A COEFFICIENT PROBLEM FOR FUNCTIONS REGULAR IN AN ANNULUS

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1. Introduction. A solution will be given in this paper for the following problem.

Let

$$w = f(z) = \sum_{-\infty}^{\infty} a_n z^n \tag{1.1}$$

be regular and single-valued in an annulus $0 \leqslant \rho \leqslant |z| < 1$. Let C_r denote the image in the w-plane of the circle $|z| = r$, $\rho < r < 1$, and let us suppose that there exists in the w-plane a straight line which cuts C_r, for every r in the range $\rho < r < 1$, in precisely $2p$ distinct points, p being a positive integer. What are the sharp bounds enforced upon the complex coefficients a_n $(n = 0, \pm 1, \ldots)$ by these conditions?

It will be noticed at once that, because of the presence of the arbitrary constant a_0 in (1.1), and because all the coefficients may be complex numbers, no restriction will be put upon the problem if we assume in what follows that the given straight line is the real axis.

Special cases of the above problem have been solved during the past two decades. These cases have been confined to functions $f(z)$ analytic in the circle $0 \leqslant |z| < 1$ $(\rho = 0)$. If $\rho = 0$, $p = 1$, and if $f(z)$ has real coefficients in (1.1) with $a_0 = 0$, $a_1 = 1$, then $f(z)$ is typically-real for $|z| < 1$. In this case it was shown by Rogosinski [8], Dieudonné [1] and Szász [9] that

$$|a_n| \leqslant n|a_1|, \qquad n = 2, 3, \ldots. \tag{1.2}$$

For $\rho = 0, p = 1, a_0 = 0, a_n$ complex numbers, it was shown by the author [5] that

$$|a_n| \leqslant n^2|a_1|, \qquad n = 2, 3, \ldots. \tag{1.3}$$

The inequalities expressed in (1.3) were obtained again at a later date by Kakeya [4]. Both (1.2) and (1.3) are sharp. When $\rho = 0$, p arbitrary, and when $a_0 = a_1 = \ldots = a_{p-1} = 0$, $a_p \neq 0$, a_n complex for $n \geqslant p$, it was also shown [6] that

$$|a_n| \leqslant \frac{2}{(2p)!}\prod_{\nu=0}^{p-1} (n^2 - \nu^2)\cdot |a_p|, \quad n = p+1, p+2, \ldots, \tag{1.4}$$

and (1.4) is also sharp. If, further, all the coefficients are real, the inequalities (1.4) are replaced [6] by the sharp inequalities (1.5),

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(1.5) $$|a_n| \leqslant \frac{1}{n(2p-1)!}\prod_{\nu=0}^{p-1}(n^2-\nu^2)\cdot|a_p|, \qquad n = p+1, p+2, \ldots .$$

With the restriction that the first p coefficients be zero removed, the author showed [7] that

(1.6) $$\limsup_{n\to\infty}\left|\frac{a_n}{n^{2p}}\right| \leqslant 2\sum_{k=0}^{p}\frac{|a_k|}{(p+k)!(p-k)!},$$

but no sharp estimates for each individual $|a_n|$, $n > p$, were obtained. Quite recently Goodman and Robertson [3] have solved the problem for $\rho = 0$, p arbitrary, when the coefficients are all real, obtaining the sharp inequalities for $n > p$,

(1.7) $$|a_n| \leqslant \sum_{k=1}^{p}\frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)}\;|a_k|,$$

which is of particular interest since it establishes, for a large class of multivalent functions, the Goodman conjecture [2] that (1.7) holds for all functions multivalent of order p in $|z| < 1$.

In this paper we turn to the general case. Thus p is an arbitrary positive integer. The coefficients $a_n(n = 0, \pm 1, \pm 2, \ldots)$ are complex numbers and ρ may be positive or zero. We shall denote by $\bar{a}$ the complex conjugate of the number a. Our main result is stated in the following theorem.

THEOREM 1. *Let*

(1.8) $$w = f(z) = \sum_{-\infty}^{\infty} a_n z^n$$

be regular and single-valued for $0 \leqslant \rho \leqslant |z| < 1$, *the coefficients* a_n *in* (1.8) *being complex numbers. On each circle* $|z| = r$, $\rho < r < 1$, *let the imaginary part of* $f(z)$ *change sign* $2p$ *times, where* p *is a positive integer independent of* r. *Then, for* $n > p$, *the following inequalities hold and are sharp in all the variables* $|a_k - \bar{a}_{-k}|$ $(k = 0, 1, \ldots, p)$:

(1.9) $$|a_n - \bar{a}_{-n}| \leqslant \sum_{k=0}^{p}\Delta(p,k,n)|a_k - \bar{a}_{-k}|,$$

where

$$\Delta(p,0,n) \equiv \prod_{\nu=1}^{p}(n^2-\nu^2)/(p!)^2,$$

$$\Delta(p,k,n) \equiv 2\prod_{\substack{\nu=0\\ \nu\neq k}}^{p}(n^2-\nu^2)/(p+k)!(p-k)!, \qquad p \geqslant k > 0.$$

By a rotation and translation we may obtain the following Corollary to Theorem 1.

COROLLARY. *Let* $f(z)$ *be given as in* (1.8), *regular and single-valued for* $\rho \leqslant |z| < 1$. *Let* l *be any straight line in the* w-*plane. Denote by* $de^{i\gamma}$ $(d \geqslant 0)$, *the point on* l *nearest the origin. For each* r *in the range* $\rho < r < 1$ *let the image* *curve* C_r *of* $|z| = r$, *through the mapping* $w = f(z)$, *cross* l *precisely* $2p$ *times* (p *fixed*). *Then for* $n > p$,

$$(1.10) \qquad |a_n e^{-i\gamma} + \bar{a}_{-n} e^{i\gamma}| \leqslant 2\Delta(p, 0, n)|\Re(a_0 e^{i\gamma}) - d| + \sum_{k=1}^{p} \Delta(p, k, n)|a_k e^{-i\gamma} + \bar{a}_{-k} e^{i\gamma}|.$$

In the corollary, if $d = 0$ we define γ to be $\frac{1}{2}\pi + \beta$, $0 \leqslant \beta < \pi$, where β is the inclination of the line l. If l is the real axis, then $d = \beta = 0$, $\gamma = \frac{1}{2}\pi$, and (1.10) reduces to (1.9).

We call attention to the fact that for $0 < k \leqslant p$,

$$(1.11) \qquad \Delta(p, k, n) = \frac{n}{k} D(p, k, n)$$

where $D(p, k, n)$ is the coefficient of $|a_k|$ in (1.7).

