Canad. Math. Bull. Vol. **32** (2), 1989

# U-SETS IN COMPACT, 0-DIMENSIONAL, METRIC GROUPS

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ABSTRACT. This paper studies a pointwise definition of sets of uniqueness on compact, 0-dimensional, metric groups. It is shown that this definition is equivalent for closed sets to one based on supports of pseudo functions. An analog of Rajchman's theorem is given leading to examples of sets of uniqueness.

In [6], Vilenkin initiated the study of uniqueness in compact, 0-dimensional, abelian, metric groups. One of his results is that the empty set is a set of uniqueness. Since that time, investigations into the structure of sets of uniqueness on the group of integers of a $p$-series field have been carried out, for example, by Wade [7] and Yoneda [8]. The purpose of this paper is to continue the preliminary investigation in Grubb [4] in the specific case of compact, 0-dimensional, metric groups and to give methods of constructing sets of uniqueness in this case.

Let $G$ be a compact, 0-dimensional, metric group, not necessarily abelian, and let $\{H_n\}_0^\infty$ be a strictly decreasing sequence of open normal subgroups forming a neighborhood base at the identity. Let $\Sigma$, the dual object of $G$, denote the set of equivalence classes of irreducible representations of $G$. If $\sigma \in \Sigma$, we pick a irreducible representation $U^\sigma$ in the equivalence class $\sigma$. We also let $H^\sigma$ and $d_\sigma$ represent, respectively, the Hilbert space and the dimension of the Hilbert space on which the representation $U^\sigma$ acts. If $A \in B(H^\sigma)$, a bounded linear operator on $H^\sigma$, we set

$$\|A\|_1 = \Sigma|\lambda_k| \quad \text{and} \quad \|A\|_\infty = \max|\lambda_k|$$

where the $\lambda_k$ are the eigenvalues of $A$. We also set

$$C(\Sigma) = \prod_{\sigma\in\Sigma} B(H^\sigma),$$

$$F_0(\Sigma) = \{(A_\sigma) \in C(\Sigma) : \|A_\sigma\|_\infty \text{ vanishes at infinity on } \Sigma\},$$

$$F_1(\Sigma) = \left\{(A_\sigma) \in C(\Sigma) : \sum_{\sigma\in\Sigma} d_\sigma\|A_\sigma\|_1 < \infty\right\}$$

We also let $A(G)$ denote the algebra of absolutely convergent Fourier series on $G$. Thus, if $f \in A(G)$, we may write by 34.5 of [5]

$$f(x) = \sum_{\sigma\in\Sigma} d_\sigma tr(A_\sigma U^\sigma(x)) \tag{1}$$

Received by the editors March 6, 1987 and, in revised form, August 19, 1987
AMS Classification: Primary 43A46 Secondary 42C10.

where $(A_\sigma) \in F_1(\Sigma)$. Conversely, for $(A_\sigma) \in F_1(\Sigma)$, the above expression defines an element of $A(G)$. Since $F_0(\Sigma)$ is a subspace of the dual space of $F_1(\Sigma)$ (see 28.31 of [5]), we may regard elements of $F_0(\Sigma)$ as distributions acting on the test functions $A(G)$. In this guise, elements of $F_0(\Sigma)$ are called pseudofunctions on $G$ and rename $F_0(\Sigma)$ to $PF(G)$. Thus for $f \in A(G)$ as in (1) and $S \in PF(G)$ with formal expansion

$$S \sim \sum_{\sigma \in \Sigma} d_\sigma tr(B_\sigma U^\sigma)$$

where $(B_\sigma) \in F_0(\Sigma)$, we have by 28.31 of [5], the action of $S$ on $f$:

$$\langle S, f \rangle = \sum_{\sigma \in \Sigma} d_\sigma tr(B_\sigma^* A_\sigma).$$

For a pseudofunction $S$, and element $x$ in $G$, and a non-negative integer $n$, we define the $n^{th}$ partial sum of the Fourier series of $S$ at $x$ to be

$$S_n(S,x) = S_n(S)(x) = \langle S, \xi_{xH_n} \rangle / \lambda(H_n)$$

where $\lambda$ is Haar measure on $G$. Note that for our groups $\xi_{xH_n} \in A(G)$. In fact, if $H$ is an open subgroup and $x \in G$, $\|\xi_{xH}\|_A = 1$.

