On the evaluation of $\int \frac{dx}{(x-e)^{n+1}\sqrt{(ax^2+2bx+c)}}$

By R. Wilson.

It is not perhaps generally realised that the special bilinear substitution $x = \frac{lt+m}{t+1}$, used to reduce the integral

$$\int \frac{dx}{(ex^2+2fx+g)\sqrt{(ax^2+2bx+c)}}$$

to canonical form[1], can also be used to simplify the calculation of the integral $\int \frac{dx}{(x-e)^{n+1}\sqrt{(ax^2+2bx+c)}}$. At the same time this treatment forms a suitable introduction to the more difficult case of the former integral.

In the latter case l and m are chosen to satisfy the relations

$$l-e=0, \quad alm+b(l+m)+c=0,$$

so that

$$l-m = \frac{ae^2+2be+c}{ae+b} = e-m$$

and, after substitution, the integral becomes

$$\frac{-(m-l)^{-n}}{\sqrt{(ae^2+2be+c)}} \int \frac{(t+1)^n\,dt}{\sqrt{\{t^2+(ca-b^2)/(ae+b)^2\}}}.$$

The properties of the quadratic form show that (after incorporation of the numerical factor $\sqrt{(ae^2+2be+c)}$ when it is imaginary) the denominator must take one of the three forms $K\sqrt{(t^2+k^2)}$, $K\sqrt{(t^2-k^2)}$ or $K\sqrt{(k^2-t^2)}$, where K and k are both real. Consider, for example, the typical case $I_n \equiv \int \frac{(t+1)^n\,dt}{\sqrt{(t^2+k^2)}}$. The reduction formula is easily seen to be

$$nI_n-(2n-3)\,I_{n-1}+(n-1)(k^2+1)\,I_{n-2} = (t+1)^{n-1}\sqrt{(t^2+k^2)}, \ (n \geqq 2)$$

in which the last member of the chain is

$$I_1 - I_0 = \sqrt{(t^2+k^2)}$$

where

$$I_0 = \int \frac{dt}{\sqrt{(t^2+k^2)}}.$$

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[1] See for example G. H. Hardy, *Pure Mathematics* (Cambridge), 1908, pp. 246-7, Ex. 37.

This method is an improvement on that arising from the orthodox substitution $x - e = \frac{1}{z}$, in which the final reduction to canonical form with $n = 0$ is frequently tedious[1]. The following example with $n = 3$ shows that the reduction process is not necessary for low values of n.

$$\int \frac{(t+1)^3\,dt}{\sqrt{(t^2+k^2)}} = \int t\sqrt{(t^2+k^2)}\,dt + 3\int \sqrt{(t^2+k^2)}\,dt + (3-k^2)$$
$$\int \frac{t\,dt}{\sqrt{(t^2+k^2)}} + (1-3k^2)\int \frac{dt}{\sqrt{(t^2+k^2)}}$$

which may be integrated at sight.

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A dual quadratic transformation associated with the Hessian conics of a pencil

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1. The invariants and covariants of a system of two conics have been much studied[2] but little has been said about those of three conics. Three conics $f_1 \equiv a_x^2$, $f_2 \equiv b_x^2$, $f_3 \equiv c_x^2$ have a symmetrical invariant Ω_{123}, or in symbolical notation $(a\,b\,c)^2$. According to Ciamberlini[3] the vanishing of this invariant signifies that *the* Φ *conic of any two of* f_1, f_2, f_3 *is inpolar with respect to the third*; and in a previous paper[4] I have

[1] The integral $\int \frac{dx}{(x+1)\sqrt{(2x-x^2)}} = -\int \frac{dz}{\sqrt{(-1+4z-3z^2)}}$ or $-\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{(\frac{1}{9}-t^2)}}$ is a case in point.

[2] See Salmon Conic Sections, Ch. xviii, or Sommerville, Analytical Conics, Ch. xx. Taking point-coordinates x, y, z with corresponding line-coordinates l, m, n, a conic $a_x^2 \equiv a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{31}zx + 2a_{12}xy = 0$ has a tangential equation $A_{11}l^2 + A_{22}m^2 + A_{33}n^2 + 2A_{23}mn + 2A_{31}nl + 2A_{12}lm = 0$. Then the vanishing of the invariant $\Theta = b_{11}A_{11} + b_{22}A_{22} + b_{33}A_{33} + 2b_{23}A_{23} + 2b_{31}A_{31} + 2b_{12}A_{12}$ of the conics $f_1 \equiv a_x^2$, $f_2 \equiv b_x^2$ implies that there are triangles circumscribed to f_1 which are self-polar for f_2, and f_1 is said to be inpolar to f_2. The contravariant conic Φ_{12} is the envelope of a line whose intersections with f_1 harmonically separate its intersections with f_2.

[3] *Giorn. di Mat.*, Napoli, 24 (1886), 141.

[4] *Proc. Ed. Math. Soc.*, 2 iv (1935) 258.