mathcal{A}_R$ has at least two vertices and $\mathfrak{m}$ is adjacent to all vertices. Since $\mathcal{A}_R$ is a tree, for each pair of distinct non-zero proper ideals I and J, different from $\mathfrak{m}$, we have $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) \neq (0)$ and the result follows from Theorem 3.1. This completes the proof. ∎

Theorem 3.5 *Let R be a commutative ring. Then $\mathcal{A}_R$ is a bipartite graph with $\delta(\mathcal{A}_R) > (0)$ if and only if $\mathcal{A}_R$ is a star of order at least 2.*

Proof One side is clear. Suppose $\mathcal{A}_R$ is a bipartite graph without isolated vertices and let X_1 and X_2 be the partite sets of $\mathcal{A}_R$. Since $\mathcal{A}_R$ is triangle-free, $|\mathrm{Max}(R)| \leq 2$. First let $\mathrm{Max}(R) = \{\mathfrak{m}\}$. Then $\mathrm{Ann}(\mathfrak{m}) = (0)$, otherwise $\mathfrak{m}$ is an isolated vertex in $\mathcal{A}_R$, which is a contradiction. It follows that $\mathfrak{m}$ is adjacent to all vertices and hence $\mathcal{A}_R$ is a star.

Now let $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. Since $\mathfrak{m}_1 + \mathfrak{m}_2 = R$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are adjacent in $\mathcal{A}_R$. We may assume that $\mathfrak{m}_1 \in X_1$ and $\mathfrak{m}_2 \in X_2$. If $\mathrm{Ann}(\mathfrak{m}_1) = \mathrm{Ann}(\mathfrak{m}_2) = (0)$, then $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are adjacent to all vertices. Since $\mathcal{A}_R$ is a bipartite graph, we deduce that R has exactly four ideals (0), $\mathfrak{m}_1$, $\mathfrak{m}_2$, R. This yields $\mathcal{A}_R = K_2$ as desired. If $\mathrm{Ann}(\mathfrak{m}_1) = (0)$ and $\mathrm{Ann}(\mathfrak{m}_2) \neq (0)$ (the case $\mathrm{Ann}(\mathfrak{m}_1) \neq (0)$ and $\mathrm{Ann}(\mathfrak{m}_2) = (0)$ is similar), then $\mathfrak{m}_1$ is adjacent to all vertices implying that $|X_1| = 1$ and so $\mathcal{A}_R$ is a star. Finally let $\mathrm{Ann}(\mathfrak{m}_1) \neq (0)$ and $\mathrm{Ann}(\mathfrak{m}_2) \neq (0)$. Then $J(R) = (0)$, otherwise for any non-zero ideal I of R we have $\mathrm{Ann}(\mathfrak{m}_1) \subseteq \mathrm{Ann}(I) \cap \mathrm{Ann}(J(R))$ or $\mathrm{Ann}(\mathfrak{m}_2) \subseteq \mathrm{Ann}(I) \cap \mathrm{Ann}(J(R))$ that implies $J(R)$ is an isolated vertex in $\mathcal{A}_R$, a contradiction. It follows from $J(R) = (0)$ and the Chinese Remainder Theorem that R is a direct product of two fields and the result follows by Theorem 3.1. This completes the proof. ∎

4 When Is $\mathcal{A}_R$ a Finite Graph?

In this section, we prove that $\mathcal{A}_R$ is finite if and only if each vertex of $\mathcal{A}_R$ has finite degree.

Theorem 4.1 *Let R be a commutative ring such that $\mathcal{A}_R$ has no isolated vertex. Then the following statements are equivalent.*

(i) *$\mathcal{A}_R$ is a finite graph.*
(ii) *Every vertex of $\mathcal{A}_R$ has finite degree.*

Proof One side is clear. Let each vertex of $\mathcal{A}_R$ have finite degree. If $\mathrm{Ann}(I) = (0)$ for some non-zero proper ideal I, then I is adjacent to all vertices of $\mathcal{A}_R$. This implies

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that $\mathcal{A}_R$ is a finite graph. Assume $\mathrm{Ann}(I) \neq (0)$ for each non-zero proper ideal I. Since $\mathcal{A}_R$ has no isolated vertex, by Theorem 2.5 $J(R) = (0)$. By Observation 2.4, we have $|\mathrm{Max}(R)| < \infty$. Now it follows from the Chinese Remainder Theorem that $R \cong F_1 \times \cdots \times F_n$, where $n = |\mathrm{Max}(R)|$ and F_i is a field for each i. Then obviously R has finitely many ideals and the proof is complete. ∎

Corollary 4.2 *If $\mathcal{A}_R$ is a k-regular graph ($0 < k < \infty$), then $\mathcal{A}_R$ is a complete graph.*

Proof If $\mathrm{Ann}(I) = (0)$ for some non-zero proper ideal I, then I is adjacent to all vertices of $\mathcal{A}_R$. Since $\mathcal{A}_R$ is regular, we conclude that G is complete. Assume $\mathrm{Ann}(I) \neq (0)$ for each non-zero proper ideal I. Using an argument similar to that described in the proof of Theorem 4.1, we have $R \cong F_1 \times \cdots \times F_n$ where F_i is a field for each i. We claim that $n = 2$. Assume to the contrary that $n \geq 3$. Let $I = F_1 \times (0) \times \cdots \times (0)$. Since $\mathrm{Ann}(I) = (0) \times F_2 \times \cdots \times F_n$, I is adjacent to exactly one vertex $(0) \times F_2 \times \cdots \times F_n$ and hence $\deg(I) = 1$. On the other hand, each maximal ideal has degree at least two, which is a contradiction. Thus $n = 2$, and so $\mathcal{A}_R = K_2$. This completes the proof. ∎

5 Chromatic Number, Diameter of $\mathcal{A}_R$

In this section, we study the chromatic number and diameter of the co-annihilating-ideal graphs of commutative rings. In particular, we show that if $\mathcal{A}_R$ is connected, then $\mathrm{diam}(\mathcal{A}_R) \leq 3$.

