Canad. J. Math. Vol. 70 (4), 2018 pp. 773–803
http://dx.doi.org/10.4153/CJM-2017-009-4

Multiplication Formulas and Canonical Bases for Quantum Affine gl_n

Dedicated to Professor Leonard Scott on the occasion of his 75th birthday.

Jie Du and Zhonghua Zhao

Abstract. We will give a representation-theoretic proof for the multiplication formula in the Ringel–Hall algebra $\mathfrak{H}_\Delta(n)$ of a cyclic quiver $\Delta(n)$. As a first application, we see immediately the existence of Hall polynomials for cyclic quivers, a fact established by J. Y. Guo and C. M. Ringel, and derive a recursive formula to compute them. We will further use the formula and the construction of a monomial basis for $\mathfrak{H}_\Delta(n)$ given by Deng, Du, and Xiao together with the double Ringel–Hall algebra realisation of the quantum loop algebra $\mathbf{U}_v(\widehat{\mathfrak{gl}}_n)$ given by Deng, Du, and Fu to develop some algorithms and to compute the canonical basis for $\mathbf{U}_v^+(\widehat{\mathfrak{gl}}_n)$. As examples, we will show explicitly the part of the canonical basis associated with modules of Lowey length at most 2 for the quantum group $\mathbf{U}_v(\widehat{\mathfrak{gl}}_2)$.

1 Introduction

The investigation of quantum algebras associated with affine Hecke algebras has made significant progress recently. In the affine type A case, an algebraic approach was developed in [4] for the Schur–Weyl theory associated with the quantum loop algebra of $\mathfrak{gl}_n$, affine q-Schur algebras and Hecke algebras of the affine symmetric groups. This approach, motivated by the algebraic approach for quantum $\mathfrak{gl}_n$, is different from the geometric approach developed in [15,23]. Further in [10,11], new realisations for these quantum loop algebras and their integral Lusztig type form are obtained using affine q-Schur algebras. This generalises the work of Beilinson–Lusztig–MacPherson [1] to this affine case. For affine types of other than A, Fan *et al.* used affine q-Schur algebras of type C to construct in [13] various types of quantum symmetric pairs. The multiplication formulas there are much more complicated, but can be used to study the modified versions of these quantum algebras and their canonical basis. In this paper, we will see how a new multiplication formula discovered in [10] is used to compute certain slices of the canonical basis for the +-part of the quantum loop algebra of $\mathfrak{gl}_n$.

The key ingredient of the approach developed in [4] is the double Ringel–Hall algebra characterisation for the Drinfeld's quantum loop algebra of $\mathfrak{gl}_n$ [7]. In this way,

Received by the editors August 10, 2016; revised March 29, 2017.
Published electronically August 16, 2017.
Z. Zhao is the corresponding author.
The research was carried out while the second author was visiting the University of New South Wales. The hospitality and support from UNSW are gratefully acknowledged. He also thanks the China Scholarship Council for the financial support.
AMS subject classification: **16G20**, 20G42.
Keywords: Ringel–Hall algebra, quantum group, cyclic quiver, monomial basis, canonical basis.

the Ringel–Hall algebra of a cyclic quiver and its opposite algebra become the ±-part of the quantum loop algebra of $\mathfrak{gl}_n$, and their generators associated with the semisimple modules of the cyclic quiver play the role as done by usual Chevalley generators. In particular, the quantum affine Schur–Weyl duality can be described by explicit actions of these (infinitely many) generators associated with semisimple representations and a new realisation, *i.e.*, a new construction of the quantum loop algebra of $\mathfrak{gl}_n$, is achieved through a beautiful multiplication formula of a basis element by a semisimple generator. It should be pointed out that these multiplication formulas are derived in the affine q-Schur algebras with most of the computation done within the affine Hecke algebras. However, when the formulas restrict to the ±-part, they result in multiplication formulas for (generic) Ringel–Hall algebras of a cyclic quiver. Thus, a natural question arises: Is there a direct proof for these formulas as a quantumization[1] of Hall numbers associated with representations of a cyclic quiver over finite fields?

In this paper, we first provide a representation-theoretic proof for the multiplication formula in the Ringel–Hall algebra (Theorem 2.1). One key idea used in the proof is the bijective correspondence between the m-dimensional subspaces of an n-dimensional space and the reduced row echelon form of $m \times n$ matrices of rank m. We then use the multiplication formula to show in general the existence of Hall polynomials for cyclic quivers (*cf.* [12,27]). As a further application of the formula, we derive a recursive formula for computing Hall polynomials and compute the canonical basis for (the +-part of) a quantum affine $\mathfrak{gl}_n$. This requires a systematic construction of a certain monomial basis. Thanks to [6], we will use the theory there to derive a couple of algorithms on matrices and will then follow them to produce the required monomial basis. Computing canonical bases is in general very difficult. Besides some lower rank cases of finite type (see, *e.g.,* [19, §3] for types A_1 and A_2 and [30, 31] for type A_3, B_2) and certain tight monomials for quantum affine $\mathfrak{sl}_2$ ([22]), there seem to be no explicit affine examples done in the literature. We now use the multiplication formula to compute several infinite series of the canonical basis for $\mathbf{U}_v(\widehat{\mathfrak{gl}}_2)$. To ease the difficulty, we divide the basis into the so-called "slices" labelled by the Lowey length $\ell(M)$ and the periodicity $p(M)$ associated with a representation M of a cyclic quiver. We explicitly compute all slices of the canonical basis associated with modules of Lowey length at most 2 for quantum affine $\mathfrak{gl}_2$. In a forthcoming paper, we will give further applications to the theory of quantum loop algebras of $\mathfrak{sl}_n$ developed in [6].

The paper is roughly divided into two parts. The first part from Sections 2 to 4 deals with the theory of integral Hall algebras associated with finite fields, including the existence of Hall polynomials (Theorem 2.2) and a recursive formula (Corollary 4.8). The remaining sections focus on computation of canonical basis for the (generic and twisted) Ringel–Hall algebras and quantum affine $\mathfrak{gl}_n$. With a selected monomial basis, we formulate Algorithm 5.5 to compute the canonical basis. Five slices of the canonical basis for quantum affine $\mathfrak{gl}_2$ are explicitly worked out; see Propositions 6.1 and 6.4 and Theorems 7.4 and 8.1.

Notation For a positive integer n, let $M_{\triangle,n}(\mathbb{Z})$ be the set of all $\mathbb{Z} \times \mathbb{Z}$ matrices $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ with $a_{i,j} \in \mathbb{Z}$ such that

[1] See the definition on [5, p. 17].

https://doi.org/10.4153/CJM-2017-009-4 Published online by Cambridge University Press

(a) $a_{i,j} = a_{i+n,j+n}$ for $i, j \in \mathbb{Z}$, and
(b) for every $i \in \mathbb{Z}$, both the set $\{j \in \mathbb{Z} \mid a_{i,j} \neq 0\}$ and $\{j \in \mathbb{Z} \mid a_{j,i} \neq 0\}$ are finite.

Let $\Theta_\triangle(n) = M_{\triangle,n}(\mathbb{N})$ be the subset of $M_{\triangle,n}(\mathbb{Z})$ consisting of matrices with entries from $\mathbb{N}$. Let

$$\Theta^+_\triangle(n) = \left\{A \in \Theta_\triangle(n) \mid a_{ij} = 0 \text{ for } i \geqslant j\right\},$$
$$\Theta^-_\triangle(n) = \left\{A \in \Theta_\triangle(n) \mid a_{ij} = 0 \text{ for } i \leqslant j\right\}.$$

For $A \in \Theta_\triangle(n)$, write $A = A^+ + A^0 + A^-$, where A^0 is the diagonal submatrix of A, $A^+ \in \Theta^+_\triangle(n)$, and $A^- \in \Theta^-_\triangle(n)$.

The *core* of a matrix A in $\Theta^+_\triangle(n)$ is the $n \times l$ submatrix of A consisting of rows from 1 to n and columns from 1 to l, where l is the column index of the right most non-zero entry in the given n rows.

Set

$$\mathbb{Z}^n_\triangle = \left\{(\lambda_i)_{i\in\mathbb{Z}} \mid \lambda_i \in \mathbb{Z}, \lambda_i = \lambda_{i-n} \text{ for } i \in \mathbb{Z}\right\},$$
$$\mathbb{N}^n_\triangle = \left\{(\lambda_i)_{i\in\mathbb{Z}} \in \mathbb{Z}^n_\triangle \mid \lambda_i \geqslant 0 \text{ for } i \in \mathbb{Z}\right\}.$$

For each $A \in M_{\triangle,n}(\mathbb{Z})$, let

$$\mathsf{row}(A) = (\Sigma_{j\in\mathbb{Z}} a_{i,j})_{i\in\mathbb{Z}} \in \mathbb{Z}^n_\triangle, \quad \mathsf{col}(A) = (\Sigma_{i\in\mathbb{Z}} a_{i,j})_{j\in\mathbb{Z}} \in \mathbb{Z}^n_\triangle.$$

Define an order relation $\leqslant$ on $\mathbb{N}^n_\triangle$ by

$$\lambda \leqslant \mu \Longleftrightarrow \lambda_i \leqslant \mu_i \ (1 \leqslant i \leqslant n).$$

We say $\lambda < \mu$ if $\lambda \leqslant \mu$ and $\lambda \neq \mu$.

Let $\mathbb{Q}(v)$ be the fraction field of $\mathcal{Z} := \mathbb{Z}[v, v^{-1}]$. For integers N, t with $t \geqslant 0$ and $\mu \in \mathbb{Z}^n_\triangle$ and $\lambda \in \mathbb{N}^n_\triangle$, define Gaussian polynomials and their symmetric version in $\mathcal{Z}$:

$$\left[\!\left[\begin{matrix} N \\ t \end{matrix} \right]\!\right] = \frac{[\![N]\!]!}{[\![t]\!]![\![N-t]\!]!} = \prod_{1\leqslant i\leqslant t} \frac{v^{2(N-i+1)-1}}{v^{2i}-1} \qquad \text{and} \qquad \begin{bmatrix} N \\ t \end{bmatrix} = v^{-t(N-t)} \left[\!\left[\begin{matrix} N \\ t \end{matrix} \right]\!\right],$$

where $[\![t]\!] = [\![1]\!][\![2]\!]\cdots[\![t]\!]$ with $[\![m]\!] = \frac{v^{2m}-1}{v^2-1}$.

For a prime power q, we write $[\![{N \atop t}]\!]_q$ for the value of the polynomial at $v^2 = q$.

2 The Integral Hall Algebras of Cyclic Quivers and Hall Polynomials

Let $\Delta = \Delta(n)$ $(n \geqslant 2)$ be the cyclic quiver with vertex set $I := \mathbb{Z}/n\mathbb{Z} = \{1, 2, \ldots, n\}$ and arrow set $\{i \to i+1 \mid i \in I\}$, and let $k\Delta$ be the path algebra of Δ over a field k. For a representation $M = (V_i, f_i)_i$ of Δ, let $\mathbf{dim} M = (\dim V_1, \dim V_2, \ldots, \dim V_n) \in \mathbb{N}I = \mathbb{N}^n$ and $\dim M = \sum_{i=1}^n \dim V_i$ denote the dimension vector and the dimension of M, respectively, and let $[M]$ denote the isoclass (isomorphism class) of M.

A representation $M = (V_i, f_i)_i$ of Δ over k (or a $k\Delta$-module) is called *nilpotent* if the composition $f_n \cdots f_2 f_1 \colon V_1 \to V_1$ is nilpotent, or equivalently, one of the maps $f_{i-1} \cdots f_1 f_n \cdots f_i \colon V_i \to V_i$ $(2 \leqslant i \leqslant n)$ is nilpotent. By $\mathsf{Rep}^0\, \Delta(n) = \mathsf{Rep}^0_k\, \Delta(n)$ we denote the category of finite dimensional nilpotent representations of $\Delta(n)$ over k. For each vertex $i \in I$, there is a one-dimensional representation $S_i = S_{i,k}$ in $\mathsf{Rep}^0\, \Delta(n)$

satisfying $(S_i)_i = k$ and $(S_i)_j = 0$ for $j \neq i$. It is known that $\{S_i \mid i \in I\}$ forms a complete set of simple objects in $\mathsf{Rep}^0\,\Delta(n)$.

For $M \in \mathsf{Rep}^0\,\Delta(n)$, we denote by $\mathsf{rad}(M)$ the radical of M, *i.e.* the intersection of all maximal submodules of M, and by $\mathsf{top}(M) = M/\mathsf{rad}(M)$, the top of M.

Up to isomorphism, all non-isomorphic indecomposable representations in $\mathsf{Rep}^0\,\Delta(n)$ are given by $S_i[l]$ ($i \in I$ and $l \geqslant 1$) of length l with top S_i. Note that $S_i[l]$ can be described by vector spaces and linear maps around the cyclic quiver:

$$0 \xrightarrow{0} k \xrightarrow{1} k \xrightarrow{1} \cdots \xrightarrow{1} k \xrightarrow{1} k \xrightarrow{0} 0 \cdots \tag{2.1}$$

Here, the number of k's is l and the first k is at vertex i, the second at $i+1, \ldots,$ the $(n+i)$-th is again at vertex $i = n+i$, etc.

For any $A = (a_{i,j}) \in \Theta^+_\triangle(n)$, let

$$M(A) = M_k(A) = \bigoplus_{1 \leqslant i \leqslant n,\, i<j} a_{i,j} S_i[j-i].$$

Then the set $\{M_k(A) \mid A \in \Theta^+_\triangle(n)\}$ forms a complete set of all non-isomorphic finite dimensional nilpotent representations of $\Delta(n)$. If k is a finite field of $q = q_k$ elements, we write $M_q(A) = M_k(A)$.

Every element $\alpha = (\alpha_i)_{i\in\mathbb{Z}} \in \mathbb{N}^n_\triangle$ defines a semisimple representation

$$S_\alpha = S_{\alpha,k} = \oplus_{i=1}^n \alpha_i S_i.$$

A matrix $A = (a_{i,j}) \in \Theta^+_\triangle(n)$ is called *aperiodic* if, for each $l \geqslant 1$, there exists $i \in \mathbb{Z}$ such that $a_{i,i+l} = 0$. Otherwise, A is called *periodic*. A nilpotent representation $M(A)$ is called aperiodic (resp. periodic) if A is aperiodic (resp. periodic). Denote by $\Theta^{ap}_\triangle(n)$ the subset of all aperiodic elements in $\Theta^+_\triangle(n)$.

Associated with a cyclic quiver, Ringel introduced an associative algebra, the *Hall algebra*, which can be defined at two levels: the integral level and the generic level.

For $A, B, C \in \Theta^+_\triangle(n)$ and any prime power q, let $\mathfrak{h}^{M_q(A)}_{M_q(B),M_q(C)}$ be the number of submodules N of $M_q(A)$ such that

$$N \cong M_q(C) \quad \text{and} \quad M_q(A)/N \cong M_q(B).$$

More generally, given $A, B_1, B_2, \ldots, B_m \in \Theta^+_\triangle(n)$, denote by $\mathfrak{h}^{M_q(A)}_{M_q(B_1),M_q(B_2),\ldots,M_q(B_m)}$ the number of filtrations

$$M_q(A) = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_{m-1} \subseteq M_m = 0$$

such that $M_{t-1}/M_t \cong M_q(B_t)$ for $1 \leqslant t \leqslant m$.

The *(integral) Hall algebra* $\mathfrak{H}^\diamond_\triangle(n,q)$ associated with $\mathsf{Rep}^0_k\,\Delta(n)$ over a finite field k of q elements, is the free $\mathbb{Z}$-module spanned by basis $\{u_{A,q} := u_{[M_q(A)]} \mid A \in \Theta^+_\triangle(n)\}$ with multiplication[2] given by

$$u_{B,q} \diamond u_{C,q} = \sum_{A\in\Theta^+_\triangle(n)} \mathfrak{h}^{M_q(A)}_{M_q(B),M_q(C)} u_{A,q}.$$

By a result in [12, 27], the Hall numbers $\mathfrak{h}^{M_q(A)}_{M_q(B),M_q(C)}$ are polynomials in q with integral coefficients. We now provide an independent proof for the fact, building on the following multiplication formula. A generic version of this formula is given by Fu

[2]In [6] the multiplication is denoted by $\circ$.

and the first author in [10] and is obtained by using the techniques of Hecke algebras, affine q-Schur algebras, and the new realisation of the quantum loop algebra of $\mathfrak{gl}_n$.

Theorem 2.1 *For $A \in \Theta_\triangle^+(n)$, $\alpha = (\alpha_i)_{i\in\mathbb{Z}} \in \mathbb{N}_\triangle^n$, we have the following multiplication formula in the Hall algebra $\mathfrak{H}_\triangle^\diamond(n,q)$:*

$$u_{\alpha,q} \diamond u_{A,q} = \sum_{\substack{T\in\Theta_\triangle^+(n)\\ \mathrm{row}(T)=\alpha}} q^{\sum_{\substack{1\leqslant i\leqslant n\\ i<l<j}}(a_{ij}t_{il}-t_{ij}t_{i+1,l})} \prod_{\substack{1\leqslant i\leqslant n\\ j\in\mathbb{Z},j>i}} \left[\!\!\left[\begin{matrix} a_{ij}+t_{ij}-t_{i-1,j} \\ t_{ij} \end{matrix} \right]\!\!\right]_q u_{A+T-\widehat{T}^+,q},$$

where $\widehat{\ }:\Theta_\triangle(n) \to \Theta_\triangle(n), A = (a_{i,j}) \mapsto \widehat{A} = (\widehat{a}_{i,j})$ is the row-descending map defined by $\widehat{a}_{i,j} = a_{i-1,j}$ for all $i,j \in \mathbb{Z}$ and $\widehat{T}^+$ denotes the upper triangular submatrix of $\widehat{T}$.

We will prove this result in the next section. We first use the formula to prove the existence of Hall polynomials.

Let $\mathcal{M}$ be the set of all isoclasses of representation in $\mathsf{Rep}^0\,\Delta(n)$. Given two objects $M,N \in \mathsf{Rep}^0\,\Delta(n)$, there exists a unique (up to isomorphism) extension G of M by N with minimum $\dim \mathsf{End}(G)$ [2,3,5,24]. The extension G is called the *generic extension*[3] of M by N and is denoted by $G = M * N$. If we define $[M]*[N] = [M*N]$, then it is known from [24] that $*$ is associative and $(\mathcal{M},*)$ is a monoid with identity $[0]$.