If $E$ is a subset of $G$, we say that $E$ is a set of uniqueness ($U$-set) if the only pseudofunction $S$ satisfying

$$\lim_{n \to \infty} S_n(S,x) = 0 \quad \text{for} \quad x \text{ not in } E$$

is $S = 0$. It is shown in [3] that for abelian groups and $\sigma$-compact sets, this is equivalent to the definition of $U$-sets using Vilenkin's summation method.

If $S$ is a pseudofunction and $E$ is a closed set, we say that $S$ is supported on $E$ if $\langle S, f \rangle = 0$ for every $f \in A(G)$ with support disjoint from $E$. The next result shows that a closed set is a $U$-set if and only if there are no nonzero pseudofunctions supported on it.

THEOREM 1. *Let $S \in PF(G)$ and $V \subseteq G$ be open. The following are equivalent:*
i) $\lim_{n \to \infty} S_n(S,x) = 0$ *for all* $x \in V$,
ii) *the limit in i) is uniform on compacta in* $V$,
iii) $S$ *is supported on* $G \backslash V$.

The following proof works if $S$ is a pseudomeasure on $G$, i.e. a bounded linear functional on $A(G)$.

PROOF. We recall from Theorem 4.3 of [4] that if $\lim_{n \to \infty} S_n(S,x) = 0$ for all $x$ in $yH_m$, then $\langle S, \xi_{yH_m} \rangle = 0$.

i) $\Rightarrow$ iii) Let $xH_n \subseteq V$ be a basic open set. If $yH_m$ is a basic open set not necessarily contained in $V$, the above says that $\langle S, \xi_{xH_n} \xi_{yH_m} \rangle = 0$. But functions of the form $\xi_{yH_m}$ generate the space of trigonometric polynomials which is dense in $A(G)$. Thus

$$\langle S, \xi_{xH_n} f \rangle = 0 \quad \text{for all} \quad f \in A(G).$$

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If $f$ has support contained in $V$, cover the support with finitely many disjoint sets of the form $xH_n \subseteq V$. Writing $f = \sum f \xi_{xH_n}$ with the sum over these set shows $\langle S, f\rangle = 0$.

iii) $\Rightarrow$ ii) Covering compact sets with sets of the form $xH_n \subseteq V$ shows that $S_m(S, y)$ is eventually 0 on the compactum, since for such $xH_n$, $\langle S, \xi_{xH_n}\rangle = 0$.

ii) $\Rightarrow$ i) is trivial. □

In particular, for closed sets, our definition of $U$-set is equivalent to both Vilenkin's original and the more general one of not supporting non-zero pseudofunctions. See for example, Graham and McGehee [2] in the abelian case and Bozoejko [1] in the non-abelian case for the definition of $U$-sets in locally compact groups in terms of pseudofunctions. Note that the space $C_\rho^*$in [1] is our $PF(G)$.

A corollary to this theorem is that measurable $U$-sets are of (Haar) measure 0. In fact, if $K$ is a closed set of positive measure, we may regard $\xi_K$ as a non-zero pseudofunction (see below) whose support is $K$. Thus $K$ is not a $U$-set. Since any measurable set of positive measure contains such a closed set $K$, and since subsets of $U$-sets are $U$-sets, the stated corollary follows. It follows from Theorem 4.6 of [4] that a countable union of closed $U$-sets is again a $U$-set. Thus countable sets are $U$-sets.

Our definition of a $U$-set also gives a uniqueness result for integrable functions on $G$.

THEOREM 2. *Let $S$ be a pseudofunction, $F : G \longrightarrow \mathbf{C}$ a (Haar) integrable function and $E$ a closed $U$-set in $G$. If*

$$\lim_{n\to\infty} S_n(S,x) = f(x) \quad \text{for} \quad x \text{ not in } E,$$

*then $S = f$ in the sense that*

$$\langle S, g\rangle = \int_G \bar{f} g \, d\lambda \quad \text{for} \quad g \in A(G).$$