The special case of Theorem 1 where $p = 1$ states that

$$(1.12) \qquad |a_n - \bar{a}_{-n}| \leqslant |n^2 - 1| \cdot |a_0 - \bar{a}_0| + n^2 |a_1 - \bar{a}_{-1}|$$

for all integers n. (1.12) includes (1.3) as a special case. It is also seen that (1.6) follows as another special case of (1.9).

From the identity

$$(1.13) \qquad \Im\{zf'(z)\} = -\frac{\partial}{\partial\theta}\Re f(re^{i\theta}), \qquad z = re^{i\theta},$$

the following theorem is obtained immediately from Theorem 1. It shows that the Goodman conjecture (1.7) for all multivalent functions of order p in $|z| < 1$ holds for still another special class of multivalent functions of order p, this time with the coefficients complex numbers. We state the result as

THEOREM 2. *Let*

$$(1.14) \qquad f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

have complex coefficients and be regular and multivalent of order p *for* $|z| < 1$, *and for each* r *of the interval* $\rho < r < 1$ *let*

$$\frac{\partial}{\partial\theta}\Re f(re^{i\theta})$$

change sign $2p$ *times on* $|z| = r$. *Then for* $n > p$,

$$(1.15) \qquad |a_n| \leqslant \sum_{k=1}^{p} \frac{2k(n+p)!}{(p+k)!(p-k)!(n-p-1)!(n^2-k^2)} |a_k|,$$

and this bound is sharp in all the variables $|a_k|$ $(k = 1, 2, \ldots, p)$.

Theorem 2 was shown [**5**] in the special case $p = 1$, in which case $w = f(z)$ is univalent (*schlicht*) for $|z| < 1$ and maps each circle $|z| = r$, $\rho < r < 1$, onto a contour with the property that each line parallel to the imaginary axis cuts the contour in at most two points. In Theorem 2 with $p \geqslant 1$, each such line is cut in at most $2p$ points.

2. Preliminary remarks. For the sake of clarity we give here a few definitions from [**3**].

Definition 1. The harmonic function $v(r, \theta)$ is said to have a *change of sign* at $\theta = \theta_j$ if there exists an $\epsilon > 0$ such that, for $0 < \delta < \epsilon$,

$$v(r, \theta_j + \delta) \cdot v(r, \theta_j - \delta) < 0. \tag{2.1}$$

Definition 2. $\Im f(z) = v(r, \theta)$ is said to *change sign q times* on $|z| = r$ if there are q values of θ, say $\theta_1, \theta_2, \ldots, \theta_q$ such that

(a) inequality (2.1) holds for each θ_j $(j = 1, 2, \ldots, q)$,

(b) $\theta_j \not\equiv \theta_k \pmod{2\pi}$ if $j \neq k$,

(c) if θ_j is any value of θ for which $v(r, \theta)$ has a change of sign then, for one of the θ_j $(j = 1, \ldots, q)$, $\theta_s \equiv \theta_j \pmod{2\pi}$.

In proving (1.9) of Theorem 1 we may assume that $f(z)$, given by (1.8), is regular on $|z| = 1$. For if this is not the case and $f(z)$ is regular for $0 \leqslant \rho \leqslant |z| < 1$ and $f(z) \equiv f_-(z) + f_+(z)$, where

$$f_+(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad f_-(z) \equiv \sum_{n=1}^{\infty} a_{-n} z^{-n},$$

then, for t sufficiently near to, but less than, one,

$$f_t(z) \equiv f_+(tz) + f_-(z/t) \tag{2.2}$$

is regular on $|z| = 1$. Having proved (1.9) for $f_t(z)$ we have only to let $t \to 1$ to obtain (1.9) for $f(z)$. Hence in what follows we shall assume $f(z)$ to be regular on $|z| = 1$.

LEMMA 1. *Let $f(z)$ be given by* (1.8) *and be regular for $\rho \leqslant |z| \leqslant 1$. Let $\Im f(e^{i\theta})$ change sign $2p$ times at $\theta = \theta_1, \theta_2, \ldots, \theta_{2p}$, $(p \geqslant 1)$. There exist real numbers μ and ν such that, if $g(z)$ is defined by*

$$g(z) = (z + z^{-1} - 2\cos\nu) f(ze^{i\mu}) = \sum_{-\infty}^{\infty} b_n z^n, \tag{2.3}$$

then $\Im g(e^{i\theta})$ changes sign $2(p-1)$ times.

Proof. Since $\Im g(e^{i\theta}) = 2(\cos\theta - \cos\nu)\Im f(e^{i(\theta+\mu)})$ we may take

$$2\mu = \theta_i + \theta_j, \quad 2\nu = \theta_j - \theta_i, \tag{2.4}$$

where θ_i and θ_j, $i \neq j$, are any two values of θ where $\Im f(e^{i\theta})$ changes sign. Then $\Im f(e^{i(\theta+\mu)})$ changes sign at $\theta = \pm(\theta_j - \theta_i)/2 = \pm\nu$ as well as at $2(p-1)$ other values of θ. However, $(\cos\theta - \cos\nu)$ changes sign exactly twice in $-\pi \leqslant \theta < \pi$, at $\theta = \pm\nu$. Thus $\Im g(e^{i\theta})$ does not change sign at $\pm\nu$ but does change sign at the $2(p-1)$ other values of θ (nowhere else if $p = 1$).

The method of proof for (1.9) will be by induction on p by means of (2.3). Note that even in the special case where $f(z)$ is regular in the whole unit circle $|z| \leqslant 1$, $g(z)$ is not regular at $z = 0$ when $a_0 \neq 0$. This difficulty was easily avoided [**3**] when all the coefficients were real and $\mu = 0$.

The proof of (1.7) by induction on p made use of the fact that (1.7) was already known for $p = 1$, i.e., that (1.2) held. However, (1.9) is not known to be true even for $p = 1$, except in the special case (1.3). It will therefore be necessary to establish (1.12) first. In proving (1.12) we shall need the following lemma which appears in [**7**, p. 514].

