Can. J. Math., Vol. XXIV, No. 6, 1972, pp. 1100–1109

MODULAR HADAMARD MATRICES AND RELATED DESIGNS, II

O. MARRERO AND A. T. BUTSON

1. Introduction. An h by h matrix with entries ± 1 is called a *modular Hadamard matrix* if the inner product of any two distinct row vectors is a multiple of a fixed (positive) integer n; such a matrix is also referred to as an "$H(n, h)$ matrix" with parameters n and h. Modular Hadamard matrices and the related combinatorial designs were introduced in [**2**]; that paper was concerned mainly with two of the related designs, the "pseudo (v, k, λ)-designs" and the "$(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-designs" (the reader is referred to [**2**] for the definition of these designs). This paper is concerned with the construction and existence of modular Hadamard matrices, and special attention is given to some particular classes of these matrices.

The notation used in this paper follows that used in [**2**]. In addition, the left Kronecker product $C_{rs} = A_r \otimes B_s$ of two matrices $A_r = [a_{ij}]$ and B_s is the rs by rs matrix whose ijth s by s block is given by $a_{ij}B$, $1 \leqq i, j \leqq r$. An ordinary Hadamard matrix (or "$H(0, h)$ matrix") is said to be "normalized" provided that it has only $+1$'s as entries in both the first row and the first column.

The following result [**2**, Theorem 2.1] will be used several times in the sequel, and is stated here for reference purposes.

THEOREM 1.1. *Let H_h be a $(1, -1)$-matrix having the first row consisting of all $+1$'s. For each $i, j = 2, \ldots, h$ let k_i denote the number of $+1$'s in the ith row, and let λ_{ij} denote the number of times the ith and jth rows have a $+1$ in the same column, $i \neq j$. Then necessary and sufficient conditions that H be an $H(n, h)$ matrix are*

$$(1.1) \qquad \begin{aligned} 2k_i &\equiv h \pmod{n}, \\ 4\lambda_{ij} &\equiv h \pmod{n}. \end{aligned}$$

2. Construction and existence theorems for $H(n, h)$ matrices. This section contains some results which show how to construct new modular Hadamard matrices from given modular Hadamard matrices and ordinary Hadamard matrices, from several well-known combinatorial designs, and from abelian difference sets. In particular, it is shown that $H(n, h)$ matrices can always be constructed when $n|h$ or when $n|(h - 4)$. Also, the existence of $H(n, h)$ matrices will be completely determined for $n = 2$, $n = 3$, and $n = 6$.

Received October 13, 1971 and in revised form, January 13, 1972.

It is clear that an $H(n, h)$ matrix is also an $H(d, h)$ matrix for any divisor d of n. Thus, it is of interest to find ways of obtaining, from given $H(n_1, h_1)$ and $H(n_2, h_2)$ matrices, an $H(n, h)$ matrix having $n > n_1, n_2$. The left Kronecker product of two modular Hadamard matrices, when one of the factors is an ordinary Hadamard matrix, is one such construction.

THEOREM 2.1. *Let M be an $H(n_1, h_1)$ matrix and let N be an $H(n_2, h_2)$ matrix. Then $M \otimes N$ is an $H(n, h_1h_2)$ matrix, where $n = \gcd\{h_1n_2, n_1n_2, h_2n_1\}$.*

THEOREM 2.2. *If M and N are $H(0, h_1)$ and $H(n_2, h_2)$ matrices, respectively, then $M \otimes N$ is an $H(n_2h_1, h_1h_2)$ matrix.*

When $n|h$ or $n|(h - 4)$, $H(n, h)$ matrices can be constructed quite easily. One observes that J_h is an $H(n, h)$ matrix whenever $n|h$, and $J_h - 2I_h$ is an $H(h - 4, h)$ matrix, and hence it is also an $H(n, h)$ matrix when $n|(h - 4)$. (When $n|h$ it is sometimes possible to construct an $H(n, h)$ matrix different from J_h; and when $n|(h - 4)$, there are always additional ways of constructing an $H(n, h)$ matrix. These additional constructions are simple and are therefore omitted.) This proves:

THEOREM 2.3. *If $n|h$ or $n|(h - 4)$, then $H(n, h)$ matrices can be constructed.*

It was observed in [**2**] that a necessary condition for the existence of an $H(n, h)$ matrix with $h \geqq 3$ is that $h \equiv 4t \pmod n$ for some $t \in Z$. It is now shown that this condition is also sufficient in certain cases. (Note that if $n = \gcd\{4u - 2, 4t - 2\}$ and $h = 4u + 4t - 2$, then $h \equiv 4t \pmod n$.)