Besides the monoid structure, $\mathcal{M}$ has also a poset structure. For two nilpotent representations $M,N \in \mathsf{Rep}^0\,\Delta(n)$ with $\mathbf{dim}M = \mathbf{dim}N$, define

$$N \leqslant_{\mathrm{dg}} M \Longleftrightarrow \dim \mathsf{Hom}(X,N) \geqslant \dim \mathsf{Hom}(X,M), \text{ for all } X \in \mathsf{Rep}^0\,\Delta(n);$$

see [33]. This gives rise to a partial order on the set of isoclasses of representations in $\mathsf{Rep}^0\,\Delta(n)$, called the *degeneration order*. Thus, it also induces a partial order on $\Theta_\triangle^+(n)$ by setting

$$A \leqslant_{\mathrm{dg}} B \Longleftrightarrow M(A) \leqslant_{\mathrm{dg}} M(B).$$

Following [1, 9] we can define the order relation $\preccurlyeq$ on $M_{\triangle,n}(\mathbb{Z})$ as follows. For $A \in M_{\triangle,n}(\mathbb{Z})$ and $i \neq j \in \mathbb{Z}$, let

$$\sigma_{i,j}(A) = \begin{cases} \sum_{s\leqslant i, t\geqslant j} a_{s,t}, & \text{if } i<j,\\ \sum_{s\geqslant i, t\leqslant j} a_{s,t}, & \text{if } i>j. \end{cases}$$

For $A,B \in M_{\triangle,n}(\mathbb{Z})$, define

$$B \preccurlyeq A \text{ if and only if } \sigma_{i,j}(B) \leqslant \sigma_{i,j}(A) \text{ for all } i \neq j.$$

Set $B \prec A$ if $B \preccurlyeq A$, and for some (i,j) with $i \neq j$, $\sigma_{i,j}(B) < \sigma_{i,j}(A)$.

Note that restricting the order relation to $\Theta_\triangle^+(n)$ gives a poset $(\Theta_\triangle^+(n),\preccurlyeq)$. Note also from [9, Th. 6.2] that, if $A,B \in \Theta_\triangle^+(n)$, then

(2.2) $$B \leqslant_{\mathrm{dg}} A \Longleftrightarrow B \preccurlyeq A \text{ and } \mathbf{dim}M(A) = \mathbf{dim}M(B).$$

Thus, $(\Theta_\triangle^+(n),\leqslant_{\mathrm{dg}})$ is also a poset.

An element $\lambda \in \mathbb{N}_\triangle^n$ is called *sincere* if $\lambda_i > 0$ for all $i \in I$. Let

$$I^{\mathrm{sin}} = \{\text{all sincere vectors in } \mathbb{N}_\triangle^n\} \quad \text{and} \quad \widetilde{I} = I \cup I^{\mathrm{sin}}.$$

[3] There exists a geometrical description when the field k is algebraically closed; for details, see [24].

For $X \in \{I, I^{\text{sin}}, \widetilde{I}\}$, let Σ_X be the set of words on the alphabet X and let $\widetilde{\Sigma} = \Sigma_{\widetilde{I}}$.

For each $w = \boldsymbol{a}_1\boldsymbol{a}_2\cdots\boldsymbol{a}_m \in \widetilde{\Sigma}$, we set $M(w) = S_{\boldsymbol{a}_1} * S_{\boldsymbol{a}_2} * \cdots * S_{\boldsymbol{a}_m}$. Then there is a unique $A \in \Theta^+_\triangle(n)$ such that $M(w) \cong M(A)$, and we set $\wp(w) = A$, which induces a surjective map $\wp: \widetilde{\Sigma} \to \Theta^+_\triangle(n), w \mapsto \wp(w)$. Note that $\wp$ induces a surjective map $\wp: \Sigma \to \Theta^{ap}_\triangle(n)$.

For $\boldsymbol{a} \in \widetilde{I}$, set $u_{\boldsymbol{a},q} = u_{[S_{\boldsymbol{a},q}]}$. For any $w = \boldsymbol{a}_1\boldsymbol{a}_2\cdots\boldsymbol{a}_m \in \widetilde{\Sigma}$ and $A \in \Theta^+_\triangle(n)$, repeatedly applying Theorem 2.1 shows that there exists a polynomial $\varphi^A_w \in \mathbb{Z}[\boldsymbol{q}]$ such that $\varphi^A_w(q) = \mathfrak{h}^M_{M_1,M_2,\cdots,M_m}$ with $M_i \cong S_{\boldsymbol{a}_i,q}$ and $M \cong M_q(A)$.

Any word $w = \boldsymbol{a}_1\boldsymbol{a}_2\cdots\boldsymbol{a}_m \in \widetilde{\Sigma}$ can be uniquely expressed in the *tight form* $w = \boldsymbol{b}_1^{e_1}\boldsymbol{b}_2^{e_2}\cdots\boldsymbol{b}_t^{e_t}$ where $e_i = 1$ if $\boldsymbol{b}_i$ is sincere, and e_i is the number of consecutive occurrence of $\boldsymbol{b}_i$ if $\boldsymbol{b}_i \in I$. By [6, Lem. 5.1] (see also the proof of [3, Prop. 9.1]), φ^A_w is divisible by $\prod_{i=1}^t [\![e_i]\!]^!$ for every $A \le \wp(w)$. Thus, there exists $\gamma^A_w \in \mathbb{Z}[\boldsymbol{q}]$ such that

$$\varphi^A_w = \prod_{i=1}^t [\![e_i]\!]^! \gamma^A_w \in \mathbb{Z}[\boldsymbol{q}].$$

Note that the polynomials γ^A_w are also Hall polynomials. In fact, for a finite field k of q elements, we have $\gamma^A_w(q) = \mathfrak{h}^M_{N_1,N_2,\ldots,N_m}$ with $N_i \cong e_i S_{\boldsymbol{b}_i,q}$ and $M \cong M_q(A)$. A word w is called *distinguished* if the Hall polynomial $\gamma^{\wp(w)}_w = 1$.

As a first application, we now use the multiplication formula to prove the existence of Hall polynomials. This result was first given in [12], [27, 8.1].

Theorem 2.2 *The Hall numbers $\mathfrak{h}^{M_q(A)}_{M_q(B),M_q(C)}$ associated with $A, B, C \in \Theta^+_\triangle(n)$ and any prime power q are polynomials in q. In other words, there exist $\varphi^A_{B,C} \in \mathbb{Z}[\boldsymbol{q}]$ such that $\varphi^A_{B,C}(q) = \mathfrak{h}^{M_q(A)}_{M_q(B),M_q(C)}$ for all such q.*

Proof For $w = \boldsymbol{b}_1\boldsymbol{b}_2\cdots\boldsymbol{b}_t \in \widetilde{\Sigma}$, if we write in $\mathfrak{H}^\diamond_\triangle(n,q)$,

$$u_{w,q} = u_{\boldsymbol{b}_1,q} \diamond \cdots \diamond u_{\boldsymbol{b}_m,q} = \sum_{B' \le \wp(w)} \mathfrak{h}^{B'}_w u_{B',q},$$

Then, by Theorem 2.1, there exist polynomials $\varphi^{B'}_w$ such that $\varphi^{B'}_w(q) = \mathfrak{h}^{B'}_w$. Assume now that w is distinguished (see [6, Th. 6.2]) such that $B = \wp(w)$. Then $\varphi^B_w = \prod_{i=1}^t [\![e_i]\!]^!$ and $\varphi^{B'}_w/\varphi^B_w = \gamma^{B'}_w$ are all polynomials.

Now, by Theorem 2.1 again, the Hall numbers in $u_{w,q} \diamond u_{C,q} = \sum_{A \le B*C} \mathfrak{h}^A_{w,C} u_{A,q}$ are the values of certain polynomials $\varphi^A_{w,C}$ at q. On the other hand,

$$\begin{aligned} u_{w,q} \diamond u_{C,q} &= \sum_{B' \le B} \mathfrak{h}^{B'}_w (u_{B',q} \diamond u_{C,q}) \\ &= \mathfrak{h}^B_w (u_{B,q} \diamond u_{C,q}) + \sum_{B' < B} \mathfrak{h}^{B'}_w (u_{B',q} \diamond u_{C,q}) \\ &= \mathfrak{h}^B_w (u_{B,q} \diamond u_{C,q}) + \sum_{A < B*C} \Big(\sum_{B' < B} \mathfrak{h}^{B'}_w \mathfrak{h}^A_{B',C} \Big) u_{A,q}. \end{aligned}$$

https://doi.org/10.4153/CJM-2017-009-4 Published online by Cambridge University Press

By equating coefficients, we see that all polynomials $\varphi^A_{w,C}$ is divisible by φ^B_w. Thus, we have

$$\mathfrak{h}^B_w(u_{B,q} \diamond u_{C,q}) = \sum_{A \leq B*C} \mathfrak{h}^A_{w,C} u_{A,q} - \sum_{A<B*C} \Big(\sum_{B'<B} \mathfrak{h}^{B'}_w \mathfrak{h}^A_{B',C} \Big) u_{A,q}.$$

Now the assertion follows from induction on $\leq$. ∎

In Section 4, we will give algorithms to compute distinguished words w_A associated with each $A \in \Theta^+_\triangle(n)$ and to derive a recursive formula for Hall polynomials.

3 Proof of Theorem 2.1

Recall that a matrix over a field in row-echelon form is said to be in *reduced row-echelon form* (RREF) if every leading column has 1 at the leading entry and 0 elsewhere.

Lemma 3.1 *Let $\mathcal{R}_{m,n} \subseteq M_{m,n}(\mathbb{F}_q)$ be the subset consisting of all $m \times n$ matrices in reduced row-echelon form and of rank m. Then*

$$|\mathcal{R}_{m,n}| = \left[\!\!\left[\begin{matrix} n \\ m \end{matrix} \right]\!\!\right]_{v^2=q}.$$

Proof Let $\mathcal{V}_{m,n}$ be the set of all dimension m subspaces of $\mathbb{F}^n_q$. Then, for $T \in \mathcal{R}_{m,n}$, the rows of T spans a subspace V_T of dimension m. Thus, we have a map

$$f: \mathcal{R}_{m,n} \longrightarrow \mathcal{V}_{m,n}, \quad T \longmapsto V_T.$$

Clearly, f is surjective. It is not hard to see that f is also injective. Now the assertion follows from the bijection. ∎

Proposition 3.2 *For $i \in I$, $a_t, d_t, m \in \mathbb{Z}$ with $a_t \geqslant d_t \geqslant 0$, $m \geqslant 1$, $t = 1, 2, \dots, m$, and representations*

$$\begin{aligned} &L = a_1 S_i \oplus a_2 S_i[2] \oplus \cdots \oplus a_m S_i[m], \quad M = (d_1 + \cdots + d_m) S_i, \text{ and} \\ &N = (a_1 - d_1) S_i \oplus \big((a_2 - d_2) S_i[2] \oplus d_2 S_{i+1} \big) \oplus \cdots \\ &\qquad\qquad \oplus \big((a_m - d_m) S_i[m] \oplus d_m S_{i+1}[m-1] \big), \end{aligned}$$

in $\mathrm{Rep}^0_k(\Delta(n))$, the Hall number $\mathfrak{h}^L_{M,N}$ is a polynomial in $q = q_k$:

$$\mathfrak{h}^L_{M,N} = q^{\sum_{1 \leqslant k < l \leqslant m} d_k(a_l - d_l)} \left[\!\!\left[\begin{matrix} a_1 \\ d_1 \end{matrix} \right]\!\!\right]_q \left[\!\!\left[\begin{matrix} a_2 \\ d_2 \end{matrix} \right]\!\!\right]_q \cdots \left[\!\!\left[\begin{matrix} a_m \\ d_m \end{matrix} \right]\!\!\right]_q.$$

Proof Without loss of generality, we may assume $i = 1$. Represent the modules L, N by vector spaces and linear maps around the cyclic quiver as follows (*cf.* (2.1)):

$$L: k^{a_1+a_2+\cdots+a_m} \xrightarrow{p_1} k^{a_2+a_3+\cdots+a_m} \xrightarrow{p_2} k^{a_3+\cdots+a_m} \xrightarrow{p_3} \cdots \xrightarrow{p_{m-2}} k^{a_{m-1}+a_m} \xrightarrow{p_{m-1}} k^{a_m}$$

$$\begin{aligned} N: k^{a_1-d_1+a_2-d_2+\cdots+a_m-d_m} &\xrightarrow{f} k^{a_2+a_3+\cdots+a_m} \xrightarrow{p_2} k^{a_3+\cdots+a_m} \xrightarrow{p_3} \cdots \\ &\xrightarrow{p_{m-2}} k^{a_{m-1}+a_m} \xrightarrow{p_{m-1}} k^{a_m}. \end{aligned}$$

Here p_i is the projection map defined by the matrix $(0_{a_i}, I_{\widetilde{a}_{i+1}})$, where

$$\widetilde{a}_i := a_i + \cdots + a_m$$

and 0_{a_i} is the $\widetilde{a}_{i+1} \times a_i$ zero matrix, while f is the restriction of p_1. Thus, f projects the component $k^{a_1-d_1}$ to 0 and imbeds the component $k^{a_i-d_i}$ for $i \geq 2$ into the component k^{a_i} via the $a_i \times (a_i - d_i)$ matrix $J_i = \binom{I_{a_i-d_i}}{0}$. In other words, f is defined by the $\widetilde{a}_2 \times (\widetilde{a}_1 - \widetilde{d}_1)$ matrix A whose first $a_1 - d_1$ columns are zero columns and having blocks $J_1, J_2, \ldots, J_m$ on the diagonal of the remaining submatrix.

Let $U \leq L$ be a submodule such that $U \cong N$, $L/U \cong M$. Then $U = \mathsf{Ker}(g)$ for some module epimorphism $g: L \to M$. Thus, the short exact sequence $0 \to U \to L \to M \to 0$ gives the following commutative diagram:

$$\begin{tikzcd}
U \arrow[d, "\iota"] & \mathsf{Ker}\, g_1 \arrow[r, "p_1"] \arrow[d, "\iota_1"] & k^{a_2+a_3+\cdots+a_m} \arrow[r, "p_2"] \arrow[d, "\mathrm{id}"] & k^{a_3+\cdots+a_m} \arrow[r, "p_3"] \arrow[d, "id"] & \cdots \arrow[r, "p_{m-2}"] & k^{a_{m-1}+a_m} \arrow[r, "p_{m-1}"] \arrow[d, "id"] & k^{a_m} \arrow[d, "id"] \\
L \arrow[d, "g"] & k^{a_1+a_2+\cdots+a_m} \arrow[r, "p_1"] \arrow[d, "g_1"] & k^{a_2+a_3+\cdots+a_m} \arrow[r, "p_2"] \arrow[d] & k^{a_3+\cdots+a_m} \arrow[r, "p_3"] \arrow[d] & \cdots \arrow[r, "p_{m-2}"] & k^{a_{m-1}+a_m} \arrow[r, "p_{m-1}"] \arrow[d] & k^{a_m} \arrow[d] \\
M \arrow[d] & k^{d_1+d_2+\cdots+d_m} \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & 0 \\
0 & 0
\end{tikzcd}$$

Since g is surjective, it is easy to see $\mathsf{Ker}\, g_1 \cong k^{a_1-d_1+\cdots+a_m-d_m}$ as vector spaces. Represent the linear map $g_1: k^{a_1+\cdots+a_m} \to k^{d_1+\cdots+d_m}$ by a $\widetilde{d}_1 \times \widetilde{a}_1$ matrix T_U in reduced row-echelon form. Since g_1 is onto, T_U is an upper triangular matrix with $\widetilde{d}_1$ leading columns and $\ell = \widetilde{a}_1 - \widetilde{d}_1$ non-leading columns, corresponding to ℓ free variables $x_{i_1}, x_{i_2}, \ldots, x_{i_\ell}$. Let v_j be the solution to $T_U x = 0$ obtained by setting $x_{i_j} = 1$ and other free variables to 0. Then, $\mathsf{Ker}\, g_1$ has a basis $v_1, v_2, \ldots, v_\ell$.

Since $U \cong N$, there exists a linear isomorphism $\phi = (\phi_1, \phi_2, \cdots, \phi_m)$ making the following diagram commute

$$\begin{tikzcd}
U \arrow[d, "\phi", "\cong"'] & \mathsf{Ker}\, g_1 \arrow[r, "p_1"] \arrow[d, "\phi_1"] & k^{a_2+a_3+\cdots+a_m} \arrow[r, "p_2"] \arrow[d, "\phi_2"] & k^{a_3+\cdots+a_m} \arrow[r, "p_3"] \arrow[d, "\phi_3"] & \cdots \arrow[r, "p_{m-2}"] & k^{a_{m-1}+a_m} \arrow[r, "p_{m-1}"] \arrow[d, "\phi_{m-1}"] & k^{a_m} \arrow[d, "\phi_m"] \\
N & k^{a_1-d_1+a_2-d_2+\cdots+a_m-d_m} \arrow[r, "f"] & k^{a_2+a_3+\cdots+a_m} \arrow[r, "p_2"] & k^{a_3+\cdots+a_m} \arrow[r, "p_3"] & \cdots \arrow[r, "p_{m-2}"] & k^{a_{m-1}+a_m} \arrow[r, "p_{m-1}"] & k^{a_m}
\end{tikzcd}$$

Hence, the images of $p_i \cdots p_2 p_1$ in the top row maps must have the same dimension as that of the map $p_i \cdots p_2 f$ below. Since the dimension of $\mathsf{Im}(f)$ is $\widetilde{a}_2 - \widetilde{d}_2$, p_1 must send $v_1, \ldots, v_{a_1-d_1}$ to 0. This forces the first a_1 columns contains d_1 leading columns. Similarly, $\dim \mathsf{Im}(p_2 p_1) = \dim \mathsf{Im}(p_2 f)$ forces the next a_2 columns in T_U contains d_2 leading columns, and so on. This proves that, if T_U is divided in $d_i \times a_j$ blocks, then T_U is upper triangular with m $(d_i \times a_i)$-blocks on the diagonal each of which has rank d_i.

Let $\mathcal{T}$ be the subset of all $T \in M_{\widetilde{d}_1, \widetilde{a}_1}(\mathbb{F}_q)$ such that T is in RREF and T has m $(d_i \times a_i)$-blocks B_i on the diagonal each of which has rank d_i. The argument above shows that the map $U \mapsto T_U$ is a bijection from the set $\{U \subseteq L \mid U \cong N, L/U \cong M\}$ to $\mathcal{T}$. Hence, $\mathfrak{h}^L_{M,N} = |\mathcal{T}|$.