LEMMA 2. *Let*

$$F(z) = \sum_{-\infty}^{\infty} A_n z^n \tag{2.5}$$

be analytic and single-valued for $\rho < |z| < 1$. *If* $\Re F(re^{i\theta}) \geqslant 0$ *for* $\rho < r < 1$, *then*

$$|A_n + \bar{A}_{-n}| \leqslant 2\Re A_0, \tag{2.6}$$

where $\bar{A}_{-n}$ *denotes the complex conjugate of* A_{-n}. *If* $F(z)$ *is regular on* $|z| = 1$, *and if* $\Re F(e^{i\theta}) \geqslant 0$, *then* (2.6) *again holds.*

3. Proof of (1.12). Let

$$\phi(z) = c_0 + c_1 z + \ldots + c_n z^n + \ldots \tag{3.1}$$

be regular for $|z| \leqslant 1$ and let $\Im\phi(z)$ change sign exactly twice on $|z| = 1$. By Lemma 1, μ and ν exist so that

$$\begin{aligned} g(z) &= (z + z^{-1} - 2\cos\nu)\,\phi(ze^{\mu i}) \\ &= b_{-1}z^{-1} + b_0 + b_1 z + \ldots + b_n z^n + \ldots \end{aligned} \tag{3.2}$$

is regular on $|z| = 1$ and $\Im g(z)$ does not change sign on $|z| = 1$. Since $ig(z)$ (or $-ig(z)$) satisfies the conditions of Lemma 2, we have

$$|b_1 - \bar{b}_{-1}| \leqslant |\Im(b_0 - \bar{b}_0)|, \tag{3.3}$$

$$|b_n| \leqslant |\Im(b_0 - \bar{b}_0)|, \qquad n > 1. \tag{3.4}$$

From (3.1) and (3.2) a comparison of coefficients in the two power series gives

$$b_{-1} = c_0, \qquad b_0 = c_1 e^{\mu i} - 2c_0\cos\nu, \tag{3.5}$$

$$b_n = c_{n+1}e^{(n+1)\mu i} - 2c_n e^{n\mu i}\cos\nu + c_{n-1}e^{(n-1)\mu i}, \qquad n \geqslant 1, \tag{3.6}$$

$$c_n e^{n\mu i} = \sum_{k=-1}^{n-1} b_k \frac{\sin(n-k)\nu}{\sin\nu}. \tag{3.7}$$

Substituting for b_{-1} and b_0 from (3.5) in (3.7) we have

$$\begin{aligned} c_n e^{n\mu i} &= c_0\frac{\sin(n+1)\nu}{\sin\nu} + (c_1 e^{\mu i} - 2c_0\cos\nu)\frac{\sin n\nu}{\sin\nu} + \sum_{k=1}^{n-1} b_k\frac{\sin(n-k)\nu}{\sin\nu} \\ &= c_0\left\{\frac{\sin(n+1)\nu}{\sin\nu} - 2\cos\nu\frac{\sin n\nu}{\sin\nu}\right\} + c_1 e^{\mu i}\frac{\sin n\nu}{\sin\nu} + \sum_{k=1}^{n-1} b_k\frac{\sin(n-k)\nu}{\sin\nu} \\ &= -c_0\frac{\sin(n-1)\nu}{\sin\nu} + c_1 e^{\mu i}\frac{\sin n\nu}{\sin\nu} + \sum_{k=1}^{n-1} b_k\frac{\sin(n-k)\nu}{\sin\nu}. \end{aligned} \tag{3.8}$$

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From (3.8) we have the inequalities

$$|c_n| \leqslant (n-1)|c_0| + n|c_1| + \sum_{k=1}^{n-1} (n-k)|b_k|, \qquad n > 1. \tag{3.9}$$

From (3.3), (3.4), and (3.5) we have

$$|b_1| \leqslant |b_{-1}| + |\mathfrak{I}(b_0 - \bar{b}_0)| \leqslant 5|c_0| + 2|c_1|, \tag{3.10}$$

$$|b_k| \leqslant 2|b_0| \leqslant 4|c_0| + 2|c_1|, \qquad k > 1. \tag{3.11}$$

Substituting (3.10) and (3.11) in (3.9) we obtain

$$|c_n| \leqslant 6(n-1)|c_0| + (3n-2)|c_1| + \sum_{k=2}^{n-1} (n-k)(4|c_0| + 2|c_1|)$$

$$= |c_0|\left\{6n - 6 + 4\sum_{k=2}^{n-1} (n-k)\right\} + |c_1|\left\{3n - 2 + 2\sum_{k=2}^{n-1} (n-k)\right\}.$$

Thus for all integers n,

$$|c_n| \leqslant 2|n^2 - 1| \cdot |c_0| + n^2|c_1|, \tag{3.12}$$

and we now have proved a special case of (1.12).

Let now

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n \tag{3.13}$$

be regular on $|z| = 1$ and such that $\mathfrak{I}f(z)$ changes sign exactly twice on $|z| = 1$. We may write $f(z)$ as

$$f(z) = \sum_{1}^{\infty} (a_{-n}z^{-n} + \bar{a}_{-n}z^n) + a_0 + \sum_{n=1}^{\infty} (a_n - \bar{a}_{-n})z^n \tag{3.14}$$

where both series in (3.14) converge on $|z| = 1$. On $|z| = 1$,

$$\mathfrak{I}\left\{\sum_{n=1}^{\infty} (a_{-n}z^{-n} + \bar{a}_{-n}z^n) + \mathfrak{R}a_0\right\} \equiv 0. \tag{3.15}$$

In this case, from (3.14) and (3.15) it follows that on $|z| = 1$ the imaginary part of the function

$$\phi^*(z) = (\mathfrak{I}a_0)i + \sum_{n=1}^{\infty} (a_n - \bar{a}_{-n})z^n \tag{3.16}$$

changes sign exactly twice. Thus $\phi^*(z)$ behaves like $\phi(z)$ in (3.1) and inequalities corresponding to (3.12) must hold for $\phi^*(z)$. This then gives for all integers n,

$$|a_n - \bar{a}_{-n}| \leqslant |n^2 - 1| \cdot |a_0 - \bar{a}_0| + n^2|a_1 - \bar{a}_{-1}|. \tag{3.17}$$

We note that in (3.17) not both the terms $|a_0 - \bar{a}_0|$ and $|a_1 - \bar{a}_{-1}|$ can vanish simultaneously. For otherwise the function

$$\phi^*(z) = (a_2 - \bar{a}_{-2})z^2 + \ldots + (a_n - \bar{a}_{-n})z^n + \ldots,$$

regular for $|z| \leqq 1$, would map $|z| = r$, for every r in $0 < r \leqslant 1$, into a contour which cuts the real axis at least four times. This, however, is contrary to the hypothesis for $\phi^*(z)$ since $p = 1$. The proof of (1.12) is now complete.

4. Proof of Theorem 1. We now let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \tag{4.1}$$

be regular and single-valued for $0 \leqslant \rho \leqslant |z| \leqslant 1$. We suppose that $\Im f(z)$ changes sign exactly $2p$ times on $|z| = 1$. By Lemma 1 there exist real numbers μ and ν such that

$$g(z) = (z + z^{-1} - 2\cos\nu) f(ze^{i\mu}) = \sum_{-\infty}^{\infty} b_n z^n \tag{4.2}$$

is regular and single-valued for $\rho \leqslant |z| \leqslant 1$ and $\Im g(z)$ changes sign $2(p-1)$ times on $|z| = 1$.