THEOREM 2.4. *Given $H(0, 4u)$ and $H(0, 4t)$ matrices, then an $H(n, 4u + 4t - 2)$ matrix can be constructed, where $n = \gcd\{4u - 2, 4t - 2\}$.*

Let M and N be the matrices obtained from normalized $H(0, 4u)$ and $H(0, 4t)$ matrices, respectively, by removing from each the first row and the first column. Then

$$\begin{bmatrix} M_{4u-1} & J_{4u-1,4t-1} \\ -J_{4t-1,4u-1} & N_{4t-1} \end{bmatrix}$$

is an $H(n, 4u + 4t - 2)$ matrix, where $n = \gcd\{4u - 2, 4t - 2\}$.

The special case where $M = N$ is a normalized $H(0, 4t)$ matrix yields a class of $H(n, h)$ matrices with $n \equiv 2 \pmod 4$.

COROLLARY 2.1. *An $H(4t - 2, 8t - 2)$ matrix can be constructed using an $H(0, 4t)$ matrix.*

Certain well-known combinatorial designs can be used to construct modular Hadamard matrices, as follows. If M is the matrix obtained from the incidence matrix of a (v, k, λ)-design [**1**; **3**] by replacing all 0's by -1's, then M is an $H(n, v)$ matrix, where n is a positive divisor of $v - 4(k - \lambda)$. Also,

$$\begin{bmatrix} M & J_v \\ -J_v & M \end{bmatrix}$$

is an $H(2v - 4(k - \lambda), 2v)$ matrix. This proves:

THEOREM 2.5. *Given a* (v, k, λ)*-design, then* $H(v - 4(k - \lambda), v)$ *and* $H(2v - 4(k - \lambda), 2v)$ *matrices can be constructed.*

According to Fisher's inequality, a (b, v, r, k, λ)-design has $b \geqq v$ [**3**]. If M is the matrix obtained from the transpose of the incidence matrix of a (b, v, r, k, λ)-design by first replacing all 0's by -1's and then adjoining $b - v$ rows of $+1$'s, then M is an $H(n, b)$ matrix, where $n = \gcd\{b, 2r, 4\lambda\}$. This is stated as:

THEOREM 2.6. *If a* (b, v, r, k, λ)*-design is given, then an* $H(n, b)$ *matrix can be constructed, where* $n = \gcd\{b, 2r, 4\lambda\}$.

For the PBIB designs [**4**] and GD designs [**1**], cases need to be considered depending upon whether $b > v$, $b = v$, or $b < v$, but the same technique may be applied to the incidence matrices of these designs to obtain modular Hadamard matrices. This yields:

THEOREM 2.7. *Given a* PBIB *design with* m *associate classes, an* $H(n, h)$ *matrix can be constructed, where*

(1) $n = \gcd\{v - 4(k - \lambda_1), \ldots, v - 4(k - \lambda_m)\}$ *and* $h = v$, *if* $b = v$,
(2) $n = \gcd\{b, 2r, 4\lambda_1, \ldots, 4\lambda_m\}$ *and* $h = b$, *if* $b > v$, *or*
(3) $n = \gcd\{v - 4(r - \lambda_1), \ldots, v - 4(r - \lambda_m)\}$ *and* $h = v$, *if* $b < v$.

COROLLARY 2.2. *Given a* GD *design, then an* $H(n, h)$ *matrix can be constructed, where*

(1) $n = \gcd\{v - 4(k - \lambda_1), v - 4(k - \lambda_2)\}$ *and* $h = v$, *when* $b = v$,
(2) $n = \gcd\{b, 2r, 4\lambda_1, 4\lambda_2\}$ *and* $h = b$, *when* $b > v$, *and*
(3) $n = \gcd\{v - 4(r - \lambda_1), v - 4(r - \lambda_2)\}$ *and* $h = v$, *when* $b < v$.

THEOREM 2.8. *Given two abelian difference sets* D_1 *and* D_2 *with parameters* v_1, k_1, λ_1 *and* v_2, k_2, λ_2, *respectively, then an* $H(n, h)$ *matrix can be constructed, where*
$n = \gcd\{v_1v_2 - 4(k_1k_2 - \lambda_1\lambda_2), v_1v_2 - 4(k_1k_2 - k_1\lambda_2), v_1v_2 - 4(k_1k_2 - k_2\lambda_1)\}$
and $h = v_1v_2$.