Now, to form such a matrix T, by Lemma 3.1, the number of the $(d_1 \times a_1)$-block B_1 is $\left[\!\left[{a_1 \atop d_1} \right]\!\right]$ and the number of other $(d_1 \times a_i)$-blocks for $i \geq 2$ in the first d_1 rows is $q^{d_1(a_2-d_2+a_3-d_3+\cdots+a_m-d_m)}$. Counting the number of the blocks in the next d_2 rows, d_3 rows, ..., similarly, yields

$$|\mathcal{T}| = q^{d_1(a_2-d_2+a_3-d_3+\cdots+a_m-d_m)} \left[\!\!\left[\begin{matrix} a_1 \\ d_1 \end{matrix} \right]\!\!\right]_q \times q^{d_2(a_3-d_3+\cdots+a_m-d_m)} \left[\!\!\left[\begin{matrix} a_2 \\ d_2 \end{matrix} \right]\!\!\right]_q \times \cdots \\ \times q^{d_{m-1}(a_m-d_m)} \left[\!\!\left[\begin{matrix} m \\ d_m \end{matrix} \right]\!\!\right]_q,$$

as desired. ∎

Remark 3.3 A dual version of the above result, where the roles M and N are swapped, is known in [28, §2.2] and was used in [14, Lem. 2.3.5]. Unlike the representation-theoretic proof above, the proof in loc. cit. involves the geometry of the Grassmanian variety.

Lemma 3.4 *For nilpotent representations L, M, N of $\triangle(n)$, if $N \leq L$ and $L/N \cong M$ is semisimple, then there exist submodules $L_i \leq L$, $N_i \leq N$, and $M_i \leq M$ such that $L = \bigoplus_{i=1}^n L_i$, $N = \bigoplus_{i=1}^n N_i$, $M = \bigoplus_{i=1}^n M_i$, and*

$$\mathfrak{h}^L_{M,N} = \prod_{i=1}^n \mathfrak{h}^{L_i}_{M_i,N_i}.$$

Proof Let $\mathrm{top}(L)_i$ denote the isotypic component of $\mathrm{top}(L)$ associated with S_i. Then $L = \oplus_{i=1}^n L_i$ where $\mathrm{top}(L_i) = \mathrm{top}(L)_i$. Thus, if M_i denotes the isotypic component of M associated with S_i and $\pi: L \to M$ denotes the quotient map, then restriction defines an epimorphism $\pi_i = \pi|_{L_i}: L_i \to M_i$. Let $N_i = \pi_i^{-1}(M_i)$. Then $N_i = L_i \cap N$ and $N = \oplus_{i=1}^n N_i$. Now, our assertion follows from the following bijection:

$$\prod_{i+1}^n \{U_i \leq L_i \mid U_i \cong N_i, L_i/U_i \cong M_i\} \longrightarrow \{U \leq L \mid U \cong N, L/U \cong M\}, \\ (U_1, \ldots, U_n) \longmapsto U_1 + \cdots + U_n,$$

noting that $U = (U \cap L_1) + \cdots + (U \cap L_n)$. ∎

We are now ready to give a representation-theoretic proof for the multiplication formula in [10, Th. 4.5]. As mentioned in the introduction, this formula is the restriction of certain multiplication formulas to the positive part for the quantum loop algebra of $\mathfrak{gl}_n$ [10, Prop. 4.2], which is obtained from lifting some multiplication formulas in the affine q-Schur algebras associated with the affine Hecke algebra. See [14, Prop. 2.3.6] for a geometric proof building on the Hall polynomials computed in [28, §2.2].

Proof of Theorem 2.1 We first claim that if L is an extension of the semisimple representation S_α by $N = M(A)$, then $L \cong M(A + T - \widehat{T}^+)$ for some $T \in \Theta^+_\triangle(n)$

https://doi.org/10.4153/CJM-2017-009-4 Published online by Cambridge University Press

with $\alpha = \mathrm{row}(T)$. Indeed, suppose $L \cong M(C)$ for some $C = (c_{i,j})$ and decompose $L = \oplus_{i=1}^{n} L_i$ as in Lemma 3.4. If $U \leq L$ is a submodule isomorphic to N, then there exist $t_{ij} \in \mathbb{N}$ such that $U_i = U \cap L_i \cong \bigoplus_{i<j}((c_{ij} - t_{ij})S_i[j-i] \oplus t_{ij}S_{i+1}[j-i-1])$, where $\sum_{i<j} t_{ij} = \alpha_i$. Thus, $U \cong N$ becomes

$$\bigoplus_{i=1}^{n} \bigoplus_{i<j} \big((c_{ij} - t_{ij})S_i[j-i] \oplus t_{ij}S_{i+1}[j-i-1] \big) \cong \bigoplus_{i=1}^{n} \bigoplus_{i<j} a_{ij}S_i[j-i].$$

By the Krull–Remak–Schmidt theorem, we have

$$c_{ij} - t_{ij} + t_{i-1,j} = a_{ij} \quad \text{for all } i<j \text{ with } i = 1,2,\cdots,n. \tag{3.1}$$

Hence, if we form the upper triangular matrix $T = (t_{i,j}) \in \Theta_{\triangle}^{+}(n)$, then $C = A + T - \widehat{T}^{+}$, proving the claim.

For $C = A + T - \widehat{T}^{+}$, by Lemma 3.4, we have

$$\mathfrak{h}_{S_\alpha,A}^{C} = \prod_{i=1}^{n} \mathfrak{h}_{M_i,N_i}^{L_i},$$

where

$$L_i \cong \bigoplus_{j>i} (a_{ij} + t_{ij} - t_{i-1,j})S_i[j-i], \quad M_i \cong \bigoplus_{j>i} t_{ij}S_i, \quad \text{and}$$
$$N_i \cong \bigoplus_{j>i} (a_{ij} - t_{i-1,j})S_i[j-i] \oplus t_{ij}S_{i+1}[j-i-1].$$

Applying Proposition 3.2 with $a_l = a_{i,i+l} + t_{i,i+l} - t_{i-1,i+l}$, $d_l = t_{i,i+l}$ yields

$$\mathfrak{h}_{M_i,N_i}^{L_i} = q^{\sum_{\substack{l,j\in\mathbb{Z}\\ i<l<j}} t_{il}(a_{ij}-t_{i-1,j})} \prod_{j\in\mathbb{Z},i<j} \begin{bmatrix} a_{ij} + t_{ij} - t_{i-1,j} \\ t_{ij} \end{bmatrix}_q \qquad (q = q_k).$$

Finally, it remains to prove

$$\sum_{i=1}^{n} \sum_{i<l<j} t_{il}(a_{ij} - t_{i-1,j}) = \sum_{i=1}^{n} \sum_{i<l<j} (a_{ij}t_{il} - t_{ij}t_{i+1,l}), \tag{3.2}$$

or, equivalently, to prove

$$\sum_{\substack{1\leqslant i\leqslant n\\ i<l<j}} t_{il}t_{i-1,j} = \sum_{\substack{1\leqslant i\leqslant n\\ i<l<j}} t_{ij}t_{i+1,l}.$$

This follows from the fact that the sets $J_1 = \{t_{il}t_{i-1,j} \neq 0 \mid 1 \leqslant i \leqslant n, i<l<j\}$ and $J_2 = \{t_{ij}t_{i+1,l} \neq 0 \mid 1 \leqslant i \leqslant n, i<l<j\}$ are identical. To see this, take $t_{il}t_{i-1,j} \in J_1$ where $i<l<j$. If $2 \leqslant i \leqslant n$, then $t_{i-1,j}t_{(i-1)+1,l} \in J_2$. If $i = 1$, then $t_{1,l}t_{0,j} = t_{n,n+j}t_{n+1,l+n} \in J_2$. Hence, $J_1 \subseteq J_2$. Similarly, $J_2 \subseteq J_1$ and so $J_1 = J_2$. ∎

Corollary 3.5 (i) *By the extension of modules, we have*

$$t_{ij} \in \begin{cases} [0, \min\{\alpha_i, a_{i+1,j}\}], & \text{if } |j-i| > 1, \\ [0, \alpha_i], & \text{if } |j-i| = 1, \end{cases}$$

and for any $i = 1,2,\dots,n$, $\sum_{j>i} t_{ij} = \alpha_i$.

(ii) *The power of* $\boldsymbol{q}$, $\sum_{\substack{1\leqslant i\leqslant n\\ i<l<j}}(a_{ij}t_{il} - t_{ij}t_{i+1,l})$, *is non-negative.*

Proof Since $c_{ij} \geqslant t_{ij}$, it follows from (3.1) that $a_{ij} \geqslant t_{i-1,j}$, proving (i). Then (ii) follows from (3.2). ∎

4 Distinguished Words and a Recursive Formula

For $A \in \Theta^+_\triangle(n)$, denote by $\ell(A) = \ell(M(A))$ the *Loewy length* of $M(A)$ and define the *periodicity* of $M(A)$ by

$$p(A) = \begin{cases} \max\{l \in \mathbb{N} \mid a_{i,i+l} \neq 0 \text{ for all } 1 \leqslant i \leqslant n\}, & \text{if } A \text{ is periodic,} \\ 0, & \text{if } A \text{ is aperiodic.} \end{cases}$$

Clearly, $0 \leqslant p(A) \leqslant \ell(A)$. Thus, $p(A) = 0$ means that A is aperiodic. If $p(A) = \ell(A)$, A is called *strongly periodic*.

We now record several results in [6] stated in multisegments in terms of matrices. Note that if Π is the set of all multisegments, then there is a bijection

$$\Pi \longrightarrow \Theta^+_\triangle(n), \quad \pi = \sum_{i\in I, l\geqslant 1} \pi_{i,l}[i;l) \longmapsto A_\pi = (a_{i,i+l})_{i\in I, l\geqslant 1} \text{ with } a_{i,i+l} = \pi_{i,l}.$$

Proposition 4.1 ([6, §4]) (i) *For any $A \in \Theta^+_\triangle(n)$, there exists uniquely a pair (A', A'') associated with A such that A' is strongly periodic, A'' is aperiodic, and $M(A) \cong M(A'') * M(A')$.*

(ii) *For aperiodic part A'', there exists a distinguished word $w_{A''} = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \Sigma_I \cap \wp^{-1}(A'')$.*

(iii) *For strongly periodic part A', there exists a distinguished word $w_{A'} = \boldsymbol{a}_1\boldsymbol{a}_2\cdots\boldsymbol{a}_p \in \Sigma_{I^{sin}} \cap \wp^{-1}(A')$, moreover, $S_{\boldsymbol{a}_s} \cong \mathrm{soc}^{p-s+1} M(A')/\mathrm{soc}^{p-s} M(A'), 1 \leqslant s \leqslant p = p(A)$.*

(iv) *$w_{A''}w_{A'} = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t}\boldsymbol{a}_1\boldsymbol{a}_2\cdots\boldsymbol{a}_p$ is a distinguished word of A.*

A construction of distinguished words of the strongly periodic part and aperiodic part has been given in [6]. Building on this, we now introduce some matrix algorithms to compute certain distinguished words in order to provide a monomial basis for computing the canonical basis.

If we take $A = (a_{i,j})$, then $M(A) = \bigoplus_{i=1}^n \bigoplus_{j>i} a_{ij} S_i[j-i]$ and $\mathrm{soc}(S_i[j-i]) = S_{j-1}$, $\mathrm{soc}^2(S_i[j-i]) = S_{j-2}[2], \ldots, \mathrm{soc}^l(S_i[j-i]) = S_{j-l}[l]$. Here we understand $j - l \equiv j' (\mathrm{mod}\, n)$ and if $l \geqslant j-i$, $\mathrm{soc}^l(S_i[j-i]) = S_i[j-i]$.

We review the construction of producing the unique pair (A', A'') in Proposition 4.1(i). For $A \in \Theta^+_\triangle(n)$ with $p = p(A)$, then $\mathrm{soc}^p(M(A)) = M(A')$ and $M(A'') \cong M(A)/M(A')$.

Definition 4.2 For $A \in \Theta^+_\triangle(n)$ with $p = p(A)$, define the *distinguished pair* (A', A'') as follows.

(i) The matrix $A' = (a'_{i,j})$, called the *strongly periodic part* of A, is obtained by setting

$$a'_{i,j} = \begin{cases} a_{i,j}, & \text{if } j < i + p, \\ \sum_{i_0 \leqslant i} a_{i_0,j}, & \text{if } j = i + p. \end{cases}$$

https://doi.org/10.4153/CJM-2017-009-4 Published online by Cambridge University Press

In other words, A' is the matrix obtained by replacing the "p-th-diagonal" $(a_{i,i+p})_{i\in\mathbb{Z}}$ by $\mathrm{col}(B)$, where B is the matrix obtained from A by vanishing all the entries below the p-th-diagonal.

(ii) The matrix $A'' = (a''_{i,j})$, called the *aperiodic part*, is obtained by setting

$$a''_{i,j} = a_{i,j+p}.$$

First, based on the structure of $\mathrm{soc}^t\, M(A)$ for strongly aperiodic $A \in \Theta^+_\triangle(n)$, $t \in \mathbb{N}$, we give a matrix algorithm of [6, Lemma 4.2] as follows.

Algorithm 4.3 (for the strongly periodic part) Suppose A' is strongly periodic. Then $p = p(A') = \ell(A')$ and the algorithm runs p steps:

put $B = (b_{i,j}) := A'$
for j from 1 to p do
$T := \sum_{i=1}^{n} b_{i,i+p-j+1}E_{i,i+p-j+1}$, $B := B - T + \widehat{T}^+$, $\boldsymbol{a}_j = \mathrm{row}(T)$ enddo
output

$$w_{A'} = \boldsymbol{a}_1\boldsymbol{a}_2\cdots\boldsymbol{a}_p.$$

Remark 4.4 Every $\boldsymbol{a}_i$ is sincere and is uniquely determined by A. For $\lambda = (\lambda_i)_{i\in\mathbb{Z}} \in \mathbb{N}^n_\triangle$, set $\lambda^{[1]} = (\lambda^{[1]}_i)_{i\in\mathbb{Z}}$, where $\lambda^{[1]}_i = \lambda_{i-1}$ for all $i \in \mathbb{Z}$. It is easy to prove that there is one to one correspondence between strongly periodic matrix A with $\ell(A) = p$ and a sincere sequence $\boldsymbol{a}_1\boldsymbol{a}_2\cdots\boldsymbol{a}_p$ with $\boldsymbol{a}^{[1]}_i \leqslant \boldsymbol{a}_{i+1}$, for $1 \leqslant i \leqslant p-1$.

Second, for $B = (b_{i,j}) \in \Theta^{ap}_\triangle(n)$ and $i \in I$, we set $M(B) = \oplus_{i\in I}M_i(B)$ and $M_i(B) = \oplus_{j>i}b_{i,j}S_i[j-i]$. We take the maximal index in every step in [6, Prop. 4.3]; then we give the following matrix algorithm.

Algorithm 4.5 (for the aperiodic part) Suppose A'' is aperiodic with $l = \ell(A'')$; consider the following run:

put $B = (b_{i,j}) := A''$; for i from 1 to l, do
if the $(l-i+1)th$ diagonal $b_{1,1+l-i+1}, b_{2,2+l-i+1}, \ldots, b_{n,n+l-i+1}$ is nonzero, choose the rightmost $b_{j,j+l-i+1} \neq 0$ such that $b_{j+1,j+1+l-i+1} \neq 0$; choose the minimal $j' \leq l-i+1$ such that $b_{j,j+j'} \neq 0$ and $j' > \ell(M_{j+1}(B))$;
do

$$T := \sum_{k=j'}^{l-i+1} b_{j,j+k}E_{j,j+k}, \quad B := B - T + \widehat{T}^+, \quad e_{i,j} := \sum_{k=j'}^{l-i+1} b_{j,j+k}, \quad \boldsymbol{x}_{i,j} = j^{e_{i,j}};$$

enddo; loop until the $(l-i+1)th$ diagonal is zero.
next i; enddo;
output

$$w_{A''} = \boldsymbol{x}_{1,j_1}\cdots\boldsymbol{x}_{1,j_a}\cdots\boldsymbol{x}_{l,k_1}\cdots\boldsymbol{x}_{l,k_b}$$

The two algorithms give a *distinguished section*

$$\mathscr{W}(n) = \left\{ w_A = w_{A''}w_{A'} \in \wp^{-1}(A) \cap \widetilde{\Sigma} \mid A \in \Theta^+_\triangle(n) \right\}.$$

When restricting to $\Theta^{ap}_\triangle(n)$, we obtain a distinguished section of Σ over $\Theta^{ap}_\triangle(n)$.

https://doi.org/10.4153/CJM-2017-009-4 Published online by Cambridge University Press

We explain the algorithms by the following example. Recall that every matrix in $\Theta^+_\triangle(n)$ is identified as its core. Sometimes, we indicate the diagonal with boldface entries for clarity.

Example 4.6 Suppose $n = 3$ and

$$A = \begin{pmatrix} \mathbf{0} & 1 & 1 & 0 & 3 & 1 & 2 & 1 & 3 \\ 0 & \mathbf{0} & 0 & 2 & 3 & 1 & 0 & 1 & 1 \\ 0 & 0 & \mathbf{0} & 3 & 0 & 1 & 1 & 1 & 0 \end{pmatrix};$$

then $p(A) = 4$, $\ell(A) = 8$ and

$$A' = \begin{pmatrix} \mathbf{0} & 1 & 1 & 0 & 6 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 2 & 3 & 6 & 0 \\ 0 & 0 & \mathbf{0} & 3 & 0 & 1 & 3 \end{pmatrix}, \quad A'' = \begin{pmatrix} \mathbf{0} & 1 & 2 & 1 & 3 \\ 0 & \mathbf{0} & 0 & 1 & 1 \\ 0 & 0 & \mathbf{0} & 1 & 0 \end{pmatrix},$$

with $\ell(A') = \ell(A'') = 4$.