PROOF. We first note that the case when $E$ is empty is a special case of Theorem 4.2 of [4]. For the general case, we define $\xi_U S \in PF(G)$ for $U$ a basic open set by

$$\langle \xi_U S, g\rangle = \langle S, \xi_U g\rangle.$$

Since $\xi_U$ is a trigonometric polynomial, it is easy to see that $\xi_U S$ is, in fact, a pseudofunction. Also,

$$\begin{aligned} S_n(\xi_U S, x) &= \langle \xi_U S, \xi_{xH_n}\rangle / \lambda(H_n) \\ &= \langle S, \xi_{U \cap xH_n}\rangle / \lambda(H_n) \\ &= \xi_U(x) \langle S, \xi_{xH_n}\rangle / \lambda(H_n) \\ &= \xi_U(x) S_n(S, x), \end{aligned}$$

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for sufficiently large $n$ depending on $U$.

For $x$ not in $E$, pick $m$ such that $xH_m$ is disjoint from $E$. Then

$$\lim_{n\to\infty} S_n(\xi_{xH_m}S, y) = \xi_{xH_m}(y) \lim_{n\to\infty} S_n(S, y) = \xi_{xH_m}(y)f(y)$$

for all $y \in G$. By the case when $E$ is empty, $\xi_{xH_m}S = \xi_{xH_m}f$. If we now consider the pseudofunction $S - f$, we find that for $xH_m$ disjoint from $E$,

$$\begin{aligned} S_m(S-f,x) &= \langle S-f, \xi_{xH_m}\rangle/\lambda(H_m) \\ &= \langle \xi_{xH_m}(S-f), 1\rangle/\lambda(H_m) \\ &= \langle \xi_{xH_m}S - \xi_{xH_m}f, 1\rangle/\lambda(H_m) \\ &= 0. \end{aligned}$$

Thus

$$\lim_{n\to\infty} S_n(S-f,x) = 0 \quad \text{for} \quad x \text{ not in } E.$$

Since $E$ is a $U$-set, $S = f$. □

We now turn to methods for constructing examples of $U$-sets. The sets produced have a family resemblence to the classical $H$-sets (see 6.3 of [9]) as well as the $pH^{(m)}$-sets of Wade [7]. We start out with a "Rajchman" Theorem.

THEOREM 3. *Let $E$ be a closed subset of $G$ and $\{f_n\}_0^\infty$ a sequence in $A(G)$ such that*

i) *$E$ is disjoint from* supp$f_n$ *for all $n \geqq 0$,*

ii) *$\|f_n\|_A$ is bounded in $n$,*

iii) *If we write $f_n(x) = \sum_{\sigma\in\Sigma} d_\sigma tr(A_{\sigma,n}U^\sigma(x))$, then $A_{1,n} \to 1$, and for $\sigma \neq 1$, $A_{\sigma,n} \to 0$ as $n \to \infty$. Both limits are in $B(H_\sigma)$.*

*Then $E$ is a $U$-set.*

PROOF. Assume $S \in PF(G)$ is supported on $E$ and has formal expansion

$$S \sim \sum_{\sigma\in\Sigma} d_\sigma tr(B_\sigma U^\sigma)$$

with $(B_\sigma) \in F_0(\Sigma)$. We wish to show that $B_0 = 0$ for all $\sigma \in \Sigma$. For any given $\sigma$, set $f_\sigma(x) = d_\sigma tr(B_\sigma U^\sigma(x))$. Then

$$\langle S, f_\sigma\rangle = d_\sigma tr(B_\sigma^* B_\sigma)$$

which is 0 if and only if $B_\sigma = 0$. Since $\langle S, f_\sigma\rangle = \langle \bar{f}_\sigma S, 1\rangle$ and since $\bar{f}_\sigma S$ is also a pseudofunction supported on $E$, it is enough to show $B_1^* = \langle S, 1\rangle = 0$.