Suppose $D_1 \subseteq (G_1, +), D_2 \subseteq (G_2, +)$, and consider the direct sum $G = G_1 \oplus G_2$. Let the $(1, -1)$-matrix $H = [h_{ij}]$ be defined by taking $h_{ij} = 1$ if $g_j \in (D_1 \oplus D_2) + g_i$, and $h_{ij} = -1$ if $g_j \notin (D_1 \oplus D_2) + g_i$, for each pair $g_i, g_j \in G$. It then follows that H is an $H(n, v_1v_2)$ matrix, where n satisfies the hypothesis of the theorem.

For each of these preceding results there exists a class of parameters of the design used in the construction which yields non-trivial (in the sense that $n \geqq 2$) $H(n, h)$ matrices.

The preceding Theorem 2.2 and Theorem 2.3 together with Theorem 2.2 and Corollary 2.1 in [**2**] may be used to determine completely the existence of $H(n, h)$ matrices when n is 2, 3, or 6, as stated in the following theorems.

https://doi.org/10.4153/CJM-1972-114-7 Published online by Cambridge University Press

THEOREM 2.9. *A necessary and sufficient condition for the existence of an* $H(2, h)$ *matrix is that* h *be even.*

THEOREM 2.10. *A necessary and sufficient condition for the existence of an* $H(3, h)$ *matrix is that* $h \equiv 0, 1 \pmod 3$, *or* $h \equiv 2 \pmod{12}$, *or* $h \equiv 0 \pmod 4$.

THEOREM 2.11. *A necessary and sufficient condition for the existence of an* $H(6, h)$ *matrix is that* h *be even.*

3. $H(4q + 1, h)$ and $H(4q + 3, h)$ matrices. The matrices which are studied in this section are the $H(n, h)$ matrices for $n = 4q + 1$ and $h = n + 1$, $2n + 1$, $3n + 1$, and $4n + 1$, and the $H(n, h)$ matrices for $n = 4q + 3$ and $h = n + 1$, $2n + 1$, and $3n + 1$. The method utilized in studying these matrices is as follows. A given $H(n, h)$ matrix is taken to be in standard form. Then, using the congruences (1.1), one finds the possible values for k_i and λ_{ij}. Next, each -1 in the given $H(n, h)$ matrix is replaced by 0, and the first row of all $+1$'s is removed. Thus, one obtains the incidence matrix of some combinatorial design having the k_i's for possible "row sums" and the λ_{ij}'s for possible "row intersections". Finally, by analyzing the combinatorial design thus obtained, one gets information about the corresponding $H(n, h)$ matrix. This method is similar to that which establishes a connection between $H(0, 4t)$ matrices and $(4t - 1, 2t - 1, t - 1)$-designs, where $t \geqq 2$. In studying the $H(n, h)$ matrices considered in this section, one is led to pseudo (v, k, λ)-designs and $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-designs. Thus, some of the results obtained in [**2**] will find application in the sequel.

Multiplication of the elements of any row by -1 preserves the orthogonality modulo n property of the rows of an $H(n, h)$ matrix. Consequently, the standard form of an $H(n, h)$ matrix may be assumed to have, in addition to the first row consisting of $+1$'s, $2k_i \leqq h$, $i = 2, \ldots, h$. Thus, when considering solutions of the congruences (1.1) in the sequel, only those values of k_i satisfying $2k_i \leqq h$ will be considered.

Let $n = 4q + 1$. When h is $n + 1$ or $2n + 1$, the congruences (1.1) have no possible solution. When $h = 3n + 1$, the congruences (1.1) yield $k_i = 6q + 2$ and $\lambda_{ij} = 3q + 1$ for $i, j \geqq 2$, $i \neq j$.

Let $n = 4q + 3$ and $h = n + 1$, then solving the congruences (1.1) yields $k_i = 2q + 2$ and $\lambda_{ij} = q + 1$. These results are collected in:

THEOREM 3.1. $H(4q + 1, 4q + 2)$ *matrices and* $H(4q + 1, 8q + 3)$ *matrices do not exist. The only* $H(4q + 1, 12q + 4)$ *matrices are the* $H(0, 12q + 4)$ *matrices, and the only* $H(4q + 3, 4q + 4)$ *matrices are the* $H(0, 4q + 4)$ *matrices.*

When $n = 4q + 1$ and $h = 4n + 1$, solving the congruences (1.1) yields $k_i = 6q + 2$ and $\lambda_{ij} = 3q + 1$ for $i, j \geqq 2$, $i \neq j$. Let $A_{16q+4,16q+5}$ be the matrix obtained from an $H(4q + 1, 16q + 5)$ matrix in standard form by first replacing all -1's by 0's, and then deleting the first row of all $+1$'s.