Applying Algorithm 4.3 to A' gives

$$\begin{aligned}
&i=1: && T = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 3 \end{pmatrix}, && B = \begin{pmatrix} \mathbf{0} & 1 & 1 & 3 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 2 & 9 & 0 \\ 0 & 0 & \mathbf{0} & 3 & 0 & 7 \end{pmatrix}, && \boldsymbol{a}_1 = (6,6,3),\\
&i=2: && T = \begin{pmatrix} \mathbf{0} & 0 & 0 & 3 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 & 9 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 & 7 \end{pmatrix}, && B = \begin{pmatrix} \mathbf{0} & 1 & 8 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 5 & 0 \\ 0 & 0 & \mathbf{0} & 3 & 9 \end{pmatrix}, && \boldsymbol{a}_2 = (3,9,7),\\
&i=3: && T = \begin{pmatrix} \mathbf{0} & 0 & 8 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 5 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 9 \end{pmatrix}, && B = \begin{pmatrix} \mathbf{0} & 10 & 0 & 0 \\ 0 & \mathbf{0} & 8 & 0 \\ 0 & 0 & \mathbf{0} & 8 \end{pmatrix}, && \boldsymbol{a}_3 = (8,5,9),\\
&i=4: && T = \begin{pmatrix} \mathbf{0} & 10 & 0 & 0 \\ 0 & \mathbf{0} & 8 & 0 \\ 0 & 0 & \mathbf{0} & 8 \end{pmatrix}, && B = 0, && \boldsymbol{a}_4 = (10,8,8).
\end{aligned}$$

The algorithm stops with the output $w_{A'} = \boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3\boldsymbol{a}_4$.

Applying Algorithm 4.5 to A'' gives

$$\begin{aligned}
&i=1: && T = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 & 3 \\ 0 & \mathbf{0} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 \end{pmatrix}, && B = \begin{pmatrix} \mathbf{0} & 1 & 2 & 1 & 0 \\ 0 & \mathbf{0} & 0 & 1 & 4 \\ 0 & 0 & \mathbf{0} & 1 & 0 \end{pmatrix}, && \boldsymbol{x}_{1,1} = 1^3,\\
&i=2: && T = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 1 & 4 \\ 0 & 0 & \mathbf{0} & 0 & 0 \end{pmatrix}, && B = \begin{pmatrix} \mathbf{0} & 1 & 2 & 1 & 0 \\ 0 & \mathbf{0} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 2 & 4 \end{pmatrix}, && \boldsymbol{x}_{2,2} = 2^5,\\
& && T = \begin{pmatrix} \mathbf{0} & 1 & 2 & 1 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 \end{pmatrix}, && A = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & 2 & 1 & 0 \\ 0 & 0 & \mathbf{0} & 2 & 4 \end{pmatrix}, && \boldsymbol{x}_{2,1} = 1^4,\\
&i=3: && T = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & 2 & 4 \end{pmatrix}, && B = \begin{pmatrix} \mathbf{0} & 4 & 0 & 0 & 0 \\ 0 & \mathbf{0} & 2 & 1 & 0 \\ 0 & 0 & \mathbf{0} & 0 & 0 \end{pmatrix}, && \boldsymbol{x}_{3,3} = 3^6,\\
& && T = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & \mathbf{0} & 2 & 1 \\ 0 & 0 & \mathbf{0} & 0 \end{pmatrix}, && B = \begin{pmatrix} \mathbf{0} & 4 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{0} & 1 \end{pmatrix}, && \boldsymbol{x}_{3,2} = 2^3,\\
&i=4: && T = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{0} & 1 \end{pmatrix}, && B = \begin{pmatrix} \mathbf{0} & 4 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 \end{pmatrix}, && \boldsymbol{x}_{4,3} = 3^1,\\
& && T = \begin{pmatrix} \mathbf{0} & 4 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 \end{pmatrix}, && B = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & \mathbf{0} & 0 \end{pmatrix}, && \boldsymbol{x}_{4,1} = 1^4.
\end{aligned}$$

The algorithm has output $w_{A''} = \boldsymbol{x}_{1,1}\boldsymbol{x}_{2,2}\boldsymbol{x}_{2,1}\boldsymbol{x}_{3,3}\boldsymbol{x}_{3,2}\boldsymbol{x}_{4,3}\boldsymbol{x}_{4,1}$. Thus, it produces the following distinguished word associated with A:

$$w_A = w_{A''}w_{A'} = \boldsymbol{x}_{1,1}\boldsymbol{x}_{2,2}\boldsymbol{x}_{2,1}\boldsymbol{x}_{3,3}\boldsymbol{x}_{3,2}\boldsymbol{x}_{4,3}\boldsymbol{x}_{4,1}\boldsymbol{a}_1\boldsymbol{a}_2\boldsymbol{a}_3\boldsymbol{a}_4.$$

Remark 4.7 In [18], a different matrix algorithm is used to get a certain triangular relation similar to [1, Prop. 3.9] for the affine q-Schur algebra $\mathcal{S}$. However, it is not clear if such a relation can be lifted to a relation similar to [1, 5.4(c)]. Thus, it is not clear how their algorithm produces a monomial basis for the Ringel–Hall algebra $\mathfrak{H}_\triangle(n)$ (or the $+$-part of the quantum affine $\mathfrak{gl}_n$).

For a fixed $A \in \Theta^+_\triangle(n)$, let

$$\Theta_A = (0,A] := \left\{ B \in \Theta^+_\triangle(n) \mid B \leqslant_{\rm dg} A \right\} \quad \text{and} \quad \Theta_{\prec A} = \left\{ B \in \Theta_A \mid B \prec A \right\}. \tag{4.1}$$

The proof of Theorem 2.2 shows that every $\varphi^A_{w_B,C}$ is divisible by $\varphi^B_{w_B}$. Let $\gamma^A_{w_B,C} = \varphi^A_{w_B,C}/\varphi^B_{w_B}$. The following result shows that the Hall polynomials $\varphi^A_{B,C}$ can be computed by a recursive formula.

Corollary 4.8 *For any $A,B,C \in \Theta^+_\triangle(n)$, let w_B be the distinguished word obtained by applying Algorithms 4.3 and 4.5 to B and, for any $B' \leqslant_{\rm dg} B$, let $\gamma^{B'}_{w_B}$ and $\gamma^A_{w_B,C}$ be obtained by the multiplication formula given in Theorem 2.1. Then the Hall polynomial $\varphi^A_{B,C}$ can be computed by the recursive formula*

$$\varphi^A_{B,C} = \begin{cases} \gamma^A_{w_B,C} - \sum_{B':B' \prec B} \gamma^{B'}_{w_B} \varphi^A_{B',C}, & \text{if } A \in \bigcup_{B' \prec B} \Theta_{\prec B' * C}; \\ \gamma^A_{w_B,C}, & \text{if } A \in \Theta_{B*C} \setminus \bigcup_{B' \prec B} \Theta_{\prec B' * C}. \end{cases}$$

5 Ringel–Hall Algebras, Quantum Affine $\mathfrak{gl}_n$, and their Canonical Bases

The generic Hall algebra $\mathfrak{H}^\diamond_\triangle(n)$ of $\Delta(n)$ is by definition the free $\mathbb{Z}[\boldsymbol{q}]$-module with basis $\{u_A := u_{[M(A)]} \mid A \in \Theta^+_\triangle(n)\}$ and multiplication given by

$$u_B \diamond u_C = \sum_{A \in \Theta^+_\triangle(n)} \varphi^A_{B,C} u_A.$$

For a finite field k of q elements, by specializing $\boldsymbol{q}$ to q, we obtain the integral Hall algebra $\mathfrak{H}^\diamond_\triangle(n,q)$ associated with $\mathsf{Rep}^0\,\Delta(n)$, as discussed in Sections 2–4.

C. M. Ringel [25, 26] further twisted the multiplication, using the Euler form, to obtain the Ringel–Hall algebra that connects to the corresponding quantum group.

For $\mathbf{a} = (a_i) \in \mathbb{Z}^n_\triangle$ and $\mathbf{b} = (b_i) \in \mathbb{Z}^n_\triangle$, the Euler form associated with the cyclic quiver $\Delta(n)$ is the bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle : \mathbb{Z}^n_\triangle \times \mathbb{Z}^n_\triangle \to \mathbb{Z}$ defined by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i \in I} a_i b_i - \sum_{i \in I} a_i b_{i+1}.$$

The (generic) Ringel–Hall algebra $\mathfrak{H}_\triangle(n)$ of $\Delta(n)$ is by definition the algebra over $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ ($v^2 = \boldsymbol{q}$) with basis $\{u_A = u_{[M(A)]} \mid A \in \Theta^+_\triangle(n)\}$ and the multiplication is twisted by the Euler form

$$u_B u_C = v^{\langle \mathbf{dim} M(B), \mathbf{dim} M(C) \rangle} \sum_{A \in \Theta^+_\triangle(n)} \varphi^A_{B,C} u_A.$$

It is well known that for two $A, B \in \Theta^+_\triangle(n)$, there holds

$$\left\langle \mathbf{dim} M(A), \mathbf{dim} M(B) \right\rangle = \dim_k \mathsf{Hom}\big(M(A), M(B)\big) - \dim_k \mathsf{Ext}^1\big(M(A), M(B)\big).$$

The $\mathcal{Z}$-subalgebra $\mathfrak{C}_\triangle(n)$ of $\mathfrak{H}_\triangle(n)$ generated by $u_i^{(m)} = u_i^m/[m]!$, $i \in I$ and $m \geqslant 1$, is called the *(generic) composition subalgebra*. Then $\mathfrak{C}_\triangle(n)$ is also generated by $u_{[mS_i]}$, since $u_i^{(m)} = v^{m(m-1)} u_{[mS_i]}$. Clearly, $\mathfrak{H}_\triangle(n)$ and $\mathfrak{C}_\triangle(n)$ admit natural $\mathbb{N}^n$-grading by dimension vectors:

$$\mathfrak{H}_\triangle(n) = \bigoplus_{\mathbf{d}\in\mathbb{N}^n} \mathfrak{H}_\triangle(n)_{\mathbf{d}} \quad \text{and} \quad \mathfrak{C}_\triangle(n) = \bigoplus_{\mathbf{d}\in\mathbb{N}^n} \mathfrak{C}_\triangle(n)_{\mathbf{d}},$$

where $\mathfrak{H}_\triangle(n)_{\mathbf{d}}$ is spanned by all u_A with $\mathbf{dim}M(A) = \mathbf{d}$ and $\mathfrak{C}_\triangle(n)_{\mathbf{d}} = \mathfrak{C}_\triangle(n) \cap \mathfrak{H}_\triangle(n)_{\mathbf{d}}$.

Base change gives the $\mathbb{Q}(v)$-algebra $\boldsymbol{\mathfrak{H}}_\triangle(n) = \mathfrak{H}_\triangle(n) \otimes_{\mathcal{Z}} \mathbb{Q}(v)$ and $\boldsymbol{\mathfrak{C}}_\triangle(n) = \mathfrak{C}_\triangle(n) \otimes_{\mathcal{Z}} \mathbb{Q}(v)$. Denote by $\boldsymbol{\mathfrak{H}}_\triangle^-(n)$ the opposite algebra of $\boldsymbol{\mathfrak{H}}_\triangle^+(n)$ $(= \boldsymbol{\mathfrak{H}}_\triangle(n))$.

By extending $\boldsymbol{\mathfrak{H}}_\triangle(n)$ to Hopf algebras

$$\boldsymbol{\mathfrak{H}}_\triangle(n)^{\geqslant 0} = \boldsymbol{\mathfrak{H}}_\triangle^+(n) \otimes \mathbb{Q}(v)[K_1^{\pm 1}, \dots, K_n^{\pm 1}],$$
$$\boldsymbol{\mathfrak{H}}_\triangle(n)^{\leqslant 0} = \mathbb{Q}(v)[K_1^{\pm 1}, \dots, K_n^{\pm 1}] \otimes \boldsymbol{\mathfrak{H}}_\triangle^-(n),$$

we define the double Ringel–Hall algebra $\mathfrak{D}_\triangle(n)$ (*cf.* [4, 32]) to be a quotient algebra of the free product $\boldsymbol{\mathfrak{H}}_\triangle(n)^{\geqslant 0} * \boldsymbol{\mathfrak{H}}_\triangle(n)^{\leqslant 0}$ via a certain skew Hopf paring $\psi\colon \boldsymbol{\mathfrak{H}}_\triangle(n)^{\geqslant 0} \times \boldsymbol{\mathfrak{H}}_\triangle(n)^{\leqslant 0} \to \mathbb{Q}(v)$. In particular, there is a triangular decomposition

$$\mathfrak{D}_\triangle(n) = \mathfrak{D}_\triangle^+(n) \otimes \mathfrak{D}_\triangle^0(n) \otimes \mathfrak{D}_\triangle^-(n),$$

where $\mathfrak{D}_\triangle^+(n) \cong \boldsymbol{\mathfrak{H}}_\triangle^+(n)$, $\mathfrak{D}_\triangle^0(n) \cong \mathbb{Q}[K_1^{\pm 1}, \dots, K_n^{\pm 1}]$ and $\mathfrak{D}_\triangle^-(n) \cong \boldsymbol{\mathfrak{H}}_\triangle^-(n)$.

Theorem 5.1 ([4, Th. 2.5.3]) *Let $\mathbf{U}_v(\widehat{\mathfrak{gl}}_n)$ be the quantum loop algebra of $\mathfrak{gl}_n$ defined in* [7] *or* [4, §2.5]*. Then there is a Hopf algebra isomorphism $\mathfrak{D}_\triangle(n) \cong \mathbf{U}_v(\widehat{\mathfrak{gl}}_n)$.*

Let $\mathbf{U} = \mathbf{U}(n) = \mathbf{U}_v(\widehat{\mathfrak{sl}}_n)$ be the quantum affine $\mathfrak{sl}_n (n \geqslant 2)$ over $\mathbb{Q}(v)$, and let $E_i, F_i, K_i^{\pm}(i \in I)$ be the generators; for details see [16, 21]. Then $\mathbf{U}$ admits a triangular decomposition $\mathbf{U} = \mathbf{U}^-\mathbf{U}^0\mathbf{U}^+$, where $\mathbf{U}^+$(resp. $\mathbf{U}^-, \mathbf{U}^0$) is the subalgebra generated by the E_i (resp. F_i, $K_i^{\pm}$ $(i \in I)$). Denote by $U_{\mathcal{Z}}^+$ the Lusztig integral form of $\mathbf{U}^+$, which is generated by all the divided powers $E_i^{(m)} = \frac{E_i^m}{[m]!}$. The relation of Ringel–Hall algebras and quantum affine $\mathfrak{sl}_n$ is described in the following theorem.

Theorem 5.2 ([27]) *There is a $\mathcal{Z}$-algebra isomorphism*

$$\mathfrak{C}_\triangle(n) \xrightarrow{\sim} U_{\mathcal{Z}}^+(n), \quad u_i^{(m)} \longmapsto E_i^{(m)}, \quad i \in I, \quad m \geqslant 1,$$

and by base change to $\mathbb{Q}(v)$, there is an algebra isomorphism $\boldsymbol{\mathfrak{C}}_\triangle(n) \xrightarrow{\sim} \mathbf{U}^+(n)$.

We now review an algorithm for computing the canonical basis. The first ingredient required in the algorithm is the following modified multiplication formula.

For $A \in \Theta_\triangle^+(n)$, let $\delta(A) = \dim \mathrm{End}(M(A)) - \dim M(A)$ and

$$\widetilde{u}_A = v^{\delta(A)} u_A = v^{\dim \mathrm{End}(M(A)) - \dim M(A)} u_A.$$

Lemma 5.3 ([10, p. 14]) *For $\alpha \in \mathbb{N}_\triangle^n$, $A \in \Theta_\triangle^+(n)$, the twisted multiplication formula in the Ringel–Hall algebra $\mathfrak{H}_\triangle(n)$ over $\mathcal{Z}$ is given by*

$$\widetilde{u}_\alpha \widetilde{u}_A = \sum_{\substack{T\in\Theta_\triangle^+(n) \\ \mathrm{row}(T)=\alpha}} v^{f_{A,T}} \prod_{\substack{1\leqslant i\leqslant n \\ j\in\mathbb{Z}, j>i}} \overline{\left[\!\!\left[\begin{matrix} a_{ij} + t_{ij} - t_{i-1,j} \\ t_{ij} \end{matrix} \right]\!\!\right]} \widetilde{u}_{A+T-\widehat{T}^+},$$

https://doi.org/10.4153/CJM-2017-009-4 Published online by Cambridge University Press

where

$$f_{A,T}=\sum_{\substack{1\leqslant i\leqslant n\\ j\geqslant l\geqslant i+1}}a_{i,j}t_{i,l}-\sum_{\substack{1\leqslant i\leqslant n\\ j>l\geqslant i+1}}a_{i+1,j}t_{i,l}-\sum_{\substack{1\leqslant i\leqslant n\\ j\geqslant l\geqslant i+1}}t_{i-1,j}t_{i,l}+\sum_{\substack{1\leqslant i\leqslant n\\ j>l\geqslant i+1}}t_{i,j}t_{i,l}.$$

For each $w=\boldsymbol{a}_1\boldsymbol{a}_2\cdots\boldsymbol{a}_m\in\widetilde{\Sigma}$ with tight form $w=\boldsymbol{b}_1^{e_1}\boldsymbol{b}_2^{e_2}\cdots\boldsymbol{b}_t^{e_t}$, define a monomial associated with w in $\mathfrak{H}_\triangle(n)$:

$$m^{(w)}=\widetilde{u}_{e_1\boldsymbol{b}_1}\cdots\widetilde{u}_{e_t\boldsymbol{b}_t}.$$

The monomials associated with the distinguished words $w_A=w_{A''}w_{A'}$ produced by Algorithms 4.3 and 4.5 will be denoted simply by

$$m^{(A)}=m^{(w_A)}=m^{(w_{A''})}m^{(w_{A'})}.$$

We now apply [6, Th. 6.2] to this particularly selected monomial set.

Lemma 5.4 (i) *For $A\in\Theta_\triangle^+(n)$, we have a triangular relation*

$$m^{(A)}=\widetilde{u}_A+\sum_{\substack{T\prec A,\,T\in\Theta_\triangle^+(n)\\ \mathbf{dim}M(A)=\mathbf{dim}M(T)}}v^{\delta(A)-\delta(T)}\gamma_{w_A}^T(v^2)\widetilde{u}_T,\tag{5.1}$$

In particular, $\mathfrak{H}_\triangle(n)$ is generated by $\{u_i^{(m)},u_\alpha=u_{[S_\alpha]}\mid i\in I,\alpha\in I_{\rm sin},m\in\mathbb{N}\}$, where $S_\alpha=\oplus_{i=1}^n\alpha_iS_i$ is the semisimple representation of $\Delta(n)$ associated with α.