By i) we get that for $n \geqq 0$,

$$\begin{aligned} 0 = \langle S, f_n\rangle &= \sum_{\sigma\in\Sigma} d_\sigma tr(B_\sigma^* A_{\sigma,n}) \\ &= tr(B_1^* A_{1,n}) + \sum_{\sigma\neq 1} d_\sigma tr(B_\sigma^* A_{\sigma,n}). \end{aligned}$$

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If $\epsilon > 0$, find a finite subset $F$ of $\sum$ such that $\|B_\sigma\|_\infty < \epsilon$ for $\sigma$ not in $F$. Then

$$\begin{aligned}|tr(B_1^*A_{1,n})| &\leqq \sum_{\sigma\neq 1} d_\sigma |tr(B_\sigma^* A_{\sigma,n})| \\ &\leqq \sum_{1\neq\sigma\in F} d_\sigma |tr(B_\sigma^* A_{\sigma,n})| + \sum_{\sigma\notin F} d_\sigma |tr(B_\sigma^* A_{\sigma,n})| \\ &\leqq \sum_{1\neq\sigma\in F} d_\sigma \|B_\sigma\|_\infty \cdot \|A_{\sigma,n}\|_1 + \sum_{\sigma\notin F} d_\sigma \|B_\sigma\|_\infty \cdot \|A_{\sigma,n}\|_1 \\ &\leqq \|S\|_{PF} \sum_{1\neq\sigma\in F} d_\sigma \|A_{\sigma,n}\|_1 + \epsilon \|f_n\|_A.\end{aligned}$$

As $n \longrightarrow \infty$, $tr(B_{1,n}^* A_{1,n})$ converges to $B_1^*$ by iii), while $\sum_{1\neq\sigma\in F} d_\sigma \|A_{\sigma,n}\|_q \longrightarrow 0$. Thus

$$|B_1^*| \leqq \epsilon M \quad \text{where} \quad M = \sup \|f_n\|_A.$$

Since $\epsilon > 0$ is arbitrary, the theorem is proved. □

We now apply this result to two cases; the first of which is essentially that of groups of bounded order, the second that of unbounded order. If $H$ is a closed subgroup of $G$ let $H^\perp = \{\sigma \in \Sigma\colon \hat{\lambda}_H(\sigma) \neq 0\}$ where $\lambda_H$ is the Haar measure on $H$. This is a possibly larger set than the annihilator of $H$ as defined in 28.72 of [5]. If $\{E_n\} \subseteq \Sigma$ is a sequence of subsets, write $E_n \longrightarrow \infty$ as $n \longrightarrow \infty$ if for every finite set $F \subseteq \Sigma$, $E_n$ is eventually disjoint from $F$.

THEOREM 4. *Let $E$ be a closed subset of $G$. Let $\{K_n\}$ be a sequence of open subgroups of $G$ such that*

a) *$EK_n \neq G$ for all $n$,*
b) *$K_n^\perp \backslash \{1\} \longrightarrow \infty$ as $n \longrightarrow \infty$,*
c) *the index of $K_n$ in $G$ is bounded.*

*Then $E$ is a $U$-set.*

Notice that the $K_n$ need not be normal.

PROOF. For each $n$, pick $x_n$ such that $x_nK_n \cap E = \emptyset$, and define $f_n = \xi_{x_nK_n}/\lambda(K_n)$. Since $K_n$ is open, $f_n$ is a trigonometric polynomial and so is in $A(G)$. Also, writing $f_n(x) = \sum_{\sigma\in\Sigma} d_\sigma tr(A_{\sigma,n} U^\sigma(x))$, we have $\{\sigma \in \Sigma\colon A_{\sigma,n} \neq 0\} \subseteq K_n^\perp$. Thus $A_{\sigma,n} \longrightarrow 0$ for $\sigma \neq 1$ by b). Since $A_{1,n} = \int f_n d\lambda = 1$ for all $n$, parts i) and iii) of the previous theorem are satisfied. Finally, $\|f_n\|_A = \|\xi_{x_nK_n}\|_A/\lambda(K_n) = 1/\lambda(K_n) = [G\colon K_n]$ is bounded. Theorem 3 gives the result. □

An example may be obtained by taking $G = \prod_1^\infty F$ to be a countable product of some finite group $F$ and $E = \prod_1^\infty (F\backslash\{e\})$. If $\pi_n\colon G \longrightarrow F$ is the $n^{th}$ projection, $K_n = ker\,\pi_n$ satisfies the hypotheses of the theorem. If, however, we take a product of different finite groups, care must be made to insure condition c) above holds, else the set $E$ may have positive measure and so could not be a $U$-set. In an attempt to

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weaken c) to include such products, some leeway must be made in condition a) to avoid this possibility. This leads to the next result.