A comparison of coefficients in (4.1) and (4.2) gives

$$b_n = a_{n+1}e^{(n+1)\mu i} - 2a_n e^{n\mu i}\cos\nu + a_{n-1}e^{(n-1)\mu i}, \quad n = 0, \pm 1, \pm 2, \dots, \tag{4.3}$$

$$a_n e^{n\mu i} = \sum_{k=-\infty}^{n-1} b_k \frac{\sin(n-k)\nu}{\sin\nu} = S_1 + S_2, \tag{4.4}$$

where

$$S_1 = \sum_{k=-\infty}^{p-1} b_k \frac{\sin(n-k)\nu}{\sin\nu}, \tag{4.5}$$

$$S_2 = \sum_{k=p}^{n-1} b_k \frac{\sin(n-k)\nu}{\sin\nu}. \tag{4.6}$$

Making use of (4.3) in (4.5) we simplify S_1 as follows:

$$\begin{aligned} S_1 &= \sum_{k=-\infty}^{p-1} (a_{k+1}e^{(k+1)\mu i} - 2a_k e^{k\mu i}\cos\nu + a_{k-1}e^{(k-1)\mu i})\frac{\sin(n-k)\nu}{\sin\nu} \\ &= a_p e^{p\mu i}\frac{\sin(n-p+1)\nu}{\sin\nu} + a_{p-1}e^{(p-1)\mu i}\left\{\frac{\sin(n-p+2)\nu}{\sin\nu} - 2\cos\nu\frac{\sin(n-p+1)\nu}{\sin\nu}\right\} \\ &\quad + \sum_{k=-\infty}^{p-2} a_k e^{k\mu i}\left\{\frac{\sin(n-k+1)\nu}{\sin\nu} - 2\cos\nu\frac{\sin(n-k)\nu}{\sin\nu} + \frac{\sin(n-k-1)\nu}{\sin\nu}\right\}, \end{aligned}$$

so that

$$S_1 = a_p e^{p\mu i}\frac{\sin(n-p+1)\nu}{\sin\nu} - a_{p-1}e^{(p-1)\mu i}\frac{\sin(n-p)\nu}{\sin\nu}. \tag{4.7}$$

Thus (4.4) simplifies to

$$a_n e^{n\mu i} = a_p e^{p\mu i}\frac{\sin(n-p+1)\nu}{\sin\nu} - a_{p-1}e^{(p-1)\mu i}\frac{\sin(n-p)\nu}{\sin\nu} + \sum_{k=p}^{n-1} b_k \frac{\sin(n-k)\nu}{\sin\nu}. \tag{4.8}$$

Next, consider the expression

$$(4.9)\quad T \equiv \sum_{k=p}^{n-1} \bar{b}_{-k}\frac{\sin (n-k)\nu}{\sin \nu}$$

$$= \sum_{k=p}^{n-1} \{\bar{a}_{1-k}e^{(k-1)\mu i} - 2\bar{a}_{-k}e^{k\mu i}\cos \nu + \bar{a}_{-k-1}e^{(k+1)\mu i}\}\frac{\sin (n-k)\nu}{\sin \nu}$$

$$= \bar{a}_{1-p}e^{(p-1)\mu i}\frac{\sin (n-p)\nu}{\sin \nu}$$

$$+ \bar{a}_{-p}e^{p\mu i}\left\{\frac{\sin (n-p-1)\nu}{\sin \nu} - 2\cos \nu\frac{\sin (n-p)\nu}{\sin \nu}\right\} + \bar{a}_{-n}e^{n\mu i}$$

$$+ \sum_{k=p+1}^{n-1} \bar{a}_{-k}e^{k\mu i}\left\{\frac{\sin (n-k-1)\nu}{\sin \nu} - 2\cos \nu\frac{\sin (n-k)\nu}{\sin \nu} + \frac{\sin (n-k+1)\nu}{\sin \nu}\right\}$$

$$(4.10)\quad T = \bar{a}_{1-p}e^{(p-1)\mu i}\frac{\sin (n-p)\nu}{\sin \nu} - \bar{a}_{-p}e^{p\mu i}\frac{\sin (n-p+1)\nu}{\sin \nu} + \bar{a}_{-n}e^{n\mu i}.$$

Adding to the right-hand side of (4.8) the expression for T in (4.10) and then subtracting the expression for T in (4.9) we may rewrite (4.8) in the form

$$(4.11)\quad a_n e^{n\mu i} = \left\{\bar{a}_{1-p}e^{(p-1)\mu i}\frac{\sin (n-p)\nu}{\sin \nu} - \bar{a}_{-p}e^{p\mu i}\frac{\sin (n-p+1)\nu}{\sin \nu} + \bar{a}_{-n}e^{n\mu i}\right\}$$

$$+ \left\{a_p e^{p\mu i}\frac{\sin (n-p+1)\nu}{\sin \nu} - a_{p-1}e^{(p-1)\mu i}\frac{\sin (n-p)\nu}{\sin \nu}\right\}$$

$$+ \sum_{k=p}^{n-1} (b_k - \bar{b}_{-k})\frac{\sin (n-k)\nu}{\sin \nu},$$

$$(4.12)\quad (a_n - \bar{a}_{-n})e^{n\mu i} = (a_p - \bar{a}_{-p})e^{p\mu i}\frac{\sin (n-p+1)\nu}{\sin \nu}$$

$$- (a_{p-1} - \bar{a}_{1-p})e^{(p-1)\mu i}\frac{\sin (n-p)\nu}{\sin \nu}$$

$$+ \sum_{k=p}^{n-1} (b_k - \bar{b}_{-k})\frac{\sin (n-k)\nu}{\sin \nu},$$

$$(4.13)\quad |a_n - \bar{a}_{-n}| \leqslant (n-p+1)|a_p - \bar{a}_{-p}| + (n-p)|a_{p-1} - \bar{a}_{1-p}|$$

$$+ \sum_{k=p}^{n-1} (n-k)|b_k - \bar{b}_{-k}|.$$

We have already seen that (1.9) holds for $p = 1$ and all integers $n > 1$ because of (3.17). We now assume that the coefficients b_n of $g(z)$ in (4.2) satisfy (1.9) with $p - 1$ replacing p and b_n replacing a_n. Then from (4.13) we shall be able to prove by induction that (1.9) holds for all p and all $n > p$. Thus we assume, for $k > p - 1$,

$$(4.14)\quad |b_k - \bar{b}_{-k}| \leqslant \sum_{s=0}^{p-1} \Delta(p-1, s, k)|b_s - \bar{b}_{-s}|.$$

From (4.3) we have the inequalities

$$(4.15)\qquad |b_0 - \bar{b}_0| \leqslant 2|a_0 - \bar{a}_0| + 2|a_1 - \bar{a}_{-1}|,$$

$$(4.16)\qquad |b_s - \bar{b}_{-s}| \leqslant |a_{s+1} - \bar{a}_{-s-1}| + 2|a_s - \bar{a}_{-s}| + |a_{s-1} - \bar{a}_{1-s}|,$$
$$s = 1, 2, \ldots, p-1.$$