(ii) *The set*

(5.2)

$$\mathscr{M}(\widehat{\mathfrak{gl}}_n)_+=\{m^{(A)}\mid A\in\Theta_\triangle^+(n)\}\quad(\textit{resp.}, \mathscr{M}(\widehat{\mathfrak{gl}}_n)_{ap}=\{m^{(A)}\mid A\in\Theta_\triangle^{ap}(n)\})$$

forms a $\mathcal{Z}$-basis for $\mathfrak{H}_\triangle(n)$ (resp., $U_{\mathcal{Z}}^+(n)$).

The ingredients to define a canonical basis of an algebra include a basis with index set P, a bar involution on the algebra, and a poset structure on P that satisfies a certain triangular condition when applying the bar to a basis element. In the current case, the basis is $\{\widetilde{u}_A\mid A\in\Theta_\triangle^+(n)\}$, the poset is $(\Theta_\triangle^+(n),\leqslant_{\rm dg})$, and the bar involution (see, *e.g.*, [29, Proposition 7.5]) is given by

$$^-:\mathfrak{H}_\triangle(n)\longrightarrow\mathfrak{H}_\triangle(n),\quad m^{(A)}\longmapsto m^{(A)},v\longmapsto v^{-1}.$$

We now use the selected monomials $m^{(A)}$ to verify the triangular relation.

Restricting to $A\in\Theta_\triangle^+(n)_{\mathbf{d}}$, $\mathbf{d}\in\mathbb{N}_\triangle^n$, by (5.1)

$$m^{(A)}=\widetilde{u}_A+\sum_{B\prec A,B\in\Theta_\triangle^+(n)_{\mathbf{d}}}h_{B,A}\widetilde{u}_B,\quad h_{B,A}=v^{\delta(A)-\delta(T)}\gamma_{w_A}^B(v^2).\tag{5.3}$$

Solving the above gives

$$\widetilde{u}_A=m^{(A)}+\sum_{B\prec A,B\in\Theta_\triangle^+(n)_{\mathbf{d}}}g_{B,A}m^{(B)}.$$

Applying the bar involution, we obtain

$$\overline{\widetilde{u}_A}=m^{(A)}+\sum_{B\in\Theta_\triangle^+(n)_{\mathbf{d}},B\prec A}\overline{g_{B,A}}m^{(B)}=\widetilde{u}_A+\sum_{B\in\Theta_\triangle^+(n)_{\mathbf{d}},B\prec A}r_{B,A}\widetilde{u}_B.$$

Now, by [19, 7.10] (or [5, §0.5],[8]), the system

$$p_{B,A} = \sum_{B \preccurlyeq C \preccurlyeq A} r_{B,C} \overline{p_{C,A}} \quad \text{for } B \preccurlyeq A, A, B \in \Theta_\triangle^+(n)_{\mathbf{d}}$$

has a unique solution satisfying $p_{A,A} = 1$, $p_{B,A} \in v^{-1}\mathbb{Z}[v^{-1}]$ for $B \prec A$. Moreover, the elements

$$\mathrm{c}_A = \sum_{B \preccurlyeq A, B \in \Theta_\triangle^{ap}(n)} p_{B,A} \widetilde{u}_B, \quad A \in \Theta_\triangle^+(n)_{\mathbf{d}},$$

satisfying $\overline{\mathrm{c}_A} = \mathrm{c}_A$, form a $\mathcal{Z}$-basis for $\mathfrak{H}_\triangle(n)_{\mathbf{d}}$. The basis

$$\mathscr{C}(\widehat{\mathfrak{gl}}_n)_+ = \{\mathrm{c}_A \mid A \in \Theta_\triangle^+(n)\}$$

is called the *canonical basis* of $\mathfrak{H}_\triangle(n)$ with respect to the PBW type basis $\{\widetilde{u}_A\}_{A \in \Theta_\triangle^+(n)}$, the bar involution, and the poset $(\Theta_\triangle^+(n), \leqslant_{\mathrm{dg}})$.

In practice, if relation (5.3) can be computed explicitly, then we can follow the following algorithm to compute the c_A (or $p_{B,A}$) inductively on the poset ideal Θ_A defined in (4.1). Write

$$\Theta_{\prec A} = \Theta_{\prec A}^1 \cup \Theta_{\prec A}^2 \cup \cdots \cup \Theta_{\prec A}^t \text{ for some } t \in \mathbb{N},$$

where

$$\Theta_{\prec A}^1 = \{\text{maximal elements of } \Theta_{\prec A}\},$$
$$\Theta_{\prec A}^i = \{\text{maximal elements of } \Theta_{\prec A} \setminus \bigcup_{j=1}^{i-1} \Theta_{\prec A}^j\}$$

for $2 \leqslant i \leqslant t$. Let

$${}'\Theta_{\prec A}^a = \{B \in \Theta_{\prec A}^a \mid h_{B,A} \notin v^{-1}\mathbb{Z}[v^{-1}]\}.$$

In the summation (5.3), assume ${}'\Theta_{\prec A}^a \neq \varnothing$ with a minimal. Then $p_{B,A} := h_{B,A} \in v^{-1}\mathbb{Z}[v^{-1}]$ for all $B \in \Theta_{\prec A}^i$ with $i < a$ or $B \in \Theta_{\prec A}^a \backslash {}'\Theta_{\prec A}^a$. For each $B \in {}'\Theta_{\prec A}^a$, $h_{B,A} \notin v^{-1}\mathbb{Z}[v^{-1}]$ has a unique decomposition $h_{B,A} = h'_{B,A} + p_{B,A}$ with $\overline{h'_{B,A}} = h'_{B,A}$ and $p_{B,A} \in v^{-1}\mathbb{Z}[v^{-1}]$. Then

$$m^{(A)} - \sum_{B \in {}'\Theta_{\prec A}^a} h'_{B,A} m^{(B)} = \widetilde{u}_A + \sum_{\substack{B \in \Theta_{\prec A}^i \\ i \leqslant a}} p_{B,A} \widetilde{u}_B + \sum_{\substack{B \in \Theta_{\prec A}^i \\ i > a}} g_{B,A} \widetilde{u}_B.$$

Continue this argument with $g_{B,A}$ if necessary; we eventually obtain

$$m^{(A)} - \sum_{B \in {}'\Theta_{\prec A}} h'_{B,A} m^{(B)} \in \widetilde{u}_A + \sum_{\substack{B <_{\mathrm{dg}} A \\ B \in \Theta_\triangle^+(n)}} v^{-1}\mathbb{Z}[v^{-1}] \widetilde{u}_B,$$

where ${}'\Theta_{\prec A}$ is a union of those ${}'\Theta_{\prec A}^a$. Since

$$\overline{m^{(A)} - \sum_{B \in {}'\Theta_{\prec A}} h'_{B,A} m^{(B)}} = m^{(A)} - \sum_{B \in {}'\Theta_{\prec A}} h'_{B,A} m^{(B)},$$

by the uniqueness of the canonical basis of $\mathfrak{H}_\triangle(n)$ with respect to the PBW type basis $\widetilde{u}_A$, we have proved the following algorithm.

Algorithm 5.5 For $A \in \Theta^+_\triangle(n)$, there exist a recursively constructed subset ${}'\Theta_{\prec A}$ of Θ_A and elements $h'_{B,A} \in \mathbb{Z}[v, v^{-1}]$ for all $B \in {}'\Theta_{\prec A}$ such that $\overline{h'_{B,A}} = h'_{B,A}$ and

$$\mathrm{C}_A = m^{(A)} - \sum_{B \in {}'\Theta_{\prec A}} h'_{B,A} m^{(B)}$$

is the canonical basis element associated with A.

If ${}'\Theta_{\prec A} = \varnothing$, then $\mathrm{C}_A = m^{(A)}$. Such a C_A is called a *tight monomial*, following [22].

6 Slices of the Canonical Basis

In certain finite type cases, the canonical bases can be explicitly computed. See, for example, Lusztig [19, §3] for types A_1 and A_2 and [30,31] for types A_3 and B_2. It is natural to expect that this is the case for quantum affine $\mathfrak{gl}_2$. However, this is much more complicated. In the next three sections, we present explicit formulas of the canonical basis for five "slices". We will see that if a module's Loewy length increases, the computation becomes more difficult.

The slices of the canonical basis are defined according to the Loewy length and periodicity of modules. In other words, for $(l, p) \in \mathbb{N}^2$ with $l \geq 1, l \geq p \geq 0$, let

$$\mathscr{C}(\widehat{\mathfrak{gl}}_n)_{(l,p)} = \{\mathrm{C}_A \mid \ell(A) = l, p(A) = p\}$$
$$(\text{resp., } \mathscr{M}(\widehat{\mathfrak{gl}}_n)_{(l,p)} = \{m^{(A)} \mid \ell(A) = l, p(A) = p\}),$$

which is called a *canonical* (resp., *monomial*) *slice*. Clearly, each of the canonical and monomial bases is a disjoint union of slices.

In the sequel, we will compute the slices $\mathscr{C}(\widehat{\mathfrak{gl}}_2)_{(l,p)}$ for $l \leq 2$. We first compute the cases $(l, p) \in \{(1,0), (1,1), (2,0)\}$, which are relatively easy.

Proposition 6.1 *For $(l, p) = (1,0)$ or $(1,1)$, we have*

$$\mathscr{C}(\widehat{\mathfrak{gl}}_2)_{(1,0)} = \mathscr{M}(\widehat{\mathfrak{gl}}_2)_{(1,0)} = \{\widetilde{u}_{aS_1}, \widetilde{u}_{bS_2} \mid a, b \in \mathbb{N} - 0\},$$
$$\mathscr{C}(\widehat{\mathfrak{gl}}_2)_{(1,1)} = \mathscr{M}(\widehat{\mathfrak{gl}}_2)_{(1,1)} = \{\widetilde{u}_{aS_1 \oplus bS_2} \mid a, b \in \mathbb{N}, ab \neq 0\}.$$

For $(l, p) = (2, 0)$, all modules are aperiodic. If we put

$$\mathscr{M}(\widehat{\mathfrak{gl}}_n)_{ap} = \{m^{(A)} \mid A \in \Theta^{ap}_\triangle(n)\}$$

(*cf.* (5.2)), then the structure of the monomial basis $\mathscr{M}(\widehat{\mathfrak{gl}}_2)_{ap}$ for the +-part $U^+_{\mathcal{Z}}(2)$ of quantum affine $\mathfrak{sl}_2$ has a very simple description.

A sequence $(a_1, a_2, \ldots, a_l) \in \mathbb{N}^l$ is called a *pyramidic* if there exists k, $1 \leq k \leq l$, such that

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k, \quad a_k \geqslant a_{k+1} \geqslant \cdots \geqslant a_l.$$

We identify the positive part $U^+_{\mathcal{Z}}(n)$ with the composition algebra under the isomorphism $\mathfrak{C}_\triangle(n) \xrightarrow{\sim} U^+_{\mathcal{Z}}(n)$, $u_i^{(m)} \mapsto E_i^{(m)}$ as given in Theorem 5.2.

Lemma 6.2 *We have*

$$\mathscr{M}(\widehat{\mathfrak{gl}}_2)_{ap} = \Big\{ E_i^{(a_1)} E_{i+1}^{(a_2)} E_i^{(a_3)} E_{i+1}^{(a_4)} \cdots E_{i'}^{(a_l)} \;\Big|\; i \in \mathbb{Z}_2, (a_1, a_2, \dots, a_l) \text{ is pyramidic, } \forall\, l \in \mathbb{N} \Big\},$$

where $i' = i$ if l is odd and $i' = i + 1$ if l is even.

Proof Applying Algorithm 4.5 to $A \in \Theta_\triangle^{ap}(2)$, we know that $m^{(A)}$ has the desired form.

Conversely, for a given

$$E(i, \boldsymbol{a}) = E_i^{(a_1)} E_{i+1}^{(a_2)} E_i^{(a_3)} E_{i+1}^{(a_4)} \cdots E_k^{(a_k)} \cdots E_l^{(a_l)},$$

where

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k, \quad a_k \geqslant a_{k+1} \geqslant \cdots \geqslant a_l > a_{l+1} = 0,$$

we construct an $A \in \Theta_\triangle^{ap}(2)$ such that $m^{(A)} = E(i, \boldsymbol{a})$. Since there are 8 cases for (i, k, l), we only prove the case where $(i, k, l) = (1, 1, 1)$. The proof for other cases is similar.

First, the matrix giving $E_1^{(a_k)} \cdots E_1^{(a_l)}$ by the algorithm has the form

$$\begin{pmatrix} 0 & a_k - a_{k+1} & a_{k+1} - a_{k+2} & \cdots & a_{l-1} - a_l & a_l \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

For a_{k-1}, there exists a unique $i_0 \in \mathbb{N}$ such that $a_{k+i_0} \geqslant a_{k-1} > a_{k+i_0+1}$, and so $a_{k+i_0} - a_{k+i_0+1} = (a_{k+i_0} - a_{k-1}) + (a_{k-1} - a_{k+i_0+1})$. Now, the matrix giving $E_2^{(a_{k-1})} E_1^{(a_k)} \cdots E_1^{(a_l)}$ has the form

$$\begin{pmatrix} 0 & a_k - a_{k+1} & a_{k+1} - a_{k+2} & \cdots & a_{k+i_0} - a_{k-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_{k-1} - a_{k+i_0+1} & \cdots & a_{l-1} - a_l & a_l \end{pmatrix}.$$

Continuing this pattern for $a_{k-2}, \dots, a_2, a_1$ eventually yields the required matrix A. ∎

We give an example to illustrate the construction.

Example 6.3 Consider

$$E_1^{(2)} E_2^{(3)} E_1^{(5)} E_2^{(8)} E_1^{(9)} E_2^{(6)} E_1^{(4)} E_2^{(3)} E_1^{(1)}.$$

First, the matrix giving $E_1^{(9)} E_2^{(6)} E_1^{(4)} E_2^{(3)} E_1^{(1)}$ is

$$\left(\begin{smallmatrix} 0 & 9-6 & 6-4 & 4-3 & 3-1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 3 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right).$$

Since $9 > 8 > 6$, the matrix giving $E_2^{(8)} E_1^{(9)} E_2^{(6)} E_1^{(4)} E_2^{(3)} E_1^{(1)}$ is

$$\left(\begin{smallmatrix} 0 & 9-8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8-6 & 6-4 & 4-3 & 3-1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 1 & 2 & 1 \end{smallmatrix}\right).$$

Due to $6 > 5 > 4$, the matrix giving $E_1^{(5)} E_2^{(8)} E_1^{(9)} E_2^{(6)} E_1^{(4)} E_2^{(3)} E_1^{(1)}$ is

$$\left(\begin{smallmatrix} 0 & 9-8 & 0 & 0 & 5-4 & 4-3 & 3-1 & 1 \\ 0 & 0 & 0 & 8-6 & 6-5 & 0 & 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \end{smallmatrix}\right).$$

Since $4 > 3 \geqslant 3$, the matrix giving $E_2^{(3)} E_1^{(5)} E_2^{(8)} E_1^{(9)} E_2^{(6)} E_1^{(4)} E_2^{(3)} E_1^{(1)}$ is

$$\left(\begin{smallmatrix} 0 & 9-8 & 0 & 0 & 5-4 & 4-3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8-6 & 6-5 & 0 & 0 & 3-3 & 3-1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 2 & 1 \end{smallmatrix}\right).$$

Finally, since $3 > 2 > 1$, the matrix giving $E_1^{(2)} E_2^{(3)} E_1^{(5)} E_2^{(8)} E_1^{(9)} E_2^{(6)} E_1^{(4)} E_2^{(3)} E_1^{(1)}$ has the form

$$\begin{pmatrix} 0 & 9-8 & 0 & 0 & 5-4 & 4-3 & 0 & 0 & 2-1 & 1 \\ 0 & 0 & 0 & 8-6 & 6-5 & 0 & 0 & 3-3 & 3-2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now we are ready to describe the slice $\mathscr{C}(\widehat{\mathfrak{gl}}_2)_{(2,0)}$, which is similar to the slices in Proposition 6.1.

Proposition 6.4 *For* $(l, p) = (2, 0)$, *we have*

$$\begin{aligned} \mathscr{C}(\widehat{\mathfrak{gl}}_2)_{(2,0)} &= \mathscr{M}(\widehat{\mathfrak{gl}}_2)_{(2,0)} \\ &= \left\{ E_1^{(a+b)} E_2^{(b)}, E_2^{(b)} E_1^{(a+b)}, E_1^{(b)} E_2^{(a+b)}, E_2^{(a+b)} E_1^{(b)} \mid a, b \in \mathbb{N}, b > 0 \right\}. \end{aligned}$$

Proof Suppose $A \in \Theta_\triangle^+(2)$ with $(\ell(A), p(A)) = (2, 0)$; then A is one of the following matrices

$$\begin{pmatrix} \mathbf{0} & a & b \\ 0 & \mathbf{0} & 0 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & \mathbf{0} & a & b \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0} & 0 & b \\ 0 & \mathbf{0} & a \end{pmatrix}, \quad \begin{pmatrix} \mathbf{0} & a & 0 & 0 \\ 0 & \mathbf{0} & 0 & b \end{pmatrix}, \quad \forall\, a, b \in \mathbb{N}, b > 0.$$

Applying Algorithm 4.5 to these matrices or by Lemma 6.2, the monomial $m^{(A)}$ has the following form

$$E_1^{(a+b)} E_2^{(b)}, \quad E_2^{(a+b)} E_1^{(b)}, \quad E_1^{(b)} E_2^{(a+b)}, \quad E_2^{(b)} E_1^{(a+b)}.$$

We now prove that these monomials are tight monomials. We only look at the first case; the other cases are similar. We now apply the formula in Lemma 5.3 to compute

$$m^{\left(\begin{smallmatrix} 0 & a & b \\ 0 & 0 & 0 \end{smallmatrix}\right)} = E_1^{(a+b)} E_2^{(b)} = \widetilde{u}_{(a+b)S_1} \widetilde{u}_{bS_2}.$$

Since $\alpha = (a + b, 0)$, the matrix T in the sum must be of the form $\left(\begin{smallmatrix} 0 & a+b-t & t \\ 0 & 0 & 0 \end{smallmatrix}\right)$. Thus,

$$\begin{aligned} m^{\left(\begin{smallmatrix} 0 & a & b \\ 0 & 0 & 0 \end{smallmatrix}\right)} &= \sum_{t \leqslant b} v^{-(a+b-t)(b-t)} \widetilde{u}_{\left(\begin{smallmatrix} 0 & a+b-t & t \\ 0 & 0 & b-t \end{smallmatrix}\right)} \\ &= \widetilde{u}_{\left(\begin{smallmatrix} 0 & a & b \\ 0 & 0 & 0 \end{smallmatrix}\right)} + \sum_{t < b} v^{-(a+b-t)(b-t)} \widetilde{u}_{\left(\begin{smallmatrix} 0 & a+b-t & t \\ 0 & 0 & b-t \end{smallmatrix}\right)}, \end{aligned}$$

which is the canonical basis element associated with $\left(\begin{smallmatrix} 0 & a & b \\ 0 & 0 & 0 \end{smallmatrix}\right)$, since $v^{-(a+b-t)(b-t)} \in v^{-1}\mathbb{Z}[v^{-1}]$ for all $t < b$. ∎

In the three slices above, the recursively constructed subset $'\Theta_{<A}$ in Algorithm 5.5 is empty. So they consist of tight monomials.

