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Multiplicative dependence in linear recurrence sequences

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Abstract. **For a wide class of integer linear recurrence sequences $(u(n))_{n=1}^{\infty}$, we give an upper bound on the number of s-tuples $(n_1, \ldots, n_s) \in (\mathbb{Z} \cap [M+1, M+N])^s$ such that the corresponding elements $u(n_1), \ldots, u(n_s)$ in the sequence are multiplicatively dependent.**

1 Introduction

1.1 Motivation and set-up

Let $\vec{u} = (u(n))_{n=1}^{\infty}$ be an integer linear recurrence sequence of order $d \geq 1$, that is, a sequence of integers satisfying a relation of the form

$$u(n+d) = c_{d-1}u(n+d-1) + \ldots + c_0 u(n), \qquad n = 1, 2, \ldots,$$

and not satisfying any shorter relation. In this case

$$f(X) = X^d - c_{d-1}X^{n+d-1} - \ldots - c_0 \in \mathbb{Z}[X]$$

is called the characteristic polynomial of $\vec{u}$.

Recently there have been several works [3, 4, 5, 6, 13, 9, 10, 11] investigating multiplicative relations of the form

$$u(n_1)^{k_1} \ldots u(n_s)^{k_s} = 1. \tag{1.1}$$

However, these papers consider certain special cases. The works [6, 11, 13] are limited to the case of binary (that is, of order $d = 2$) linear recurrence sequences and also assume that the exponents $k_1, \ldots, k_s$ are *fixed* non-zero integers, while the papers [3, 4, 9, 10] concern specific sequences. Under these restrictions, the mentioned papers contain several finiteness results. Finally, the recent work [5] concerns linear recurrence sequences of arbitrary order – however, under a rather restrictive condition on the coefficients c_i defining the generating relation.

Here we are interested in the case of general sequences of arbitrary order $d \geq 2$ and also we do not fix the exponents $k_1, \ldots, k_s$. Thus, we study s-tuples $(u(n_1), \ldots, u(n_s))$, which are *multiplicatively dependent* (m.d.), where, as usual, we say that the nonzero complex numbers $\gamma_1, \ldots, \gamma_s$ are m.d. if there exist integers $k_1, \ldots, k_s$, not all zero, such that

$$\gamma_1^{k_1} \ldots \gamma_s^{k_s} = 1.$$

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However, instead of finiteness results, we give an upper bound on the density of such s-tuples.

More precisely, for $M \geq 0$ and $N \geq 1$, we are interested in the following quantity

$$\mathrm{M}_s(M,N) = \sharp\{(n_1,\ldots,n_s) \in (\mathbb{Z}\cap[M+1,M+N])^s : \\ u(n_1),\ldots,u(n_s) \text{ are m.d.}\}.$$

To estimate $\mathrm{M}_s(M,N)$ we also study

$$\mathrm{M}^*_s(M,N) = \sharp\{(n_1,\ldots,n_s) \in (\mathbb{Z}\cap[M+1,M+N])^s : \\ u(n_1),\ldots,u(n_s) \text{ are m.d. of maximal rank}\},$$

where the maximality of the rank for m.d. of $(u(n_1),\ldots,u(n_s))$ means that no subtuple is m.d. In particular, this implies that if one has a m.d. (1.1) of maximal rank, then $k_1\cdots k_s \neq 0$.

We can then estimate $\mathrm{M}_s(M,N)$ via the inequality

$$\mathrm{M}_s(M,N) \leq \sum_{t=1}^{s}\binom{s}{t}\mathrm{M}^*_t(M,N)N^{s-t}. \tag{1.2}$$

1.2 Notation

We recall that the notations $U = O(V)$, $U \ll V$ and $V \gg U$ are equivalent to $|U| \leqslant cV$ for some positive constant c, which throughout this work, may depend only on the integer parameter s and the sequence $\vec{u}$.