Making use of (4.15) and (4.16) in (4.14) we have

$$(4.17)\qquad |b_k - \bar{b}_{-k}| \leqslant 2\Delta(p-1, 0, k)(|a_0 - \bar{a}_0| + |a_1 - \bar{a}_{-1}|) + R,$$

where

$$(4.18)\quad R \equiv \sum_{s=1}^{p-1} \Delta(p-1, s, k)(|a_{s+1} - \bar{a}_{-s-1}| + 2|a_s - \bar{a}_{-s}| + |a_{s-1} - \bar{a}_{1-s}|).$$

Collecting terms in (4.17) and (4.18), we may write

$$(4.19)\quad \begin{aligned} |b_k - \bar{b}_{-k}| \leqslant{}& \frac{2k(k+p-1)!}{(k-p)!}\left[\frac{1}{((p-1)!)^2k^2} + \frac{1}{p!(p-2)!(k^2-1)}\right]|a_0 - \bar{a}_0| \\ &+ \frac{2k(k+p-1)!}{(k-p)!}\left[\frac{1}{(2p-2)!(k^2-(p-1)^2)}\right]|a_p - \bar{a}_{-p}| \\ &+ \frac{2k(k+p-1)!}{(k-p)!}\left[\frac{1}{(2p-3)!(k^2-(p-2)^2)} + \frac{2}{(2p-2)!(k^2-(p-1)^2)}\right]|a_{p-1} - \bar{a}_{1-p}| \\ &+ \sum_{s=1}^{p-1} \frac{2k(k+p-1)!}{(k-p)!}\left[\frac{|a_s - \bar{a}_{-s}|}{(p-2+s)!(p-s)!(k^2-(s-1)^2)} + \frac{2|a_s - \bar{a}_{-s}|}{(p-1+s)!(p-1-s)!(k^2-s^2)} + \frac{|a_s - \bar{a}_{-s}|}{(p+s)!(p-2-s)!(k^2-(s+1)^2)}\right]. \end{aligned}$$

Substituting (4.19) in (4.13), we obtain

$$(4.20)\qquad |a_n - \bar{a}_{-n}| \leqslant \sum_{\mu=0}^{p} D_\mu |a_\mu - \bar{a}_{-\mu}|,$$

where

$$(4.21)\ D_0 = \sum_{k=p}^{n-1} \frac{2k(n-k)(k+p-1)!}{(k-p)!}\left[\frac{2}{((p-1)!)^2k^2} + \frac{1}{p!(p-2)!(k^2-1)}\right],$$

$$(4.22)\quad D_{p-1} = \frac{4}{(2p-2)!}\sum_{k=p-2}^{n-1} \frac{k(n-k)(k+p-2)!}{(k+p-2)(k-p+2)!} \cdot [(p-1)(k^2-(p-1)^2) + k^2 - (p-2)^2],$$

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(4.23) $$D_p = \frac{2}{(2p-2)!}\sum_{k=p-1}^{n-1}\frac{k(n-k)(k+p-2)!}{(k-p+1)!},$$

while for $\mu = 1, 2, \ldots, p-2$ we have

(4.24) $$D_\mu = \sum_{k=p}^{n-1}\frac{2k(n-k)(k+p-1)!}{(k-p)!}A_k$$

where

$$A_k = \Bigg[\frac{1}{(p-2+\mu)!(p-\mu)!(k^2-(\mu-1)^2)} + \frac{2}{(p-1+\mu)!(p-1-\mu)!(k^2-\mu^2)} + \frac{1}{(p+\mu)!(p-2-\mu)!(k^2-(\mu+1)^2)}\Bigg].$$

To complete the proof of (1.9) by induction it is then sufficient to evaluate D_μ ($\mu = 0, 1, \ldots, p$) and, indeed, to show that

(4.25) $$D_\mu \equiv \Delta(p, \mu, n).$$

5. Formulae for D_μ. We shall prove first (4.25) in the case $\mu = p$, that is,

(5.1) $$\frac{2}{(2p-2)!}\sum_{k=p-1}^{n-1}\frac{k(n-k)(k+p-2)!}{(k-p+1)!} = \frac{2n(n+p)!}{(2p)!(n-p-1)!(n^2-p^2)}.$$

Equation (5.1) is equivalent to

(5.2) $$\sum_{k=p-1}^{n-1}\frac{k(n-k)(k+p-2)!}{(k-p+1)!} = \frac{n(n+p-1)!}{2p(2p-1)(n-p)!}$$

which is easily seen to be true when $n = p$. We shall prove (5.2) by induction on n, making use of the formula

(5.3) $$\sum_{s=0}^{m}\frac{(s+q)!}{s!} = \frac{(m+q+1)!}{(q+1)\cdot(m!)}.$$

Assuming (5.2) for an integer n, we then have

(5.4) $$\sum_{k=p-1}^{n}\frac{k(n+1-k)(k+p-2)!}{(k-p+1)!}$$

$$= \frac{n(n+p-2)!}{(n-p+1)!} + \sum_{k=p-1}^{n-1}\frac{k(n+1-k)(k+p-2)!}{(k-p+1)!}$$

$$= \frac{n(n+p-2)!}{(n-p+1)!} + \frac{n(n+p-1)!}{2p(2p-1)(n-p)!} + \sum_{k=p-1}^{n-1}\frac{k(k+p-2)}{(k-p+1)!}$$

$$= \frac{n(n+p-2)!}{(n-p+1)!} + \frac{n(n+p-1)!}{2p(2p-1)(n-p)!} + \sum_{k=p-1}^{n-1}\frac{(k+p-1)!}{(k-p+1)!}$$

$$- (p-1)\sum_{k=p-1}^{n-1}\frac{(k+p-2)!}{(k-p+1)}$$

$$= \frac{n(n+p-2)!}{(n-p+1)!} + \frac{n(n+p-1)!}{2p(2p-1)(n-p)!} + \frac{(n+p-1)!}{(2p-1)(n-p)!} - \frac{(p-1)(n+p-2)!}{(2p-2)(n-p)!}$$

$$= \frac{(n+1)(n+p)!}{2p(2p-1)(n+1-p)!}.$$

Thus (5.2) holds when n is replaced by $n+1$. This completes the proof by induction of (5.1).