7 Computing the Slice $\mathscr{C}(\widehat{\mathfrak{gl}}_2)_{(2,1)}$

For computing the slices $\mathscr{C}(\widehat{\mathfrak{gl}}_2)_{(2,1)}$ and $\mathscr{C}(\widehat{\mathfrak{gl}}_2)_{(2,2)}$ in this and next sections, we consider a matrix of the form

$$A = \begin{pmatrix} 0 & a & c & 0 \\ 0 & 0 & b & d \end{pmatrix} \in \Theta_\triangle^+(2)$$

satisfying $\ell(A) = 2$, $p(A) > 0$, where $a, b, c, d \in \mathbb{N}$. Then $c + d \neq 0$ and $ab + cd \neq 0$.

https://doi.org/10.4153/CJM-2017-009-4 Published online by Cambridge University Press

Lemma 7.1 *For the A as given above, we have*

$$\Theta_A = (0, A] = \left\{ A_{(k_1,k_2)} \mid k_1, k_2 \in \mathbb{N}, (k_1, k_2) \leqslant (c, d) \right\},$$

where

$$A_{(k_1,k_2)} = \begin{pmatrix} 0 & a+c+d-k_1-k_2 & k_1 & 0 \\ 0 & 0 & b+c+d-k_1-k_2 & k_2 \end{pmatrix}.$$

Proof The proof is straightforward by (2.2). Note also that

$$A_{(t_1,t_2)} \leqslant_{\mathrm{dg}} A_{(k_1,k_2)} \iff (t_1, t_2) \leqslant (k_1, k_2). \qquad \blacksquare$$

For $cd \neq 0$, the poset ideal can be described by its Hasse diagram $H(c, d)$; see Figure 1.

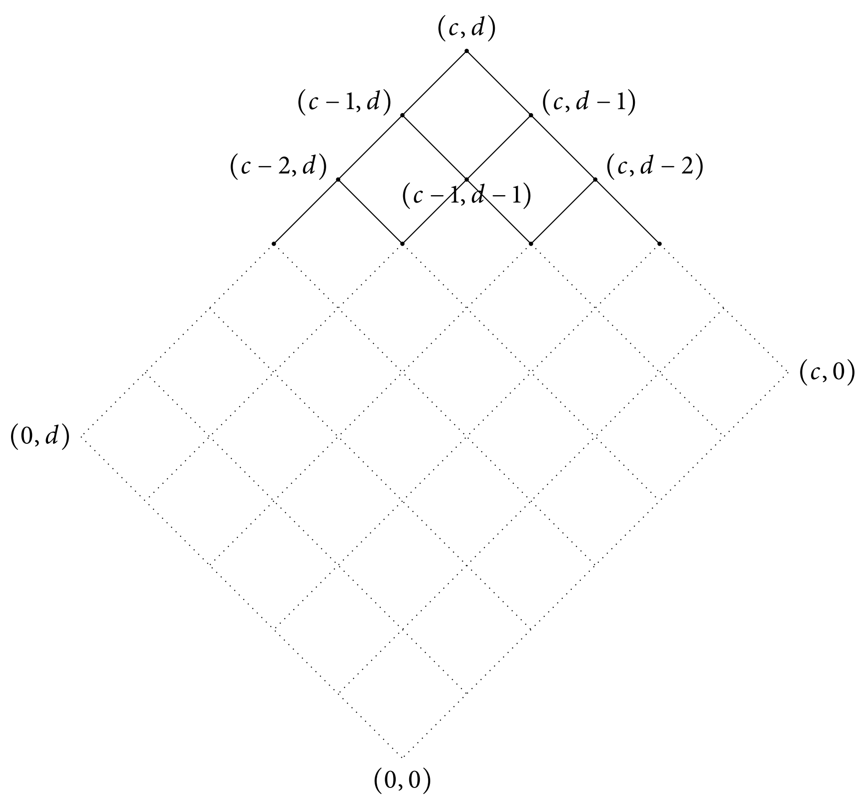

Figure 1: $H(c, d)$

For $B = A_{(k_1,k_2)}$, by Definition 4.2, we have $B' = \left(\begin{smallmatrix} 0 & a+c+d-k_1 & 0 \\ 0 & 0 & b+c+d-k_2 \end{smallmatrix}\right)$ and $B'' = \left(\begin{smallmatrix} 0 & k_1 & 0 \\ 0 & 0 & k_2 \end{smallmatrix}\right)$. The following follows immediately from Lemma 5.3.

Lemma 7.2 *Putting* $\widetilde{u}_{(k_1,k_2)} = \widetilde{u}_{A_{(k_1,k_2)}}$ *and* $m^{(k_1,k_2)} = m^{(A_{(k_1,k_2)})}$, *we have*

$$\begin{aligned} m^{(k_1,k_2)} &= \widetilde{u}_{\left(\begin{smallmatrix} 0 & k_1 & 0 \\ 0 & 0 & k_2 \end{smallmatrix}\right)} \widetilde{u}_{\left(\begin{smallmatrix} 0 & a+c+d-k_1 & 0 \\ 0 & 0 & b+c+d-k_2 \end{smallmatrix}\right)} \\ &= \sum_{t_1 \leqslant k_1, t_2 \leqslant k_2} v^{(a-b-k_1+k_2+t_1-t_2)(k_1-k_2-t_1+t_2)} \overline{\left[\!\!\left[\begin{smallmatrix} a+c+d-t_1-t_2 \\ k_1-t_1 \end{smallmatrix} \right]\!\!\right] \left[\!\!\left[\begin{smallmatrix} b+c+d-t_1-t_2 \\ k_2-t_2 \end{smallmatrix} \right]\!\!\right]} \widetilde{u}_{(t_1,t_2)}. \end{aligned}$$

We now compute the canonical basis elements for those A with $c = 0$ or $d = 0$ (but not both zero). In other words, $p(A) = 1$. We need the following identities for symmetric Gaussian polynomials.

Lemma 7.3 ([30, Section 3.1]) (i) *Assume that* $m \geqslant k \geqslant 0, \delta \in \mathbb{N}$. *Then*

$$\sum_{i=0}^{\delta} (-1)^i v^{i(m-k)} \begin{bmatrix} k-1+i \\ k-1 \end{bmatrix} \begin{bmatrix} m \\ \delta - i \end{bmatrix} = v^{-k\delta} \begin{bmatrix} m-k \\ \delta \end{bmatrix}.$$

(ii) *Assume that* $m \geqslant k \geqslant 0, \delta, n \in \mathbb{N}$. *Then*

$$\sum_{i=0}^{\delta} (-1)^i v^{i(m-k-n)} \begin{bmatrix} k-1+i \\ k-1 \end{bmatrix} \begin{bmatrix} m+n \\ \delta - i \end{bmatrix} = \sum_{t=0}^{\min\{\delta,n\}} v^{-k(\delta-t)-n\delta+t(m+n)} \begin{bmatrix} m-k \\ \delta-t \end{bmatrix} \begin{bmatrix} n \\ t \end{bmatrix}.$$

We now perform Algorithm 5.5 to compute the slice $\mathscr{C}(\widehat{\mathfrak{gl}}_2)_{(2,1)}$. In this case, the recursively constructed subset $'\Theta_{<A}$ in the Algorithm 5.5 is $'\Theta_{<A} = \Theta_{<A}$.

Theorem 7.4 *If* $A \in \Theta_\vartriangle^+(2)$ *with* $(\ell(A), p(A)) = (2,1)$, *then A is of the form*

$$\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \end{pmatrix} \quad or \quad \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & d \end{pmatrix} \ (a,b,c,d \in \mathbb{N}_{\geqslant 1}).$$

(i) *For* $A = \left(\begin{smallmatrix} 0 & a & c \\ 0 & 0 & b \end{smallmatrix}\right)$, $c_A = m^{(A)}$ *is a tight monomial if and only if* $a \leqslant b$. *The canonical basis element associated with A with* $a > b$ *has the form*

$$c_A = \sum_{k=0}^{c} (-1)^{c-k} \begin{bmatrix} a-b-1+c-k \\ a-b-1 \end{bmatrix} m^{(k,0)} = \sum_{t=0}^{c} v^{-t(a-b+t)} \overline{\left[\!\!\left[\begin{matrix} b+t \\ t \end{matrix} \right]\!\!\right]} \widetilde{u}_{(c-t,0)},$$

where $\widetilde{u}_{(k,0)} = \widetilde{u}_{A_{(k,0)}}$ *and* $A_{(k,0)} = \left(\begin{smallmatrix} 0 & a+c-k & k \\ 0 & 0 & b+c-k \end{smallmatrix}\right)$.

(ii) *For* $A = \left(\begin{smallmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & d \end{smallmatrix}\right)$, $c_A = m^{(A)}$ *is a tight monomial if and only if* $a \geqslant b$. *The canonical basis element associated with A with* $a < b$ *has the form*

$$c_A = \sum_{l=0}^{d} (-1)^{d-l} \begin{bmatrix} b-a-1+d-l \\ b-a-1 \end{bmatrix} m^{(0,l)} = \sum_{t=0}^{d} v^{-t(b-a+t)} \overline{\left[\!\!\left[\begin{matrix} a+t \\ t \end{matrix} \right]\!\!\right]} \widetilde{u}_{(0,d-t)},$$

where $\widetilde{u}_{(0,l)} = \widetilde{u}_{A_{(0,l)}}$ *and* $A_{(0,l)} = \left(\begin{smallmatrix} 0 & a+d-l & 0 & 0 \\ 0 & 0 & b+d-l & l \end{smallmatrix}\right)$.

Proof We only prove (i); the proof for (ii) is similar. In this case, the Hasse diagram $H(c,0)$ is a linear figure. In other words, we have $A = A_{(c,0)} >_{\mathrm{dg}} A_{(c-1,0)} >_{\mathrm{dg}} \cdots >_{\mathrm{dg}} A_{(1,0)} >_{\mathrm{dg}} A_{(0,0)}$. Note that in this case $A' = \left(\begin{smallmatrix} 0 & a & 0 \\ 0 & 0 & b+c \end{smallmatrix}\right)$ and $A'' = \left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right)$. Thus,

$$m^{(A)} = m^{(A'')} m^{(A')} = \widetilde{u}_{A''} \widetilde{u}_{A'}.$$

We now apply the formula in Lemma 5.3. We have here $\alpha = (c, 0)$. If $T \in \Theta^+_\triangle(2)$ satisfying $A' - T + \widehat{T}^+ \in \Theta^+_\triangle(2)$ and $\mathrm{row}(T) = \alpha$, then $T = \left(\begin{smallmatrix} 0 & c-t & t \\ 0 & 0 & 0 \end{smallmatrix}\right)$ for some $0 \le t \le c$. Thus, $A' - T + \widehat{T}^+ = A_{(t,0)}$ and

$$f_{A',T} = a(c-t) - (b+c)(c-t) + t(c-t) = (c-t)(a-b-c+t).$$

Hence,

$$\begin{aligned} m^{(A)} &= \sum_{0 \leqslant t \leqslant c} v^{(c-t)(a-b-c+t)} \overline{\left[\!\!\left[\begin{matrix} a+c-t \\ c-t \end{matrix} \right]\!\!\right]} \widetilde{u}_{(t,0)} \\ &= \widetilde{u}_A + \sum_{0 \leqslant t \leqslant c-1} v^{(c-t)(a-b-c+t)} \overline{\left[\!\!\left[\begin{matrix} a+c-t \\ c-t \end{matrix} \right]\!\!\right]} \widetilde{u}_{(t,0)}. \end{aligned}$$

Consequently, $m^{(A)}$ becomes a canonical basis element (or a tight monomial) if $a \leqslant b$.

By the calculation above, we have $A_{(k,0)} = \left(\begin{smallmatrix} 0 & a+c-k & k \\ 0 & 0 & b+c-k \end{smallmatrix}\right)$ for $k = 0, 1, 2, \dots, c$, and so, by Lemma 7.2,

$$\begin{aligned} m^{(k,0)} = \widetilde{u}_{\left(\begin{smallmatrix} 0 & k \\ 0 & 0 \end{smallmatrix}\right)} \widetilde{u}_{\left(\begin{smallmatrix} 0 & a+c-k & 0 \\ 0 & 0 & b+c \end{smallmatrix}\right)} &= \sum_{0 \leqslant t \leqslant k} v^{(k-t)(a+t-b-k)} \overline{\left[\!\!\left[\begin{matrix} a+c-t \\ k-t \end{matrix} \right]\!\!\right]} \widetilde{u}_{(t,0)} \\ &= \sum_{0 \leqslant t \leqslant k} v^{(k-t)(t-b-c)} \begin{bmatrix} a+c-t \\ k-t \end{bmatrix} \widetilde{u}_{(t,0)}. \end{aligned}$$

Assume now that $a > b$ and consider the following bar fixed sum:

$$\begin{aligned} M(c) &:= \sum_{k=0}^{c} (-1)^{c-k} \begin{bmatrix} a-b-1+c-k \\ a-b-1 \end{bmatrix} m^{(k,0)} \\ &= \sum_{k=0}^{c} (-1)^{c-k} \begin{bmatrix} a-b-1+c-k \\ a-b-1 \end{bmatrix} \left(\sum_{t=0}^{k} v^{(k-t)(t-b-c)} \begin{bmatrix} a+c-t \\ k-t \end{bmatrix} \widetilde{u}_{(t,0)} \right) \\ &= \sum_{k=0}^{c} \sum_{t=0}^{k} (-1)^{c-k} v^{(k-t)(t-b-c)} \begin{bmatrix} a-b-1+c-k \\ a-b-1 \end{bmatrix} \begin{bmatrix} a+c-t \\ k-t \end{bmatrix} \widetilde{u}_{(t,0)} \\ &= \sum_{t=0}^{c} \left(\sum_{k=t}^{c} (-1)^{c-k} v^{(k-t)(t-b-c)} \begin{bmatrix} a-b-1+c-k \\ a-b-1 \end{bmatrix} \begin{bmatrix} a+c-t \\ k-t \end{bmatrix} \right) \widetilde{u}_{(t,0)} \\ &= \widetilde{u}_A + \sum_{t=0}^{c-1} \left(\sum_{k=t}^{c} (-1)^{c-k} v^{(k-t)(t-b-c)} \begin{bmatrix} a-b-1+c-k \\ a-b-1 \end{bmatrix} \begin{bmatrix} a+c-t \\ k-t \end{bmatrix} \right) \widetilde{u}_{(t,0)}. \end{aligned}$$

However, for fixed t,

$$\begin{aligned} f_{(t,0)} &:= \sum_{k=t}^{c} (-1)^{c-k} v^{(k-t)(t-b-c)} \begin{bmatrix} a-b-1+c-k \\ a-b-1 \end{bmatrix} \begin{bmatrix} a+c-t \\ k-t \end{bmatrix} \\ &= \sum_{k'=0}^{c'} (-1)^{c'-k'} v^{-k'(b+c')} \begin{bmatrix} a-b-1+c'-k' \\ a-b-1 \end{bmatrix} \begin{bmatrix} a+c' \\ k' \end{bmatrix} \quad (c' = c-t, k' = k-t) \\ &= v^{-c'(b+c')} \sum_{i=0}^{c'} (-1)^i v^{i(b+c')} \begin{bmatrix} a-b-1+i \\ a-b-1 \end{bmatrix} \begin{bmatrix} a+c' \\ c'-i \end{bmatrix} \quad (i = c'-k'). \end{aligned}$$

Let $k = a - b$, $m = a + c'$ and $\delta = c'$. Applying Lemma 7.3(i) gives

$$f_{(t,0)} = v^{-c'(b+c')} v^{-c'(a-b)} \begin{bmatrix} b+c' \\ c' \end{bmatrix} = v^{-c'(a+c')} \begin{bmatrix} b+c' \\ c' \end{bmatrix}$$
$$= v^{-c'(a-b+c')} \overline{\left[\!\!\left[\begin{matrix} b+c' \\ c' \end{matrix} \right]\!\!\right]} \in v^{-1}\mathbb{Z}[v^{-1}],$$

since $a > b$. Hence, $M(c) \in \widetilde{u}_A + \sum_{t=0}^{c-1} v^{-1}\mathbb{Z}[v^{-1}]\widetilde{u}_{(t,0)}$. On the other hand, $\overline{M(c)} = M(c)$. Consequently, $\mathrm{c}_A = M(c)$, as desired. ∎

8 Computing the Slice $\mathscr{C}(\widehat{\mathfrak{gl}}_2)_{(2,2)}$

In the last section, we compute the canonical basis associated with the matrix $A = \left(\begin{smallmatrix} 0 & a & c & 0 \\ 0 & 0 & b & d \end{smallmatrix}\right)$ with $\ell(A) = 2 = p(A)$ and $a, b, c, d \in \mathbb{N}$. Thus, $cd \neq 0$.