It is convenient to denote by $\log_k x$ the k-fold iterated logarithm, that is, for $x \geq 1$ we set

$$\log_1 x = \log x \qquad \text{and} \qquad \log_k = \log_{k-1}\max\{\log x, 2\}, \quad k = 2,3,\ldots.$$

1.3 Main results

We say that the sequence $\vec{u}$ is *non-degenerate* if there are no roots of unity among the ratios of distinct roots of f. We say that the sequence $\vec{u}$ has a dominant root, if its characteristic polynomial f has a root λ with

$$|\lambda| > \max\{|\mu| : f(\mu) = 0, \ \mu \neq \lambda\}.$$

Furthermore, we say that $\vec{u}$ is *simple* if f has no multiple roots.

Theorem Let $\vec{u}$ be a simple non-degenerate sequence of order $d \geq 2$. For any fixed $s \geq 1$, uniformly over $M \geq 0$, we have

$$\mathrm{M}^*_s(M,N) \leq N^{s(1-1/(4d-3))+o(1)}.$$

□

Analysing the proof of Theorem 1.1, one can see that for $M = 0$ we can drop $o(1)$ in the bound.

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Remark Considering s-tuples with $n_1 = n_2$ we see that

$$\mathrm{M}_s(M, N) \geq N^{s-1}. \tag{1.3}$$

Therefore, it is impossible to derive a bound of the same type as in Theorem 1.1 for $\mathrm{M}_s(M, N)$. □

When M is (exponentially) large compared to N, we get the following bound, which improves Theorem 1.1 for $s < 4d - 3$.

Theorem Let $\vec{u}$ be a simple non-degenerate sequence of order $d \geq 2$ with a dominant root and let

$$M \geq \exp(N \log_3 N / \log_2 N).$$

Then, for any fixed $s \geq 1$, uniformly over M, we have

$$\mathrm{M}_s^*(M, N) \leq N^{s-1+o(1)}.$$

□

Remark The condition on M in Theorem 1.3 is chosen to achieve the strongest possible bound. Examining its proof one can see that for $s < 4d - 3$ one can also improve Theorem 1.1 for $M \geq \exp(N^{\eta})$ with any $\eta > s/(4d - 3)$ (but only for sequences with a dominant root). □

From the definition of m.d. of maximal rank, we have $\mathrm{M}_1^*(M, N) = O(1)$, see [1, Lemma 2.1]. Hence, we see from (1.2) that in applying Theorem 1.1 to bounding $\mathrm{M}_s(M, N)$ the case of $s = 2$ becomes the bottleneck. Thus, we now investigate this case separately.

Theorem Let $\vec{u}$ be a simple non-degenerate sequence of order $d \geq 2$ with an irreducible characteristic polynomial having a dominant root. Uniformly over $M \geq 0$, we have

$$\mathrm{M}_2^*(M, N) = N + O(1).$$

□

Since, as we have mentioned, $\mathrm{M}_1^*(M, N) = O(1)$, the bounds of Theorems 1.1 and 1.5 inserted in (1.2) imply that if $\vec{u}$ is a simple non-degenerate sequence of order $d \geq 2$ with an irreducible characteristic polynomial having a dominant root then

$$\mathrm{M}_s(M, N) \ll N^{s-3/(4d-3)+o(1)}, \tag{1.4}$$

where the bottleneck comes from the bound on $\mathrm{M}_3^*(M, N)$. In fact in this bound the condition of irreducibility can be dropped, see Remark 3.2 below.

If $M \geq \exp(N \log_3 N / \log_2 N)$, then using instead Theorem 1.3, one obtains the upper bound

$$\mathrm{M}_s(M, N) \ll N^{s-1+o(1)},$$

which matches the trivial lower bound (1.3).