Next we shall establish (4.25) for the values $\mu = 1, 2, \ldots, p-2$. We shall omit the proof of (4.25) in the cases $\mu = 0$, $\mu = p-1$ since the method of proof in these two cases is very similar to the typical case which follows. We assume now that $1 \leqslant \mu \leqslant p-2$, $n > p$, and shall prove that

$$(5.6) \quad \sum_{k=p}^{n-1} \frac{k(n-k)(k+p-1)!}{(k-p)!}\left[\frac{(p+\mu-1)(p+\mu)}{k^2-(\mu-1)^2} + \frac{2(p^2-\mu^2)}{k^2-\mu^2} + \frac{(p-\mu-1)(p-\mu)}{k^2-(\mu+1)^2}\right] = \frac{n(n+p)!}{(n-p-1)!(n^2-\mu^2)},$$

which is equivalent to (4.25). It can be easily verified that (5.6) holds when $n = p+1$. We assume (5.6) for an integer n and prove that (5.6) holds for the integer $n+1$. Replacing n by $n+1$ in the left-hand side of equation (5.6) we have

$$(5.7) \quad \sum_{k=p}^{n} \frac{k(n+1-k)(k+p-1)!}{(k-p)!}\left[\frac{(p+\mu)^2-(p+\mu)}{k^2-(\mu-1)^2} + \frac{2(p^2-\mu^2)}{k^2-\mu^2} + \frac{(p-\mu)^2-(p-\mu)}{k^2-(\mu+1)^2}\right] = \sum_{k=n}^{n} + \sum_{k=p}^{n-1}$$

$$= \frac{n(n+p-1)!}{(n-p)!}\left[\frac{(p+\mu)^2-(p+\mu)}{n^2-(\mu-1)^2} + \frac{2(p^2-\mu^2)}{n^2-\mu^2} + \frac{(p-\mu)^2-(p-\mu)}{n^2-(\mu+1)^2}\right]$$
$$+ \frac{n(n+p)!}{(n-p-1)!(n^2-\mu^2)} + \sum_{k=p}^{n-1} \frac{k(k+p-1)!}{(k-p)!}\left[\frac{(p+\mu)^2-(p+\mu)}{k^2-(\mu-1)^2} + \frac{2(p^2-\mu^2)}{k^2-\mu^2} + \frac{(p-\mu)^2-(p-\mu)}{k^2-(\mu+1)^2}\right].$$

The right-hand side of (5.7) should equal

$$\frac{(n+1)(n+p+1)!}{(n-p)!((n+1)^2-\mu^2)}$$

if (5.6) is to be proven, and we can see that it does so, provided we show that

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$$\sum_{k=p}^{n-1} \frac{k(k+p-1)!}{(k-p)!}\left[\frac{(p+\mu)^2-(p+\mu)}{k^2-(\mu-1)^2} + \frac{2(p^2-\mu^2)}{k^2-\mu^2} + \frac{(p-\mu)^2-(p-\mu)}{k^2-(\mu+1)^2}\right] \tag{5.8}$$

$$= \frac{(n+p-1)!}{(n-p)!}\left[\frac{(n+1)(n+p+1)(n+p)}{(n+1)^2-\mu^2} - \frac{n(n^2-p^2)}{n^2-\mu^2} - n\left\{\frac{(p+\mu)^2-(p+\mu)}{n^2-(\mu-1)^2} + \frac{2(p^2-\mu^2)}{n^2-\mu^2} + \frac{(p-\mu)^2-(p-\mu)}{n^2-(\mu+1)^2}\right\}\right].$$

We shall prove (5.8) by induction on n. It is easily verified that (5.8) holds for $n = p+1$. Replacing n by $n+1$ in the left-hand side of (5.8), we have

$$\sum_{k=p}^{n} \frac{k(k+p-1)!}{(k-p)!}\left[\frac{(p+\mu)^2-(p+\mu)}{k^2-(\mu-1)^2} + \frac{2(p^2-\mu^2)}{k^2-\mu^2} + \frac{(p-\mu)^2-(p-\mu)}{k^2-(\mu+1)^2}\right] = \sum_{k=n}^{n} + \sum_{k=p}^{n-1} \tag{5.9}$$

$$= \frac{n(n+p-1)!}{(n-p)!}\left[\frac{(p+\mu)^2-(p+\mu)}{n^2-(\mu-1)^2} + \frac{2(p^2-\mu^2)}{n^2-\mu^2} + \frac{(p-\mu)^2-(p-\mu)}{n^2-(\mu+1)^2}\right]$$

$$+ \frac{(n+p-1)!}{(n-p)!}\left[\frac{(n+1)(n+p+1)(n+p)}{(n+1)^2-\mu^2} - \frac{n(n^2-p^2)}{n^2-\mu^2} - n\left\{\frac{(p+\mu)^2-(p+\mu)}{n^2-(\mu-1)^2} + \frac{2(p^2-\mu^2)}{n^2-\mu^2} + \frac{(p-\mu)^2-(p-\mu)}{n^2-(\mu+1)^2}\right\}\right]$$

$$= \frac{(n+p-1)!}{(n-p)!}\left[\frac{(n+1)(n+p+1)(n+p)}{(n+1)^2-\mu^2} - \frac{n(n^2-p^2)}{n^2-\mu^2}\right]$$

$$= \frac{(n+p)!}{(n-p)!}\left[\frac{(n+1)(n+p+1)}{(n+1)^2-\mu^2} - \frac{n(n-p)}{n^2-\mu^2}\right].$$

If (5.8) is to be proven the right-hand side of (5.9) should be the same as the expression obtained by replacing n by $n+1$ in the right-hand side of (5.8). This will be the case, provided we prove that

$$(n+1-p)\left[\frac{(n+1)(n+p+1)}{(n+1)^2-\mu^2} - \frac{n(n-p)}{n^2-\mu^2}\right] \tag{5.10}$$

$$= \left[\frac{(n+2)(n+p+2)(n+p+1)}{(n+2)^2-\mu^2} - \frac{(n+1)((n+1)^2-p^2)}{(n+1)^2-\mu^2} - (n+1)\left\{\frac{(p+\mu)^2-(p+\mu)}{(n+1)^2-(\mu-1)^2} + \frac{2(p^2-\mu^2)}{(n+1)^2-\mu^2} + \frac{(p-\mu)^2-(p-\mu)}{(n+1)^2-(\mu+1)^2}\right\}\right].$$

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(5.10) is equivalent to

$$(5.11)\quad \frac{2(n+1)^3 - 2p^2(n+1)}{(n+1)^2-\mu^2} - \frac{n(n-p)(n-p+1)}{n^2-\mu^2}$$
$$= \frac{(n+2)(n+p+2)(n+p+1)}{(n+2)^2-\mu^2}$$
$$- (n+1)\left\{\frac{(p+\mu)^2-(p+\mu)}{(n+1)^2-(\mu-1)^2} + \frac{2(p^2-\mu^2)}{(n+1)^2-\mu^2} + \frac{(p-\mu)^2-(p-\mu)}{(n+1)^2-(\mu+1)^2}\right\}.$$