Theorem 8.1 *Maintain the notation as set in Lemmas 7.1 and 7.2. Suppose $A = \left(\begin{smallmatrix} 0 & a & c & 0 \\ 0 & 0 & b & d \end{smallmatrix}\right) \in \Theta_\triangle^+(2)$ with $\ell(A) = 2 = p(A)$ and $a, b, c, d \in \mathbb{N}$. Then the canonical basis element c_A associated with A is given as follows.*

(i) *If $a = b$, then $\mathrm{c}_A = m^{(c,d)} - m^{(c-1,d-1)}$.*

(ii) *If $a > b$, then*

$$\mathrm{c}_A = \sum_{k_1=0}^{c} (-1)^{c-k_1} \begin{bmatrix} a-b-1+c-k_1 \\ a-b-1 \end{bmatrix} m^{(k_1,d)} - \sum_{l_1=0}^{c-1} (-1)^{c-1-l_1} \begin{bmatrix} a-b-2+c-l_1 \\ a-b-1 \end{bmatrix} m^{(l_1,d-1)}.$$

(iii) *If $a < b$, then*

$$\mathrm{c}_A = \sum_{k_1=0}^{d} (-1)^{d-k_1} \begin{bmatrix} b-a-1+d-k_1 \\ b-a-1 \end{bmatrix} m^{(c,k_1)} - \sum_{l_1=0}^{d-1} (-1)^{d-1-l_1} \begin{bmatrix} b-a-2+d-l_1 \\ b-a-1 \end{bmatrix} m^{(c-1,l_1)}.$$

We can see the symmetry of the three cases from the big diamond $H(c,d)$, Figure 1. The recursively constructed subset in Algorithm 5.5 has the form:

$${}'\Theta_{<A} = \begin{cases} \{A_{(c-1,d-1)}\}, & \text{in (i)}, \\ \{A_{(i,d)}, A_{(j,d-1)} \mid 0 \leqslant i, j \leqslant c, i < c\}, & \text{in (ii)}, \\ \{A_{(c,i)}, A_{(c-1,j)} \mid 0 \leqslant i, j \leqslant d, i < d\}, & \text{in (iii)}. \end{cases}$$

https://doi.org/10.4153/CJM-2017-009-4 Published online by Cambridge University Press

Proof We first prove (i), and thus assume $a = b$. Then the formula in Lemma 7.2 with $(k_1, k_2) = (c, d)$ becomes

$$\begin{aligned} m^{(c,d)} &= \sum_{t_1 \leqslant c, t_2 \leqslant d} v^{-(c-d-t_1+t_2)^2} \overline{\left[\!\!\left[{a+c+d-t_1-t_2 \atop c-t_1} \right]\!\!\right] \left[\!\!\left[{a+c+d-t_1-t_2 \atop d-t_2} \right]\!\!\right]} \widetilde{u}_{(t_1,t_2)} \\ &= \sum_{\substack{t_1 \leqslant c, t_2 \leqslant d \\ c-t_1 = d-t_2}} \overline{\left[\!\!\left[{a+c+d-t_1-t_2 \atop c-t_1} \right]\!\!\right]}^2 \widetilde{u}_{(t_1,t_2)} \\ &\quad + \sum_{\substack{t_1 \leqslant c, t_2 \leqslant d \\ c-t_1 \neq d-t_2}} v^{-(c-t_1-d+t_2)^2} \overline{\left[\!\!\left[{a+c+d-t_1-t_2 \atop c-t_1} \right]\!\!\right] \left[\!\!\left[{a+c+d-t_1-t_2 \atop d-t_2} \right]\!\!\right]} \widetilde{u}_{(t_1,t_2)}. \end{aligned}$$

Since $\overline{\left[\!\!\left[{a+c+d-t_1-t_2 \atop c-t_1} \right]\!\!\right]}^2 - 1 \in v^{-1}\mathbb{Z}[v^{-1}]$ ($= 0$ if $(t_1, t_2) = (c, d)$) and the coefficients in the second sum are all in $v^{-1}\mathbb{Z}[v^{-1}]$, it follows that

$$m^{(c,d)} = \begin{cases} \widetilde{u}_{(c,d)} + \widetilde{u}_{(c-1,d-1)} + \cdots + \widetilde{u}_{(c-d,0)} + X, & \text{if } c \geq d, \\ \widetilde{u}_{(c,d)} + \widetilde{u}_{(c-1,d-1)} + \cdots + \widetilde{u}_{(0,d-c)} + Y, & \text{if } c < d, \end{cases}$$

where $X, Y \in \sum_{(t_1,t_2)<(c,d)} v^{-1}\mathbb{Z}[v^{-1}]\widetilde{u}_{(t_1,t_2)}$.

Similarly, we have

$$\begin{aligned} m^{(c-1,d-1)} &= \sum_{t_1 \leqslant c-1, t_2 \leqslant d-1} v^{-(c-d-t_1+t_2)^2} \overline{\left[\!\!\left[{a+c+d-t_1-t_2 \atop c-1-t_1} \right]\!\!\right] \left[\!\!\left[{a+c+d-t_1-t_2 \atop d-1-t_2} \right]\!\!\right]} \widetilde{u}_{(t_1,t_2)} \\ &= \sum_{\substack{t_1 \leqslant c-1, t_2 \leqslant d-1 \\ c-t_1 = d-t_2}} \overline{\left[\!\!\left[{a+c+d-t_1-t_2 \atop c-1-t_1} \right]\!\!\right]}^2 \widetilde{u}_{(t_1,t_2)} \\ &\quad + \sum_{\substack{t_1 \leqslant c-1, t_2 \leqslant d-1 \\ c-t_1 \neq d-t_2}} v^{-(c-d-t_1+t_2)^2} \overline{\left[\!\!\left[{a+c+d-t_1-t_2 \atop c-1-t_1} \right]\!\!\right] \left[\!\!\left[{a+c+d-t_1-t_2 \atop d-1-t_2} \right]\!\!\right]} \widetilde{u}_{(t_1,t_2)}. \end{aligned}$$

and

$$m^{(c-1,d-1)} = \begin{cases} \widetilde{u}_{(c-1,d-1)} + \widetilde{u}_{(c-2,d-2)} + \cdots + \widetilde{u}_{(c-d,0)} + X', & \text{if } c \geq d, \\ \widetilde{u}_{(c-1,d-1)} + \widetilde{u}_{(c-2,d-2)} + \cdots + \widetilde{u}_{(0,d-c)} + Y', & \text{if } c < d, \end{cases}$$

where $X', Y' \in \sum_{(t_1,t_2)<(c-1,d-1)} v^{-1}\mathbb{Z}[v^{-1}]\widetilde{u}_{(t_1,t_2)}$. Hence,

$$m^{(c,d)} - m^{(c-1,d-1)} = \widetilde{u}_{(c,d)} + Z, \text{ where } Z \in \sum_{(t_1,t_2)<(c,d)} v^{-1}\mathbb{Z}[v^{-1}]\widetilde{u}_{(t_1,t_2)}.$$

This proves that $\mathrm{c}_A = m^{(c,d)} - m^{(c-1,d-1)}$ is the canonical basis element associated with A in this case.

https://doi.org/10.4153/CJM-2017-009-4 Published online by Cambridge University Press

Next we prove (ii). Fix $a > b$ and let

$$\begin{aligned} M(c,d) &= \sum_{k_1=0}^{c} (-1)^{c-k_1} \begin{bmatrix} a-b-1+c-k_1 \\ a-b-1 \end{bmatrix} m^{(k_1,d)} \\ &\quad - \sum_{l_1=0}^{c-1} (-1)^{c-1-l_1} \begin{bmatrix} a-b-2+c-l_1 \\ a-b-1 \end{bmatrix} m^{(l_1,d-1)} \\ &= \widetilde{u}_{(c,d)} + \sum_{t_1=0}^{c-1} f^{(c,d)}_{(t_1,d)} \widetilde{u}_{(t_1,d)} + \sum_{t_2=0}^{d-1} f^{(c,d)}_{(c,t_2)} \widetilde{u}_{(c,t_2)} \\ &\quad + \sum_{(t_1,t_2)\ll(c,d)} \big(f^{(c,d)}_{(t_1,t_2)} - f^{(c-1,d-1)}_{(t_1,t_2)}\big) \widetilde{u}_{(t_1,t_2)}, \end{aligned}$$

where $(t_1,t_2) \ll (c,d)$ means $t_1 < c$ and $t_2 < d$, and

$$\begin{aligned} \sum_{k_1=0}^{c} (-1)^{c-k_1} \begin{bmatrix} a-b-1+c-k_1 \\ a-b-1 \end{bmatrix} m^{(k_1,d)} &= \sum_{(t_1,t_2)\leqslant(c,d)} f^{(c,d)}_{(t_1,t_2)} \widetilde{u}_{(t_1,t_2)}, \\ \sum_{l_1=0}^{c-1} (-1)^{c-1-l_1} \begin{bmatrix} a-b-2+c-l_1 \\ a-b-1 \end{bmatrix} m^{(l_1,d-1)} &= \sum_{(t_1,t_2)\ll(c,d)} f^{(c-1,d-1)}_{(t_1,t_2)} \widetilde{u}_{(t_1,t_2)}. \end{aligned}$$

Expanding the left-hand sides by Lemma 7.2 yields, for $(t_1,t_2) \ll (c,d)$,

$$\begin{aligned} f^{(c,d)}_{(t_1,t_2)} &= \sum_{k_1=t_1}^{c} (-1)^{c-k_1} v^{(a-b-k_1+d+t_1-t_2)(k_1-d-t_1+t_2)} \\ &\qquad \times \left[{a-b-1+c-k_1 \atop a-b-1} \right] \overline{\left[\!\left[{a+c+d-t_1-t_2 \atop k_1-t_1} \right]\!\right] \left[\!\left[{b+c+d-t_1-t_2 \atop d-t_2} \right]\!\right]}, \\ f^{(c-1,d-1)}_{(t_1,t_2)} &= \sum_{l_1=t_1}^{c-1} (-1)^{c-1-l_1} v^{(a-b-l_1+d-1+t_1-t_2)(l_1-d+1-t_1+t_2)} \\ &\qquad \times \left[{a-b-2+c-l_1 \atop a-b-1} \right] \overline{\left[\!\left[{a+c+d-t_1-t_2 \atop l_1-t_1} \right]\!\right] \left[\!\left[{b+c+d-t_1-t_2 \atop d-1-t_2} \right]\!\right]}. \end{aligned}$$

In particular, since $a > b$,

$$\begin{aligned} f^{(c,d)}_{(c,t_2)} &= v^{(a-b+d-t_2)(-d+t_2)} \overline{\left[\!\left[\begin{matrix} b+d-t_2 \\ d-t_2 \end{matrix} \right]\!\right]} \\ &= v^{-t_2'(a-b+t_2')} \overline{\left[\!\left[\begin{matrix} b+t_2' \\ t_2' \end{matrix} \right]\!\right]} \in v^{-1}\mathbb{Z}[v^{-1}] \quad (t_2' = d-t_2 \geq 0). \end{aligned}$$

and, by Lemma 7.3(i), we have as seen at the end of the proof of Theorem 7.4,

$$\begin{aligned} f^{(c,d)}_{(t_1,d)} &= \sum_{k_1=t_1}^{c} (-1)^{c-k_1} v^{(a-b-k_1+t_1)(k_1-t_1)} \begin{bmatrix} a-b-1+c-k_1 \\ a-b-1 \end{bmatrix} \overline{\left[\!\left[\begin{matrix} a+c-t_1 \\ k_1-t_1 \end{matrix} \right]\!\right]} \\ &= v^{-t_1'(a+t_1')} \begin{bmatrix} b+t_1' \\ t_1' \end{bmatrix} = v^{-t_1'(a-b+t_1')} \overline{\left[\!\left[\begin{matrix} b+t_1' \\ t_1' \end{matrix} \right]\!\right]} \in v^{-1}\mathbb{Z}[v^{-1}] \quad (t_1' = c-t_1). \end{aligned}$$

Assume now that $(t_1,t_2) \ll (c,d)$ and let

$$g^{(c,d)}_{(t_1,t_2)} := f^{(c,d)}_{(t_1,t_2)} - f^{(c-1,d-1)}_{(t_1,t_2)}.$$

https://doi.org/10.4153/CJM-2017-009-4 Published online by Cambridge University Press

If $(t_1,t_2)=(0,0)$, then $g^{(c,d)}_{(0,0)}\in v^{-1}\mathbb{Z}[v^{-1}]$. This is done in Lemma A.1 of Appendix A.

It remains to prove that $g^{(c,d)}_{(t_1,t_2)}\in v^{-1}\mathbb{Z}[v^{-1}]$ for all $(0,0)<(t_1,t_2)\ll(c,d)$. This follows from the following recursive formula: for all $(0,0)<(t_1,t_2)\le(c',d')\ll(c,d)$,

$$g^{(c'+1,d'+1)}_{(t_1,t_2)}=\begin{cases} g^{(c'+1,d')}_{(t_1,t_2-1)}, & \text{if } t_2\ge 1,\\ g^{(c',d'+1)}_{(t_1-1,0)}, & \text{if } t_2=0,\end{cases}$$

which can be seen as follows.

First, the coefficient $g^{(c'+1,d'+1)}_{(t_1,t_2)}$ of $\widetilde{u}_{(t_1,t_2)}$ in $M(c'+1,d'+1)$ has the form

$$\begin{aligned}&\sum_{k_1=t_1}^{c'+1}(-1)^{c'+1-k_1}v^{(a-b-k_1+d'+1+t_1-t_2)(k_1-d'-1-t_1+t_2)}\\ &\qquad\times\left[{a-b+c'-k_1 \atop a-b-1}\right]\overline{\left[\!\!\left[{a+c'+d'+2-t_1-t_2 \atop k_1-t_1}\right]\!\!\right]\left[\!\!\left[{b+c'+d'+2-t_1-t_2 \atop d'+1-t_2}\right]\!\!\right]}\\ &-\sum_{l_1=t_1}^{c'}(-1)^{c'-l_1}v^{(a-b-l_1+d'+t_1-t_2)(l_1-d'-t_1+t_2)}\\ &\qquad\times\left[{a-b-1+c'-l_1 \atop a-b-1}\right]\overline{\left[\!\!\left[{a+c'+d'+2-t_1-t_2 \atop l_1-t_1}\right]\!\!\right]\left[\!\!\left[{b+c'+d'+2-t_1-t_2 \atop d'-t_2}\right]\!\!\right]}.\end{aligned}$$

If $t_2\geqslant 1$, then the coefficient $g^{(c'+1,d')}_{(t_1,t_2-1)}$ of $\widetilde{u}_{(t_1,t_2-1)}$ in $M(c'+1,d')$ has the form

$$\begin{aligned}&\sum_{k_1=t_1}^{c'+1}(-1)^{c'+1-k_1}v^{(a-b-k_1+d'+t_1-t_2+1)(k_1-d'-t_1+t_2-1)}\\ &\qquad\times\left[{a-b+c'-k_1 \atop a-b-1}\right]\overline{\left[\!\!\left[{a+c'+1+d'-t_1-t_2+1 \atop k_1-t_1}\right]\!\!\right]\left[\!\!\left[{b+c'+1+d'-t_1-t_2+1 \atop d'-t_2+1}\right]\!\!\right]}\\ &-\sum_{l_1=t_1}^{c'}(-1)^{c'-l_1}v^{(a-b-l_1+d'-1+t_1-t_2+1)(l_1-d'+1-t_1+t_2-1)}\\ &\qquad\times\left[{a-b-1+c'-l_1 \atop a-b-1}\right]\overline{\left[\!\!\left[{a+c'+1+d'-t_1-t_2+1 \atop l_1-t_1}\right]\!\!\right]\left[\!\!\left[{b+c'+1+d'-t_1-t_2+1 \atop d'-1-t_2+1}\right]\!\!\right]},\end{aligned}$$

which is the same as that of $\widetilde{u}_{(t_1,t_2)}$ in $M(c'+1,d'+1)$, proving the first recursive formula.

If $t_2=0$, $t_1\geqslant 1$, the coefficient $g^{(c',d'+1)}_{(t_1-1,0)}$ of $\widetilde{u}_{(t_1-1,0)}$ in $M(c',d'+1)$ has the form

$$\begin{aligned}&\sum_{k_1=t_1-1}^{c'}(-1)^{c'-k_1}v^{(a-b-k_1+d'+1+t_1-1)(k_1-d'-1-t_1+1)}\\ &\qquad\times\left[{a-b-1+c'-k_1 \atop a-b-1}\right]\overline{\left[\!\!\left[{a+c'+d'+1-t_1+1 \atop k_1-t_1+1}\right]\!\!\right]\left[\!\!\left[{b+c'+d'+1-t_1+1 \atop d'+1}\right]\!\!\right]}\\ &-\sum_{l_1=t_1-1}^{c'-1}(-1)^{c'-1-l_1}v^{(a-b-l_1+d'+t_1-1)(l_1-d'-t_1+1)}\\ &\qquad\times\left[{a-b-2+c'-l_1 \atop a-b-1}\right]\overline{\left[\!\!\left[{a+c'+d'+1-t_1+1 \atop l_1-t_1+1}\right]\!\!\right]}\,\overline{\left[\!\!\left[{b+c'+d'+1-t_1+1 \atop d}\right]\!\!\right]}.\end{aligned}$$

https://doi.org/10.4153/CJM-2017-009-4 Published online by Cambridge University Press

Putting $k_1' = k_1 + 1$, $l_1' = l_1 + 1$, we obtain

$$
\begin{aligned}
g^{(c',d'+1)}_{(t_1-1,0)} = & \sum_{k_1'=t_1}^{c'+1} (-1)^{c'-k_1'+1} v^{(a-b-k_1'+d'+1+t_1)(k_1'-d'-1-t_1)} \\
& \times \begin{bmatrix} a-b+c'-k_1' \\ a-b-1 \end{bmatrix} \overline{\left[\!\!\left[\begin{matrix} a+c'+d'+2-t_1 \\ k_1'-t_1 \end{matrix} \right]\!\!\right] \left[\!\!\left[\begin{matrix} b+c'+d'+2-t_1 \\ d'+1 \end{matrix} \right]\!\!\right]} \\
& - \sum_{l_1'=t_1}^{c'} (-1)^{c'-l_1'} v^{(a-b-l_1'+d'+t_1)(l_1'-d'-t_1)} \\
& \times \begin{bmatrix} a-b-1+c-l_1' \\ a-b-1 \end{bmatrix} \overline{\left[\!\!\left[\begin{matrix} a+c'+d'+2-t_1 \\ l_1'-t_1 \end{matrix} \right]\!\!\right] \left[\!\!\left[\begin{matrix} b+c'+d'+2-t_1 \\ d' \end{matrix} \right]\!\!\right]},
\end{aligned}
$$

which is the same as that of $\widetilde{u}_{(t_1,0)}$ in $M(c'+1, d'+1)$, proving the second recursive formula.