Remark Analysing the proofs, one can easily see that the above results extend without any changes to m.d. in s-tuples $(u_1(n_1), \ldots, u_s(n_s))$, of s (not necessary distinct) linear recurrence sequences. □

2 Preliminaries

2.1 Arithmetic properties of linear recurrence sequences

In this section we collect various results about the arithmetic properties of a linear recurrence sequence that we need for our main results. These include:

- a lower bound of square-free parts of elements in $\vec{u}$,
- a bound for the number of elements in $\vec{u}$ that are S-units,
- various results on congruences with elements in $\vec{u}$,
- a result on the finiteness of perfect powers in $\vec{u}$.

Some of these are obtained under the condition that $\vec{u}$ has a dominant root.

We start with a lower bound of Stewart [17, Theorem 1] on the square-free part of elements in a linear recurrence.

For any integer m, we define $\mathrm{rad}(m)$ to be the largest square-free factor of m.

***Lemma* 2.1** *Let $\vec{u}$ be a simple non-degenerate sequence of order $d \geq 2$ with a dominant root. Then there exist constants C_1 and C_2, which are effectively computable only in terms of $\vec{u}$, such that if $n \geq C_2$, then*

$$\mathrm{rad}(u(n)) > n^{C_1(\log_2 n)/\log_3 n}.$$

We also need the following upper bound from [15, Theorem 1 and Corollary] on the number of terms of $\vec{u}$ composed out of primes from a given set. We note that the condition of the exponential growth of the terms of $\vec{u}$, assumed in [15], is now known to hold for non-degenerate recurrence sequences, see [8, 14]. Hence we have the following result.

Lemma Let $\vec{u}$ be a non-degenerate sequence of order $d \geq 2$ and let S be an arbitrary set of r primes. Then, for $M \geq 0$, the number $A(S; M, N)$ of terms $u(M+1), \ldots, u(M+N)$, composed exclusively of primes from S, satisfies

$$A(S; M, N) \ll \begin{cases} rNM^{-1}\log(N+M) & \text{for } M \geq 1, \\ r(\log N)^2 & \text{for } M = 0. \end{cases}$$

□

We now present two results regarding solutions to certain congruences with elements in a linear recurrence sequence. We start with a result, which follows from [15, Lemma 2 and Lemma 3].

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Lemma 2.2 *Let $\vec{u}$ be a non-degenerate sequence of order $d \geq 2$ and let $m \geq 1$ be an integer. Then we have*

$$\sharp\{n \in \mathbb{Z} \cap [M+1, M+N] : u(n) \equiv 0 \pmod{m}\} \ll N/\log m + 1.$$

The second bound that we need holds modulo primes and follows from [2, Lemma 6]. In [2] it is formulated only for the interval $[1, N]$, however the result is uniform with respect to the sequence $\vec{u}$ and hence it holds uniformly with respect to M, too.

Let $\overline{\mathbb{F}}_p$ be the algebraic closure of the finite field $\mathbb{F}_p$ of p elements.

Lemma Let $\vec{u}$ be a simple sequence of order $d \geq 2$ and for a prime p let $\lambda_1, \ldots, \lambda_d$ be the roots of the characteristic polynomial of $\vec{u}$ in $\overline{\mathbb{F}}_p$. We set $\varrho_p = 1$ if at least one root $\lambda_1, \ldots, \lambda_d$ is zero and set

$$\varrho_p = \min_{1 \leq i < j \leq d} r_{ij},$$

where r_{ij} is the multiplicative order of λ_i/λ_j in $\overline{\mathbb{F}}_p$, otherwise. Then for any integers $M \geq 0$ and $N \geq 1$ we have

$$\sharp\{n \in \mathbb{Z} \cap [M+1, M+N] : u(n) \equiv 0 \pmod{p}\} \ll N(N^{-1} + \varrho_p^{-1})^{1/(d-1)}.$$

□

The following result is certainly well-known and is based on classical ideas of Hooley [12], however for completeness we present a short proof.

Lemma For $R \geq 2$ we consider the set

$$\mathcal{W}(R) = \{p \text{ prime} : \varrho_p \leq R\}.$$

Then $\sharp\mathcal{W}(R) \ll R^2/\log R$. □

Proof Write $\lambda_1, \ldots, \lambda_q$ for the distinct roots of the characteristic polynomial of $\vec{u}$.