If all terms with denominator $(n+1)^2 - \mu^2$ in (5.11) are collected and placed on the left-hand side, these terms simplify to the simple term $(2n+2)$ so that (5.11) is equivalent to

$$(5.12)\quad 2n+2 = \frac{n(n-p)(n-p+1)}{n^2-\mu^2} + \frac{(n+2)(n+p+2)(n+p+1)}{(n+2)^2-\mu^2}$$
$$- \frac{(n+1)(p+\mu)^2-(n+1)(p+\mu)}{(n+\mu)(n+2-\mu)}$$
$$- \frac{(n+1)(p-\mu)^2-(n+1)(p-\mu)}{(n-\mu)(n+2+\mu)},$$

and (5.12) is equivalent to

$$(5.13)\quad (2n+2)(n^2-\mu^2)(n^2+4n+4-\mu^2)$$
$$= n(n-p)(n-p+1)(n^2+4n+4-\mu^2)$$
$$+ (n+2)(n+p+2)(n+p+1)(n^2-\mu^2)$$
$$- (n+1)(n-\mu)(n+\mu+2)(p+\mu)^2$$
$$+ (n+1)(n-\mu)(n+\mu+2)(p+\mu)$$
$$- (n+1)(n+\mu)(n-\mu+2)(p-\mu)^2$$
$$+ (n+1)(n+\mu)(n-\mu+2)(p-\mu).$$

Both sides of (5.13) reduce to the polynomial

$$2n^5 + 10n^4 + (16-4\mu^2)n^3 + (8-12\mu^2)n^2 + (2\mu^4-16\mu^2)n + (2\mu^4-8\mu^2).$$

Since (5.13) is therefore an identity, this completes the proof by induction of (5.6). Thus (4.25) is established.

6. Sharpness of Theorems 1 and 2. We shall show now that inequalities (1.9), (1.10), and (1.15) are sharp for all integers p. Since the quantities

$$|a_k - \bar{a}_{-k}|, \qquad k = 0, 1, \ldots, p,$$

are to be assigned arbitrary values in advance, in order to prove that (1.9) is

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sharp it will be sufficient to exhibit a function $w = f^*(z)$ of the form (1.8) which is a power series. Thus we shall take $a_{-k} = 0$ for all $k > 0$. Since the addition of a real constant to the function $f^*(z)$ does not affect its imaginary part, we may assume that a_0 is a pure imaginary number id, $d \geqslant 0$, without restricting the problem. Then, having shown that (1.9) is sharp by exhibiting a function $f^*(z)$ satisfying the conditions of Theorem 1, for which equality signs hold for all $n > p$ in (1.9), we see at once by a rotation and translation that (1.10) is also sharp; and from (1.13) that (1.15) is indeed sharp too.

Let $|a_0|, |a_1|, \ldots, |a_p|$ be p arbitrary non-negative numbers, not all zero. Define

$$a_k = (-1)^k i^{k+1}|a_k|, \qquad k = 0, 1, \ldots, p, \tag{6.1}$$

$$A_k = (2k)!\sum_{q=0}^{k} \frac{|a_q|}{(k+q)!(k-q)!}, \qquad k = 0, 1, \ldots, p, \tag{6.2}$$

$$f^*(z) = \sum_{k=0}^{p} (-1)^k i^{k+1} A_k \frac{z^k + iz^{k+1}}{(1-iz)^{2k+1}} = \sum_{k=0}^{\infty} a_n z^n. \tag{6.3}$$

We shall show that $f^*(z)$ satisfies the conditions of Theorem 1, and that equality signs hold in (1.9) for this function. We must show that there exists an interval $\rho < r < 1$ such that on each circle $|z| = r$ of this interval the imaginary part of $f^*(z)$ changes sign exactly $2p$ times. To this end it will be sufficient to consider the real part of the function $-if^*(-iz)$ for $z = re^{i\theta}$ where

$$-if^*(-iz) = \sum_{k=0}^{p} (-1)^k A_k \frac{z^k + z^{k+1}}{(1-z)^{2k+1}}. \tag{6.4}$$

Long but straightforward and elementary calculations give

$$\begin{aligned} R_k(r,\theta) &\equiv \Re\left\{\frac{z^k + z^{k+1}}{(1-z)^{2k+1}}\right\} \\ &= \frac{(2k+1)!r^k}{(1-2r\cos\theta + r^2)^{2k+1}}\left[\frac{(-1)^k r^k(1-r^2)}{k!(k+1)!} + \sum_{m=1}^{k+1} D_m^{(k)} \cos m\theta\right], \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} &(k+1-m)!(k+1+m)!D_m^{(k)} \\ &\equiv (-1)^{k-m}[(k+1-m)(r^{k-m} - r^{k+2+m}) + (k+1+m)(r^{k+m} - r^{k+2-m})]. \end{aligned} \tag{6.6}$$

Thus

$$\begin{aligned} \lim_{r\to 1} \frac{R_k(r,\theta)}{1-r^2} = \frac{(2k+1)!}{2^{2k+1}(1-\cos\theta)^{2k+1}}&\left[\frac{(-1)^k}{k!(k+1)!}\right. \\ &\left. + \sum_{m=1}^{k+1} \frac{2(-1)^{k-m-1}(m^2 - k - 1)\cos m\theta}{(k+1-m)!(k+1+m)!}\right] \end{aligned} \tag{6.7}$$

$$= \frac{(2k+1)!}{2^{2k+1}(1-\cos\theta)^{2k+1}}\left[\frac{(-2)^k(1-\cos\theta)^k(k\cos\theta+k+1)}{(2k+1)!}\right]$$

$$= \frac{(-1)^k(k\cos\theta+k+1)}{2^{k+1}(1-\cos\theta)^{k+1}}.$$

For $z = re^{i\theta}$ we now have

(6.8) $$(1-2r\cos\theta+r^2)^{2p+1}\Re\{-if^*(-iz)\}$$

$$= \sum_{k=0}^{p}(-1)^k A_k R_k(r,\theta)(1-2r\cos\theta+r^2)^{2p+1} \equiv P(r,\theta).$$

Here $P(r,\theta)$ is a polynomial in $\cos\theta$ of degree $2p$ and the number of changes of sign of $\Im f^*(z)$ is precisely the number of changes of sign of $P(r,\theta)$ on $|z| = r < 1$.