Repeatedly applying the recursive formula yields, for all $(0,0) < (t_1, t_2) \ll (c, d)$,

$$g^{(c,d)}_{(t_1,t_2)} = g^{(c-t_1,d-t_2)}_{(0,0)}.$$

By Lemma A.1 again, $g^{(c,d)}_{(t_1,t_2)} \in v^{-1}\mathbb{Z}[v^{-1}]$. This completes the proof of (ii).

The proof of (iii) can also be reduced by induction to prove that the coefficient of $\widetilde{u}_{(0,0)}$ belongs to $v^{-1}\mathbb{Z}[v^{-1}]$, which is given in Lemma A.1 of Appendix A. ■

A The Coefficient of $\widetilde{u}_{(0,0)}$

To complete the proof of Theorem 8.1, we need the following result. We first rewrite the identity in Lemma 7.3(ii) as

$$
\text{(A.1)} \quad \sum_{i=0}^{\delta} (-1)^i v^{i(2\delta-2n-i-1)+2\delta(n+k)} \overline{\left[\!\!\left[\begin{matrix} k-1+i \\ k-1 \end{matrix} \right]\!\!\right]} \overline{\left[\!\!\left[\begin{matrix} m+n \\ \delta-i \end{matrix} \right]\!\!\right]} = \sum_{t=0}^{\min\{\delta,n\}} v^{2t(\delta+n+k-t)} \overline{\left[\!\!\left[\begin{matrix} m-k \\ \delta-t \end{matrix} \right]\!\!\right]} \left[\!\!\left[\begin{matrix} n \\ t \end{matrix} \right]\!\!\right].
$$

for all $m \geqslant k \geqslant 0, \delta, n \in \mathbb{N}$.

Lemma A.1 *For the numbers $a, b, c, d \in \mathbb{N}$ with $c, d \geqslant 1$ as given in Theorem 8.1, we have*

$$g^{(c,d)}_{(0,0)} \in v^{-1}\mathbb{Z}[v^{-1}],$$

where, for $a > b$,

$$
\begin{aligned}
g^{(c,d)}_{(0,0)} = & \sum_{k_1=0}^{c} (-1)^{c-k_1} v^{(a-b-k_1+d)(k_1-d)} \begin{bmatrix} a-b-1+c-k_1 \\ a-b-1 \end{bmatrix} \overline{\left[\!\!\left[\begin{matrix} a+c+d \\ k_1 \end{matrix} \right]\!\!\right] \left[\!\!\left[\begin{matrix} b+c+d \\ d \end{matrix} \right]\!\!\right]} \\
& - \sum_{l_1=0}^{c-1} (-1)^{c-1-l_1} v^{(a-b-l_1+d-1)(l_1-d+1)} \begin{bmatrix} a-b-2+c-l_1 \\ a-b-1 \end{bmatrix} \overline{\left[\!\!\left[\begin{matrix} a+c+d \\ l_1 \end{matrix} \right]\!\!\right] \left[\!\!\left[\begin{matrix} b+c+d \\ d-1 \end{matrix} \right]\!\!\right]};
\end{aligned}
$$

while, for $a < b$,

$$g_{(0,0)}^{(c,d)} = \sum_{k_1=0}^{d} (-1)^{d-k_1} v^{(b-a-k_1+c)(k_1-c)} \left[{b-a-1+d-k_1 \atop b-a-1} \right] \overline{\left[\!\!\left[{a+c+d \atop c} \right]\!\!\right] \left[\!\!\left[{b+c+d \atop k_1} \right]\!\!\right]}$$
$$- \sum_{l_1=0}^{d-1} (-1)^{d-1-l_1} v^{(b-a-l_1+c-1)(l_1-c+1)} \left[{b-a-2+d-l_1 \atop b-a-1} \right] \overline{\left[\!\!\left[{a+c+d \atop c-1} \right]\!\!\right] \left[\!\!\left[{b+c+d \atop l_1} \right]\!\!\right]}.$$

Proof We only prove the $a > b$ case; the other case can be proved similarly. Rewrite $g_{(0,0)}^{(c,d)}$ as

$$g_{(0,0)}^{(c,d)} = \sum_{k_1=0}^{c} (-1)^{c-k_1} v^{(a-b-k_1+d)(k_1-d)+(c-k_1)(a-b-1)} \overline{\left[\!\!\left[{a-b-1+c-k_1 \atop a-b-1} \right]\!\!\right] \left[\!\!\left[{a+c+d \atop k_1} \right]\!\!\right]} \cdot \overline{\left[\!\!\left[{b+c+d \atop d} \right]\!\!\right]}$$
$$- \sum_{l_1=0}^{c-1} (-1)^{c-1-l_1} v^{(a-b-l_1+d-1)(l_1-d+1)+(a-b-1)(c-1-l_1)}$$
$$\times \overline{\left[\!\!\left[{a-b-2+c-l_1 \atop a-b-1} \right]\!\!\right] \left[\!\!\left[{a+c+d \atop l_1} \right]\!\!\right]} \cdot \overline{\left[\!\!\left[{b+c+d \atop d-1} \right]\!\!\right]}.$$

If $c \leqslant d$, then rearranging gives

$$g_{(0,0)}^{(c,d)} = (-1)^c v^{-d(a-b+d)+c(a-b-1)} \overline{\left[\!\!\left[{a-b-1+c \atop a-b-1} \right]\!\!\right] \left[\!\!\left[{b+c+d \atop d} \right]\!\!\right]}$$
$$+ \sum_{k_1=1}^{c} (-1)^{c-k_1} v^{(a-b-k_1+d)(k_1-d)+(c-k_1)(a-b-1)} \overline{\left[\!\!\left[{a-b-1+c-k_1 \atop a-b-1} \right]\!\!\right]} \Big(\overline{\left[\!\!\left[{a+c+d \atop k_1} \right]\!\!\right]}$$
$$\times \overline{\left[\!\!\left[{b+c+d \atop d} \right]\!\!\right]} - \overline{\left[\!\!\left[{a+c+d \atop k_1-1} \right]\!\!\right] \left[\!\!\left[{b+c+d \atop d-1} \right]\!\!\right]} \Big).$$

Since $a > b$ and $c \leqslant d$, $-d(a-b+d) + c(a-b-1) = (a-b-1)(c-d) - d(1+d) < 0$ and so the first term is in $v^{-1}\mathbb{Z}[v^{-1}]$. Since the difference of the product of Gaussian polynomials is in $v^{-1}\mathbb{Z}[v^{-1}]$, and $(a-b-k_1+d)(k_1-d) + (c-k_1)(a-b-1) = (a-b-1)(c-d) + (1+d-k_1)(k_1-d) \leqslant 0$, this proves $g_{(0,0)}^{(c,d)} \in v^{-1}\mathbb{Z}[v^{-1}]$ in this case.

We now assume that $c > d$. By rearranging the exponents of v, $g_{(0,0)}^{(c,d)}$ has the form

$$g_{(0,0)}^{(c,d)} = v^{-(a-b)(c+d)-c^2-d^2} \overline{\left[\!\!\left[{b+c+d \atop d} \right]\!\!\right]} \cdot S_1$$
$$- v^{2(a-b+c+d-1)-(a-b)(c+d)-c^2-d^2} \overline{\left[\!\!\left[{b+c+d \atop d-1} \right]\!\!\right]} \cdot S_2,$$

where

$$S_1 = \sum_{k_1=0}^{c} (-1)^{c-k_1} v^{(c-k_1)(c+k_1-2d-1)+2c(a-b+d)} \overline{\left[\!\!\left[{a-b-1+c-k_1 \atop a-b-1} \right]\!\!\right] \left[\!\!\left[{a+c+d \atop k_1} \right]\!\!\right]},$$
$$S_2 = \sum_{l_1=0}^{c-1} (-1)^{c-1-l_1} v^{(c-1-l_1)(c+l_1-2d)+2(c-1)(a-b+d-1)} \overline{\left[\!\!\left[{a-b-2+c-l_1 \atop a-b-1} \right]\!\!\right] \left[\!\!\left[{a+c+d \atop l_1} \right]\!\!\right]}.$$

Applying (A.1) (*i.e.*, Lemma 7.3(ii)) to S_1 with $k = a-b, m = a+c, n = d, i = c-k_1, \delta = c$ and to S_2 with $k = a-b, m = a+c+1, n = d-1, i = c-1-l_1, \delta = c-1$ yields

$$S_1 = \sum_{t=0}^{d} v^{2t(a+c+d-b-t)} \overline{\left[\!\!\left[{b+c \atop c-t} \right]\!\!\right] \left[\!\!\left[{d \atop t} \right]\!\!\right]}, \quad S_2 = \sum_{t=0}^{d-1} v^{2t(a+c+d-b-2-t)} \overline{\left[\!\!\left[{b+c+1 \atop c-1-t} \right]\!\!\right] \left[\!\!\left[{d-1 \atop t} \right]\!\!\right]}.$$

Thus,

$$\begin{aligned}
g^{(c,d)}_{(0,0)} &= v^{-(a-b)(c+d)-c^2-d^2}\overline{\left[\!\!\left[{b+c+d \atop d}\right]\!\!\right]\left[\!\!\left[{b+c \atop c}\right]\!\!\right]} \\
&\quad + v^{-(a-b)(c+d)-c^2-d^2}\overline{\left[\!\!\left[{b+c+d \atop d}\right]\!\!\right]}\Big(\sum_{t=1}^{d} v^{2t(a+c+d-b-t)}\overline{\left[\!\!\left[{b+c \atop c-t}\right]\!\!\right]\left[\!\!\left[{d \atop t}\right]\!\!\right]}\Big) \\
&\quad - v^{2(a-b+c+d-1)-(a-b)(c+d)-c^2-d^2}\overline{\left[\!\!\left[{b+c+d \atop d-1}\right]\!\!\right]} \\
&\qquad \times \Big(\sum_{t=0}^{d-1} v^{2t(a+c+d-b-2-t)}\overline{\left[\!\!\left[{b+c+1 \atop c-1-t}\right]\!\!\right]\left[\!\!\left[{d-1 \atop t}\right]\!\!\right]}\Big).
\end{aligned}$$

Changing the running index $t \in \{0,1,\dots,d-1\}$ to $t \in \{1,2,\dots,d\}$ in the last sum gives

$$\begin{aligned}
g^{(c,d)}_{(0,0)} =& v^{-(a-b)(c+d)-c^2-d^2}\overline{\left[\!\!\left[{b+c+d \atop d}\right]\!\!\right]\left[\!\!\left[{b+c \atop c}\right]\!\!\right]} \\
&+ \sum_{t=1}^{d} v^{-(a-b)(c+d)-c^2-d^2+2t(a+c+d-b-t)} \\
&\quad \times \Big(\overline{\left[\!\!\left[{b+c+d \atop d}\right]\!\!\right]\left[\!\!\left[{b+c \atop c-t}\right]\!\!\right]\left[\!\!\left[{d \atop t}\right]\!\!\right]} - \overline{\left[\!\!\left[{b+c+d \atop d-1}\right]\!\!\right]\left[\!\!\left[{b+c+1 \atop c-t}\right]\!\!\right]\left[\!\!\left[{d-1 \atop t-1}\right]\!\!\right]}\Big).
\end{aligned}$$

The first term is clear in $v^{-1}\mathbb{Z}[v^{-1}]$ since $a > b$. Now, $c > d$ implies that

$$\begin{aligned}
&-(a-b)(c+d) - c^2 - d^2 + 2t(a+c+d-b-t) \\
&\quad \leqslant -(a-b)(c+d) - c^2 - d^2 + 2d(a+c-b) \\
&\quad = -(c-d)(a-b+c-d) < 0
\end{aligned}$$

for any $t = 1,2,\dots,d$. Hence, $g^{(c,d)}_{(0,0)} \in v^{-1}\mathbb{Z}[v^{-1}]$. ■

Added in proof Lemma 3.1 has already been observed by D. E. Knuth [17].

References

[1] A. A. Beilinson, G. Lusztig, and R. MacPherson, *A geometric setting for the quantum deformation of GL_n*. Duke Math. J. **61**(1990), no. 2, 655–677. http://dx.doi.org/10.1215/S0012-7094-90-06124-1

[2] K. Bongartz, *On degenerations and extensions of finite dimensional modules*. Adv. Math. **121**(1996), 245–287. http://dx.doi.org/10.1006/aima.1996.0053

[3] B. Deng and J. Du, *Monomial bases for quantum affine $\mathfrak{sl}_n$*. Adv. Math. **191**(2005), 276–304. http://dx.doi.org/10.1016/j.aim.2004.03.008

[4] B. Deng, J. Du, and Q. Fu, *A double Hall algebra approach to affine quantum Schur-Weyl theory*. London Mathematical Society Lecture Note Series, 401, Cambridge University Press, Cambridge, 2012. http://dx.doi.org/10.1017/CBO9781139226660

[5] B. Deng, J. Du, B. Parashall, and J. Wang, *Finite dimensional algebras and quantum groups*. Mathematical Surveys and Monographs, 150, American Mathematical Society, Providence, RI, 2008. http://dx.doi.org/10.1090/surv/150

[6] B. Deng, J. Du, and J. Xiao, *Generic extensions and canonical bases for cyclic quivers*. Canad. J. Math. **59**(2007), no. 6, 1260–1283. http://dx.doi.org/10.4153/CJM-2007-054-7

[7] V. Drinfeld, *A new realization of Yangians and quantized affine algebras*. Sov. Math. Dokl. **36**(1988), no. 2, 212–216.

[8] J. Du, *IC bases and quantum linear groups*. Proc. Sympos. Pure Math. **56**(1994), 135–148.

[9] J. Du and Q. Fu, *A modified BLM approach to quantum affine $\mathfrak{gl}_n$*. Math. Z. **266**(2010), no. 4, 747–781. http://dx.doi.org/10.1007/s00209-009-0596-6

[10] ———, *Quantum affine $\mathfrak{gl}_n$ via Hecke algebras*. Adv. Math. **282**(2015), 23–46. http://dx.doi.org/10.1016/j.aim.2015.06.007

[11] ———, *The Integral Quantum loop algebra of $\mathfrak{gl}_n$*. arxiv:1404.5679

[12] J. Y. Guo, *The Hall polynomials of a cyclic serial algebra*. Comm. Algebra. **23**(1995), 743–751. http://dx.doi.org/10.1080/00927879508825245

[13] Z. Fan, C. Lai, Y. Li, L. Luo, and W. Wang. *Affine flag varieties and quantum symmetric pairs*. Memoirs of the AMS, to appear. arxiv:1602.04383

[14] Z. Fan and Y. Li, *Positivity of canonical bases under comultiplication*. arxiv:1511.02434v3

[15] V. Ginzburg and E. Vasserot, *Langlands reciprocity for affine quantum groups of type A_n*. Internat. Math. Res. Notices **1993**, 67–85. http://dx.doi.org/10.1155/S1073792893000078

[16] J. C. Jantzen, *Lectures on quantum groups*. Graduate Studies in Mathematics, 6, American Mathematical Society, Providence, RI, 1995.

[17] D. E. Knuth, *Subspaces, subsets, and partitions*. J. Combinatorial Theory Ser. A **10**(1971), 178–180.

[18] C. Lai and L. Luo, *An elementary construction of monomial bases of quantum affine $\mathfrak{gl}_n$*. arxiv:1506.07263v1

[19] G. Lusztig, *Canonical bases arising from quantized enveloping algebras*. J. Amer. Math. Soc. **3**(1990), no. 2, 447–498. http://dx.doi.org/10.1090/S0894-0347-1990-1035415-6

[20] ———, *Affine quivers and canonical bases*. Inst. Hautes Études Sci. Publ. Math. **76**(1992), 111–163.

[21] ———, *Introduction to quantum groups*. Progress in Mathematics, 110, Birkhäuser, Boston, MA, 1993.

[22] ———, *Tight monomials in quantized enveloping algebras*. In: Quantum deformations of algebras and their representations, Israel Math. Conf. Proc., 7, Bar-Ilan University, Ramat Gan, 1993, pp. 117–132.

[23] ———, *Aperiodicity in quantum affine $\mathfrak{gl}_n$*. Asian J. Math. **3**(1999), 147–177. http://dx.doi.org/10.4310/AJM.1999.v3.n1.a7

[24] M. Reineke, *Generic extensions and multiplicative bases of quantum groups at $q = 0$*. Represent. Theory. **5**(2001), 147–163. http://dx.doi.org/10.1090/S1088-4165-01-00111-X

[25] C. M. Ringel, *Hall algebras and quantum groups*. Invent. Math. **101**(1990), 583–592. http://dx.doi.org/10.1007/BF01231516

[26] ———, *Hall algebras revisited*. In: Quantum deformations of algebras and their representations, Israel Math. Conf. Proc., 7, Bar-Ilan University, Ramat Gan, 1993, pp. 171–176.

[27] ———, *The composition algebra of a cyclic quiver*. Proc. London Math. Soc. **66**(1993), 507–537. http://dx.doi.org/10.1112/plms/s3-66.3.507

[28] O. Schiffmann, *Lectures on Hall algebras*. In: Geometric methods in representation theory. II., Sémin. Congr., 24, Soc. Math. France, Paris, 2012, pp. 1–141.

[29] M. Varagnolo and E. Vasserot, *On the decomposition matrices of the quantized Schur algebra*. Duke Math. J. **100**(1999), 267–297. http://dx.doi.org/10.1215/S0012-7094-99-10010-X

[30] N. Xi, *Canonical basis for type A_3*. Comm. Algebra. **27**(1999), no. 11, 5703–5710. http://dx.doi.org/10.1080/00927879908826784

[31] ———, *Canonical basis for type B_2*. J. Algebra. **214**(1999), no. 1, 8–21. http://dx.doi.org/10.1006/jabr.1998.7688

[32] J. Xiao, *Drinfeld double and Ringel–Green theory of Hall algebras*. J. Algebra. **190**(1997), no. 1, 100–144. http://dx.doi.org/10.1006/jabr.1996.6887

[33] G. Zwara, *Degenerations for modules over representation-finite biserial algebras*. J. Algebra. **198**(1997), no. 2, 563–581. http://dx.doi.org/10.1006/jabr.1997.7160

School of Mathematics and Statistics, University of New South Wales, Sydney 2052, Australia
e-mail: j.du@unsw.edu.au

Department of Mathematics, Beijing University of Chemical Technology, Beijing 100029, China
e-mail: zhaozh@mail.buct.edu.cn