For $R \geq 2$, let

$$Q(R) = \prod_{\rho \leq R} \prod_{1 \leq i < j \leq q} \mathrm{Nm}_{K/\mathbb{Q}}(\lambda_i^\rho - \lambda_j^\rho),$$

where $\mathrm{Nm}_{K/\mathbb{Q}}$ is the norm from the splitting field K of f to $\mathbb{Q}$. Note that $Q(R) \neq 0$ because λ_i/λ_j is not a root of unity and since λ_i and λ_j are algebraic integers we also have $Q(R) \in \mathbb{Z}$.

Clearly, for any prime p which does not divide the constant coefficient of the characteristic polynomial of $\vec{u}$ and with $\varrho_p \leq R$, we have $p \mid Q(R)$, hence

$$\sharp\mathcal{W}(R) \leq \omega(Q(R)) + O(1),$$

where $\omega(k)$ is the number of prime divisors of an integer $k \geq 1$. As clearly $\omega(k)! \leq k$, by the Stirling formula we get

$$\sharp\mathcal{W}(R) \ll \frac{\log Q(R)}{\log\log Q(R)}.$$

Since obviously $\log Q(R) \ll R^2$, the result follows. □

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Finally, we need a result on the finiteness of perfect powers in linear recurrence sequences with a dominant root. The most general and convenient form for us, which is built on several previous results in this direction, is given by of Bugeaud and Kaneko [7, Theorem 1.1].

Lemma Let $\vec{u}$ be a simple non-degenerate sequence of order $d \geq 2$ with an irreducible characteristic polynomial having a dominant root. Then the equation $u(n) = m^k$ has only finitely many solutions in integer $k \geq 2, m \neq 0, n \geq 1$. □

2.2 Vertex covers

We need the following graph-theoretic result.

Lemma Let G be a graph with vertex set $\mathcal{V}$, having no isolated vertex. Put $\ell = \sharp\mathcal{V}$. Then there exists $\mathcal{V}_1 \subseteq \mathcal{V}$ with $\sharp\mathcal{V}_1 \leq \ell/2$ such that for any $v_2 \in \mathcal{V}_2 = \mathcal{V} \setminus \mathcal{V}_1$ there exists a vertex $v_1 \in \mathcal{V}_1$ which is a neighbour of v_2. □

Proof The statement must be well-known, but we give a simple proof. If $\widetilde{G}$ is a graph (without isolated vertices) obtained from G by omitting some edges, and the statement is valid for $\widetilde{G}$, then the statement is obviously valid for G. Let $\widetilde{G}$ be a forest graph (that is, a graph without cycles) obtained from G by omitting some edges, such that the number of connected components of G and $\widetilde{G}$ are the same. Then $\widetilde{G}$ is a bipartite graph, so the statement is clearly valid for it. Hence the result follows. □

3 Proofs

3.1 Proof of Theorem 1.1

Suppose that for some $n_1, \ldots, n_s \in [M+1, M+N]$ the terms $u(n_1), \ldots, u(n_s)$ are m.d. of maximal rank, that is, we have (1.1) with some nonzero integers $k_1, \ldots, k_s$.

Choose a positive real number $R \geq 2$ to be specified later, and let $\mathcal{W}(R)$ be as in Lemma 2.3.

Write t for the number of indices $i = 1, \ldots, s$ for which $u(n_i)$ has a prime divisor $p_i \notin \mathcal{W}(R)$, and let $r = s - t$ for the number of indices i with $u(n_i)$ having all prime divisors in $\mathcal{W}(R)$. Without loss of generality, we may assume that the corresponding integers are $n_1, \ldots, n_t$, and $n_{t+1}, \ldots, n_s$, respectively.