Since we shall see that

(6.9) $$|a_n| = \sum_{k=0}^{p}\Delta(p,k,n)|a_k| \sim cn^{2p},$$

$\Im f^*(z)$ cannot change sign on $|z| = r$, for an interval $\rho < r < 1$, fewer than $2p$ times because of (1.9). Thus $P(r,\theta)$ changes sign *at least* $2p$ times on $|z| = r$, $p < r < 1$. We shall show next that $P(r,\theta)$ cannot vanish more than $2p$ times on $|z| = r$ for $\rho < r < 1$, $0 \leqslant \theta < 2\pi$. Let

(6.10) $$P(\theta) = \lim_{r\to 1}\frac{P(r,\theta)}{1-r^2}$$

$$= \sum_{k=0}^{p}\frac{A_k}{2^{k+1}}\frac{(k\cos\theta+k+1)}{(1-\cos\theta)^{k+1}}\cdot 2^{2p+1}(1-\cos\theta)^{2p+1}$$

$$= 2^{2p}(1-\cos\theta)^p\sum_{k=0}^{p}\frac{A_k}{2^k}(k\cos\theta+k+1)(1-\cos\theta)^{p-k}$$

$$= 2^{2p}(1-\cos\theta)^p Q(\theta),$$

where

(6.11) $$Q(\theta) \equiv \sum_{k=0}^{p}\frac{A_k}{2^k}(k\cos\theta+k+1)(1-\cos\theta)^{p-k}$$

(6.12) $$Q(\theta) \geqslant \frac{A_p}{2^p} > 0$$

provided not all $|a_0|, |a_1|, \ldots, |a_p|$ are zero.

$P(\theta)$ is a polynomial of degree $2p$ in the variable $u = \cos\theta$, and has exactly p real zeros in the variable u in the range $-1 \leqslant u \leqslant 1$. Since $Q(\theta) > 0$, and because the zeros of any polynomial are continuous functions of its coefficients, we conclude that $P(r,\theta)$, as a polynomial of degree $2p$ in $u = \cos\theta$, also cannot have more than p real zeros u in the range $-1 \leqslant u \leqslant 1$. For, given ϵ sufficiently small but positive, and for values of r near one, $P(r,\theta)$ has exactly p complex zeros u_k in the circle $|u-1| < \epsilon$, where u is regarded as a complex number,

$k = 1,2, \ldots, p$. Let

$$P(r, \theta) \equiv P_1(u)P_2(u), \qquad P_2(u) \equiv \prod_{k=1}^{p} (u_k - u).$$

Then

$$\lim_{r\to 1} P_2(u) = (1 - u)^p, \qquad \lim_{r\to 1} P_1(u) = 2^{2p}Q(\theta),$$

and when u is real, $-1 \leqslant u \leqslant 1$, $2^{2p}Q(\theta) \geqslant 2^pA_p > 0$. Thus, there exists a range $\rho < r < 1$ for which $|P_1(u)| > 2^{p-1}A_p > 0$, for u real and $-1 \leqslant u < 1$. It follows that, for a range $\rho < r < 1$, $P(r,\theta)$ has not more than p real zeros u in $-1 \leqslant u \leqslant 1$, and therefore not more than $2p$ zeros θ in the range $0 \leqslant \theta < 2\pi$. $P(r,\theta)$ therefore changes sign exactly $2p$ times on each circle $|z| = r$ for some range $\rho < r < 1$. This shows that $f^*(z)$ satisfies the hypothesis of Theorem 1.

In addition it is necessary to show that for $f^*(z)$ (6.9) holds. We note that the coefficient of z^n in $(z^k + iz^{k+1})(1 - iz)^{-2k-1}$ is

$$\frac{2n(n + k - 1)!i^{n-k}}{(n - k)!(2k)!}, \qquad n \geqslant k.$$

Thus for

$$f^*(z) = \sum_{n=0}^{\infty} a_n z^n,$$

(6.13)
$$\begin{aligned} a_n &= i^{n+1}(2n)\sum_{k=0}^{p} \frac{(-1)^k(n + k - 1)!}{(n - k)!(2k)!}A_k \\ &= 2ni^{n+1}\sum_{k=0}^{p} \frac{(-1)^k(n + k - 1)!}{(n - k)!}\sum_{q=0}^{k} \frac{|a_q|}{(k + q)!(k - q)!} \\ &= 2ni^{n+1}\sum_{q=0}^{p}\sum_{k=q}^{p} \frac{(-1)^k(n + k - 1)!}{(n - k)!(k + q)!(k - q)!}|a_q| \\ &= 2ni^{n+1}\sum_{q=0}^{p} \frac{(-1)^p(n + p)!|a_q|}{(n^2 - q^2)(n - p - 1)!(p + q)!(p - q)!}; \end{aligned}$$

(6.14)
$$|a_n| = \sum_{q=0}^{p} \Delta(p, q, n)|a_q|,$$

where $\Delta(p, q, n)$ is defined as in (1.9). Now the proof that (1.9) is sharp has been completed.

To show that (1.15) is also sharp we form the function

(6.15)
$$\begin{aligned} F^*(z) &= \int_0^z (f^*(z) - a_0)\frac{dz}{z} = \sum_{1}^{\infty} d_n z^n = \sum_{1}^{\infty} \frac{a_n}{n} z^n \\ &= \sum_{k=1}^{p} (-1)^k i^{k+1} B_k \frac{z^k}{(1 - iz)^{2k}} \end{aligned}$$

where

(6.16)
$$B_k = A_k - \frac{(2k)!}{k!k!}|a_0| = (2k)!\sum_{q=1}^{k} \frac{q|d_q|}{(k + q)!(k - q)!}.$$

It is immediately obvious from (6.14) and (6.15) that, for $F^*(z)$,

$$|d_n| = \sum_{q=1}^{p} \frac{q}{n} \Delta(p, q, n) |d_q|, \qquad n > p, \tag{6.17}$$

which is to say that equality signs hold in (1.15). Further, since $F^*(z)$ is a polynomial of degree p in the variable $\zeta = iz/(1 - iz)^2$, and since ζ is a univalent function of z for $|z| < 1$, it follows at once that $F^*(z)$ is multivalent of order not exceeding p in $|z| < 1$. From (6.15) it is seen that for an interval $\rho < r < 1$ the derivative

$$\frac{\partial}{\partial \theta}\{\Re F^*(re^{i\theta})\} \tag{6.18}$$

changes sign exactly $2p$ times on $|z| = r$. From (6.17) it may be noted that for $F^*(z)$,

$$|d_n| \sim cn^{2p-1}.$$

Thus $F^*(z)$ cannot be multivalent of order $\nu < p$, for in that case we should conclude from a well-known result for the coefficients of a multivalent function of order ν that $d_n = O(n^{2\nu-1})$. Thus $F^*(z)$ is multivalent of order p in $|z| < 1$. $W = F^*(z)$ has the property that it maps $|z| = r$, $\rho < r < 1$, onto a contour such that every straight line parallel to the imaginary axis cuts it in at most $2p$ points.

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