THEOREM 5. *Let $E$ be a closed subset of $G$. Let $\{K_n\}$ be a sequence of open subgroups of $G$ such that*

a) *card $\{xK_n : xK_n \cap E$ is nonvoid $\}$ is bounded in $n$,*

b) $K_n^{\perp}\backslash\{1\} \longrightarrow \infty$ *as* $n \longrightarrow \infty$,

c) $\lambda(K_n) \longrightarrow 0$ *as* $n \longrightarrow \infty$,

*Then $E$ is a $U$-set.*

PROOF. Let $g_n = \Sigma\xi_{xK_n}$, the sum being over those cosets $xK_n$ which intersect $E$, and set $f_n = 1 - g_n$. Then $\|g_n\|_A \leqq \Sigma\|\xi_{xK_n}\|_A = \Sigma 1$ is bounded by a), thus $\|f_n\|_A$ is bounded. By b), we get that $A_{\sigma,n} \longrightarrow 0$ if $\sigma \neq 1$, where the notation is as in Theorem 3. Finally, by c)

$$A_{1,n} = \int f_n d\lambda = 1 - \int g_n d\lambda = 1 - \lambda(K_n) \cdot \Sigma 1 \longrightarrow 1$$

as $n \longrightarrow \infty$. □

An example of this theorem is obtained in the group $G = \prod_{n=2}^{\infty} \mathbf{Z}(n)$, which is a product of a sequence of finite cyclic groups. If $K_n = ker\pi_n$, where $\pi_n : G \longrightarrow \mathbf{Z}(n)$ is $n^{th}$ the projection, the above result shows that $E = \prod_{n=2}^{\infty}\{0, 1\} \subseteq \prod_{n=2}^{\infty} \mathbf{Z}(n) = G$ is a $U$-set.

Finally, we give a result that connects the idea of a $U$-set to non-open subgroups of $G$.

THEOREM 6. *Let $H$ be a closed subgroup of $G$ of infinite index. Then $H$ is a $U$-set.*

PROOF. Assume $S \in PF(G)$ is supported on $H$. As in the proof of Theorem 3, it is enough to show that $\langle S, 1\rangle = 0$.

For each $\sigma \in \Sigma$, there is a basis $\{y_1, \ldots, y_{d_\sigma}\}$ of $H^\sigma$ and a number $0 \leqq m_\sigma \leqq d_\sigma$ such that $u_{ij}^\sigma(x) = \delta_{ij}$ if $j \leqq m_\sigma$ and $x \in H$, where $u_{ij}^\sigma$ is the coordinate function of $U^\sigma$ in the basis $\{y_1, \ldots, y_{d_\sigma}\}$. This is proved by considering the Haar measure $\lambda_H$ on $H$, see section 28.72 of [5]. In fact, the basis is one for which $\hat{\lambda}_H(\sigma)$ is diagonal.

If $m_\sigma = 0$ except for finitely many $\sigma \in \Sigma$, we would have $\lambda_H = g\lambda_G$ where $g$ is a trigonometric polynomial. But this would show $H$ to be an open subgroup and so not of infinite index.

Thus, the set $Q = \{\sigma \in \Sigma : m_\sigma \geqq 1\}$ is infinite. If $\sigma \in Q$, and we write

$$S \sim \sum_{\sigma \in \Sigma} d_\sigma tr(B_\sigma U^\sigma),$$

we get for $j \leqq m_\sigma$,

$$\overline{(B_\sigma)_{ji}} = \langle S, u_{ij}^\sigma\rangle = \delta_{ij}\langle S, 1\rangle \qquad (*)$$

since $u_{ij}^{\sigma} = \delta_{ij}$ on a neighborhood of $H$. (In fact, if $\sigma \in H_n^{\perp}$, $u_{ij}^{\sigma} = \delta_{ij}$ on $HH_n$). But now (*) shows that $\overline{\langle S, 1 \rangle}$ is an eigenvalue of $B_\sigma$, so

$$\|B_\sigma\|_\infty \geqq |\langle S, 1 \rangle| \geqq 0.$$

Since $(B_\sigma) \in F_0(\Sigma)$ and $Q$ is infinite, $\langle S, 1 \rangle = 0$ as desired. □

Finally, it should be noted that Theorems 3–5 do not explicitly use the assumption that $G$ is totally disconnected.

Many thanks to the referee for several helpful suggestions.

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