By Lemmas 2.1 and 2.3, for $M \geq 1$, the number K_1 of such r-tuples $(n_{t+1}, \ldots, n_s) \in [M+1, M+N]^r$ satisfies

$$K_1 \ll \left(\frac{R^2 N \log(N+M)}{M \log R}\right)^r. \tag{3.1}$$

If $M = 0$, then we have the bound

$$K_1 \ll \left(\frac{R^2 (\log N)^2}{\log R}\right)^r. \tag{3.2}$$

We assume that such an r-tuple $(n_{t+1}, \ldots, n_s)$ is fixed.

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Consider the t-tuples $(n_1, \ldots, n_t) \in [M+1, M+N]^t$. Recall that for any $1 \le i \le t$, there is a prime $p_i \notin \mathcal{W}(R)$ such that $p_i \mid u(n_i)$.

Define the graph $\mathcal{G}$ whose vertices are $u(n_1), \ldots, u(n_t)$, and connect the vertices $u(n_i)$ and $u(n_j)$ precisely when $\gcd(u(n_i), u(n_j))$ has a prime divisor outside $\mathcal{W}(R)$. Observe that as $u(n_1), \ldots, u(n_s)$ are m.d. of maximal rank, $\mathcal{G}$ has no isolated vertex. Thus, by Lemma 2.5, there exists a subset $\mathcal{I}$ of $\{1, \ldots, t\}$ with

$$m = \sharp\mathcal{I} \le \lfloor t/2 \rfloor \tag{3.3}$$

such that for any j with

$$j \in \{n_1, \ldots, n_t\} \setminus \mathcal{I}$$

the vertex $u(n_j)$ is connected with some $u(n_i)$ in $\mathcal{G}$, for some $i \in \mathcal{I}$.

Without loss of generality we may assume that $\mathcal{I} = \{1, \ldots, m\}$. Trivially, the number K_2 of such m-tuples $(n_1, \ldots, n_m) \in [M+1, M+N]^m$ satisfies

$$K_2 \ll N^m. \tag{3.4}$$

We now fix such an m-tuple. For $\ell = t - m$, we now count the number K_3 of the remaining ℓ-tuples $(n_{m+1}, \ldots, n_t) \in [M+1, M+N]^\ell$. Since each $u(n_j)$ with $m+1 \le j \le t$ has a common prime factor $p \notin \mathcal{W}(R)$ with $u(n_i)$ for some $1 \le i \le m$, by Lemma 2.2 we obtain that n_j comes from a set $\mathcal{N}$ of cardinality

$$\sharp\mathcal{N} \ll N\left(N^{-1} + \varrho_p^{-1}\right)^{1/(d-1)} \le N\left(N^{-1} + R^{-1}\right)^{1/(d-1)}.$$

Thus we obtain

$$K_3 \le (\sharp\mathcal{N})^\ell \ll \left(N\left(N^{-1} + R^{-1}\right)^{1/(d-1)}\right)^{t-m}. \tag{3.5}$$

We consider now two cases based on $M \le N \log N$ or $M > N \log N$.

If $M \le N \log N$, then

$$\mathrm{M}_s^*(M, N) \le \mathrm{M}_s^*(0, 2N \log N),$$

therefore we reduce to counting s-tuples in the interval $[0, 2N \log N]^s$.

Putting together the bounds (3.2), (3.4) and (3.5) (with N replaced by $2N \log N$), for some non-negative integer $t \le s$ and $r = s - t$, we obtain

$$\begin{aligned} \mathrm{M}_s^*(M, N) &\le K_1 K_2 K_3 \\ &\ll \left(R^2 (\log N + \log\log N)^2 / \log R\right)^r (N \log N)^m \\ &\qquad \left(N \log N \left(N^{-1} + R^{-1}\right)^{1/(d-1)}\right)^{t-m} \\ &\le N^{t+o(1)} R^{2r} \left(\left(N^{-1} + R^{-1}\right)^{1/(d-1)}\right)^{t/2}, \end{aligned} \tag{3.6}$$

where the last inequality comes from (3.3).