Theorem 5.1 *If R is a reduced Noetherian ring, then the chromatic number of $\mathcal{A}_R$ is infinite or R is a direct product of finitely many fields.*

Proof Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be the set consisting of all minimal prime ideals of R. Let $I = P_i + P_j$ for some $i \neq j$. By prime avoidance theorem [10, Theorem 1.11, p. 8], $I \notin \bigcup_{k=1}^{n} P_k$. If $I \neq R$, then I contains a non-unit regular element (see [18, Theorem 1.1 (3)]). It follows from Observation 1.3 that $\chi(\mathcal{A}_R)$ is infinite. Thus we may assume $P_i + P_j = R$ for every $1 \leq i \neq j \leq n$. Since R is reduced, we have $\cap_{k=1}^{n} P_k = (0)$ and by the Chinese Remainder Theorem $R \cong \frac{R}{P_1} \times \cdots \times \frac{R}{P_n}$. If $\frac{R}{P_i}$ is not a field for some i, then it follows from Observation 1.3 that $\chi(\mathcal{A}_R)$ is infinite. Thus $\frac{R}{P_i}$ is a field for each i and the proof is complete. ∎

Theorem 5.2 *If $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring for each i, then $\chi(\mathcal{A}_R) = \omega(\mathcal{A}_R) = n$.*

Proof Let $X_1 = \{I_1 \times \cdots \times I_n \mid I_1 \lhd R_1 \text{ and } I_j \trianglelefteq R_i \text{ for } 2 \leq j \leq n\} \setminus (0)$ and $X_i = \{I_1 \times \cdots \times I_n \mid I_j = R_j \text{ for } 1 \leq j \leq i-1, I_i \lhd R_i \text{ and } I_j \trianglelefteq R_i \text{ for } j \geq i+1\}$ for $i = 2, \dots, n$. Clearly, $X_1 \cup \cdots \cup X_n$ is a partition of $V(\mathcal{A}_R)$. By Observation 1.1, X_i is independent for each i. Thus $\mathcal{A}_R$ is an n-partite graph implying that $\omega(\mathcal{A}_R) \leq \chi(\mathcal{A}_R) \leq n$.

On the other hand, since R has at least n maximal ideals, we conclude that $\omega(\mathcal{A}_R) \geq n$, and the proof is complete. ∎

Corollary 5.3 *If R is an Artinian ring, then $\chi(\mathcal{A}_R) = \omega(\mathcal{A}_R) = |\mathrm{Max}(R)|$.*

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Theorem 5.4 *Let R be a commutative ring with non-zero identity. If $\mathcal{A}_R$ is connected, then* $\operatorname{diam}(\mathcal{A}_R) \leq 3$.

Proof Since $\mathcal{A}_R$ is connected, it follows from Theorem 2.5 that $J(R)$ is the unique non-zero ideal of R, $J(R) = (0)$ or R has a non-zero proper ideal I such that $\operatorname{Ann}_R(I) = (0)$. If $J(R)$ is the unique non-zero ideal of R, then $|V(\mathcal{A}_R)| = 1$ and hence $\operatorname{diam}(\mathcal{A}_R) = 0$. If R has a non-zero proper ideal I such that $\operatorname{Ann}_R(I) = (0)$, then I is adjacent to all vertices in $\mathcal{A}_R$ implying that $\operatorname{diam}(\mathcal{A}_R) \leq 2$. Now let $J(R) = (0)$. It follows that for any non-zero proper ideal I, there is a maximal ideal $\mathfrak{m}_I$ such that $I + \mathfrak{m}_I = R$ and hence $\operatorname{Ann}(I) \cap \operatorname{Ann}(\mathfrak{m}_I) = (0)$. Thus each non-zero proper ideal is adjacent to a maximal ideal and it follows from Observation 2.4 that $\operatorname{diam}(\mathcal{A}_R) \leq 3$. This completes the proof. ∎

Corollary 5.5 *If $R \cong F_1 \times \cdots \times F_n$ ($n \geq 2$), where F_i is a field for each i, then*

$$\operatorname{diam}(\mathcal{A}_R) = \begin{cases} 1 & \text{if } n = 2, \\ 3 & \text{if } n \geq 3. \end{cases}$$

Proof If $R \cong F_1 \times F_2$, then obviously $\mathcal{A}_R = K_2$ and so $\operatorname{diam}(\mathcal{A}_R) = 1$. Let $n \geq 3$. Clearly $J(R) = (0)$ and we deduce from Theorem 2.5 that $\mathcal{A}_R$ is connected which implies $\operatorname{diam}(\mathcal{A}_R) \leq 3$ by Theorem 5.4. Now let $I = F_1 \times (0) \times \cdots \times (0)$ and $J = (0) \times F_2 \times (0) \times \cdots \times (0)$. It is not hard to see that $d_{\mathcal{A}_R}(I, J) = 3$ and hence $\operatorname{diam}(\mathcal{A}_R) = 3$. ∎

Corollary 5.5 demonstrates that the bound of Theorem 5.4 is sharp.

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