Letting $R = N^\eta$ with some $0 < \eta < 1/2$, we obtain

$$\mathrm{M}_s^*(M, N) \ll N^{t + 2\eta r - \eta t/(2(d-1)) + o(1)} = N^{2\eta s + (1-2\eta)t - \eta t/(2(d-1)) + o(1)}.$$

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Writing $t = zs$ (and noting that $0 \le z \le 1$), the exponent of the last term above (omitting the expression $o(1)$) is given by

$$f_\eta(z) = \frac{s}{2(d-1)}\left((2d - 4d\eta + 3\eta - 2)z + 4\eta(d-1)\right).$$

So taking

$$\eta = \frac{2(d-1)}{4d-3},$$

(to make $f_\eta(z)$ a constant), we obtain

$$\mathrm{M}_s^*(M,N) \ll N^{2\eta s + o(1)} = N^{s - s/(4d-3) + o(1)},$$

which concludes this case.

If $M > N \log N$, then the bound (3.1) becomes

$$K_1 \ll (R^2/\log R)^r.$$

Putting this together with (3.4) and (3.5), we obtain (3.6) without the $(\log N)^2$ factor, that is,

$$\begin{aligned}\mathrm{M}_s^*(M,N) &\le K_1K_2K_3\\ &\ll (R^2/\log R)^r N^m \left(N\left(N^{-1}+R^{-1}\right)^{1/(d-1)}\right)^{t-m}\\ &\ll N^t R^{2r}\left(\left(N^{-1}+R^{-1}\right)^{1/(d-1)}\right)^{t/2}.\end{aligned}$$

Using the same discussion and choice of η as above, we conclude the proof. □

Remark Clearly in (3.6) we can replace $t/2$ with $\lceil t/2 \rceil$ but this does not change the optimal choice of η and thus the final bound. □

3.2 Proof of Theorem 1.3

Let $(n_1,\dots,n_s) \in [M+1, M+N]^s$ such that $u(n_1),\dots,u(n_s)$ is m.d. of maximal rank, which implies that there exist integers $k_i \ne 0$, $i = 1,\dots,s$, such that (1.1) holds. We can rewrite this relation as

$$\prod_{i\in\mathcal{I}} u(n_i)^{k_i} = \prod_{j\in\mathcal{J}} u(n_j)^{k_j}, \quad k_i, k_j > 0, \tag{3.7}$$

where $\mathcal{I} \cup \mathcal{J} = \{1,\dots,s\}$, $\mathcal{I} \ne \emptyset$, $\mathcal{J} \ne \emptyset$, $\mathcal{I} \cap \mathcal{J} = \emptyset$. Let $I = \sharp\mathcal{I}$ and $J = \sharp\mathcal{J}$, and thus, $I + J = s$.

Fix one of $2^s - 2$ possible choices of the sets $\mathcal{I}$ and $\mathcal{J}$ as above. Fix n_i, $i \in \mathcal{I}$, trivially in $O(N^I)$ ways. Then, the square-free part $\mathrm{rad}(u(n_i))$ of $u(n_i)$ is fixed for each $i \in \mathcal{I}$.

We may also assume that $n_i \ge C_2$, $i \in \mathcal{I}$, with C_2 as in Lemma 2.1, since this condition is violated only for $O(N^{s-1})$ choices of $(n_1,\dots,n_s)$, which is admissible. By Lemma 2.1, for $n_i \in [M+1, M+N]$, one has

$$\mathrm{rad}(u(n_i)) > n_i^{c(\log_2 n_i)/\log_3 n_i} \gg M^{c(\log_2 M)/\log_3(M+N)}. \tag{3.8}$$

For $i \in \mathcal{I}$, from (3.7) we see

$$\operatorname{rad}(u(n_i)) \mid \prod_{j \in \mathcal{J}} u(n_j).$$

This implies that there is a factorisation $\operatorname{rad}(u(n_i)) = d_1 \cdots d_J$ such that for each positive integer d_ℓ there exists $j \in \mathcal{J}$ such that $d_\ell \mid u(n_j)$. Let ℓ, $1 \le \ell \le J$, be such that $d_\ell \ge \operatorname{rad}(u(n_i))^{1/J}$, and

$$u(n_j) \equiv 0 \pmod{d_\ell}. \tag{3.9}$$

From (3.8) we have

$$d_\ell > M^{c_0 (\log_2 M)/\log_3(M+N)} \tag{3.10}$$

with $c_0 = c/J \ge c/s$.

Using now Lemma 2.2, the inequality (3.10) and the fact that $J \le s$, the number of $n_j \in [M+1, M+N]$ satisfying the congruence (3.9) is

$$O\left(N/\log d_\ell + 1\right) = O\left(N \frac{\log_3(M+N)}{\log M \log_2 M} + 1\right).$$

Therefore, using the trivial bound N^{J-1} for the number of the remaining choices of n_j with $j \in \mathcal{J}$, we obtain that the total number of $n_j \in [M+1, M+N]$, $j \in \mathcal{J}$, is

$$O\left(N^J \frac{\log_3(M+N)}{\log M \log_2 M} + N^{J-1}\right).$$

Thus we obtain that

$$\mathrm{M}_s^*(M, N) \ll N^s \frac{\log_3(M+N)}{\log M \log_2 M} + N^{s-1}.$$

Choosing $M \ge \exp(N \log_3 N/\log_2 N)$, we conclude the proof. □

3.3 Proof of Theorem 1.5

Clearly for $s = 2$ we have to count integers $M + 1 \le m, n \le M + N$, with

$$u(m)^a = u(n)^b \tag{3.11}$$

for some positive integers a and b, where without loss of generality we can assume that $\gcd(a, b) = 1$. We also notice that since the relation (3.11) is of maximal rank, neither $u(m) = \pm 1$ nor $u(n) = \pm 1$ holds.

Since $\vec{u}$ has a dominant root, $|u(n)|$ grows monotonically with n, provided that n is large enough. Hence there are $N + O(1)$ solutions $(m, n) \in [M+1, M+N]^2$ with $a = b = 1$.

Now we count pairs (m, n) for which (3.11) holds with some $(a, b) \ne (1, 1)$.

We observe that if $a > 1$ then $u(n)$ is the a-th power and by Lemma 2.4 there are $O(1)$ such values of n. For $b > 1$ the argument also applies to m. Hence the total contribution from such solutions, over all $a, b > 1$, is $O(1)$.

If $a > 1$ and $b = 1$ then again we see that there are $O(1)$ such values of n. From this we easily derive that $a = O(1)$, and hence we obtain $O(1)$ possible values for m. So the contribution of such solutions to (3.11) is also O(1) only.

The case of $a = 1$ and $b > 1$ is completely analogous, which concludes the proof. □

Remark We note that without the irreducibility condition of the characteristic polynomial, that is, only under the condition of having a dominant root, we have boundedness of k in Lemma 2.4, see the discussion in [7, Section 1]. Thus, the above proof shows that in this case we have a version of Theorem 1.5 in the form $\mathrm{M}_2^*(M,N) \ll N$ and thus (1.4) holds only under this assumption. □

4 Possible applications of our approach

Our approach works for many other integer sequences $(a(n))_{n=1}^{\infty}$, provided the following information is available:

(i) there are good bounds on the number of solutions to congruences $a(n) \equiv 0 \pmod q$, $1 \le n \le N$, in a broad range of positive integers q (or even just prime $q = p$) and N;

(ii) there are good bounds (or known finiteness) on the number of perfect powers among $a(n)$, $1 \le n \le N$.

For example, using results of [16], coupled with the finiteness result of Lemma 2.4, one can estimate the number of multiplicatively dependent s-tuples from values of linear recurrence sequences at polynomial values of the argument $(u\left(F(n)\right))_{n=1}^{\infty}$, where $F \in \mathbb{Z}[X]$.

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