Then

$$AJ_{16q+5,1} = (6q + 2)J_{16q+4,1} \text{ and}$$
$$AA^T = (3q + 1)I_{16q+4} + (3q + 1)J_{16q+4},$$

from which it is seen that A is the incidence matrix of a pseudo $(16q + 5, 6q + 2, 3q + 1)$-design. Similarly, when $n = 4q + 3$ and $h = 2n + 1$, one is led to a $(0, 1)$-matrix $B_{8q+6,8q+7}$ which satisfies

$$BJ_{8q+7,1} = (2q + 2)J_{8q+6,1} \text{ and}$$
$$BB^T = (q + 1)I_{8q+6} + (q + 1)J_{8q+6},$$

so that B is the incidence matrix of a pseudo $(8q + 7, 2q + 2, q + 1)$-design. Thus, in each of these two instances, one is led to a pseudo (v, k, λ)-design with $v \neq 4\lambda$ and $k = 2\lambda$. Consequently, as an application of Theorem 3.4 in [**2**], it is possible to state:

THEOREM 3.2. *The existence of an* $H(4q + 1, 16q + 5)$ *matrix is equivalent to the existence of a* $(16q + 5, 4q + 1, q)$*-design. Also, if* $32q^2 + 56q + 25 = D^2$, *then the existence of an* $H(4q + 3, 8q + 7)$ *matrix is equivalent to the existence of a* $(8q + 7, (8q + 7 - D)/2, (8q + 7 - D)/2 - q - 1)$*-design.*

When $n = 4q + 3$ and $h = 3n + 1$, it will be shown that the associated design is an $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-design. But first an interesting observation will be made concerning this particular class of $H(n, h)$ matrices.

As observed in Section 2, if H is an $H(n, h)$ matrix, then it is an $H(d, h)$ matrix for any divisor d of n (but the converse does not always hold, for it is a consequence of Theorem 2.10 and Theorem 2.11 that $H(3, 9)$ matrices exist whereas $H(6, 9)$ matrices do not exist); in particular, an $H(8q + 6, 12q + 10)$ matrix is also an $H(4q + 3, 12q + 10)$ matrix. It is an interesting fact that for this particular set of parameters the converse is also true. For suppose H is an $H(4q + 3, 12q + 10)$ matrix in standard form. For these parameters, solving the congruences (1.1) one finds that the only k_i which satisfy $2k_i \leqq h$ are $k_i = 2q + 2$, or $6q + 5$, and, therefore, that $\lambda_{ij} = q + 1$, or $5q + 4$. Consequently, the following result may be stated.

THEOREM 3.3. *A* $(1, -1)$*-matrix is an* $H(4q + 3, 12q + 10)$ *matrix if and only if it is an* $H(8q + 6, 12q + 10)$ *matrix.*

Now let H be an $H(4q + 3, 12q + 10)$ matrix in standard form, and let A be the $(0, 1)$-matrix obtained from H by first replacing all -1's by 0's, and then removing the initial row of all $+1$'s. After appropriate permutations of the rows, if necessary, it may be assumed that

$$A = \begin{bmatrix} M_{f-1,12q+10} \\ S_{12q+10-f,12q+10} \end{bmatrix}$$

and

$$AA^T = \begin{bmatrix} MM^T & (q + 1)J \\ (q + 1)J & SS^T \end{bmatrix},$$

https://doi.org/10.4153/CJM-1972-114-7 Published online by Cambridge University Press

where

$$MM^T = (q+1)I_{f-1} + (q+1)J_{f-1},$$

$$SS^T = \begin{bmatrix} 6q+5 & \mu_{12} & \cdots & \mu_{1,12q+10-f} \\ \mu_{21} & 6q+5 & \cdots & \mu_{2,12q+10-f} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \mu_{12q+10-f,1} & \mu_{12q+10-f,2} & \cdots & 6q+5 \end{bmatrix},$$

μ_{rs} is either $q+1$ or $5q+4$, and f is a fixed integer, $1 \leqq f \leqq 12q+10$. Thus, some simplification of this associated incidence matrix is desirable, and this will be accomplished presently. In fact, it will be shown that all μ_{rs} may be assumed to be $5q+4$. Consider S as the incidence matrix of a combinatorial design consisting of $12q+10-f$ subsets $X_1, \ldots, X_{12q+10-f}$ of the set $X = \{x_1, \ldots, x_{12q+10}\}$, so that $\mu_{rs} = |X_r \cap X_s|$ for $r \neq s$. For notation, $\sim X_r$ will denote the set complement of X_r. It is first observed that there cannot be 3 subsets X_r, X_s, X_t satisfying $|X_r \cap X_s| = |X_r \cap X_t| = |X_s \cap X_t| = q+1$. For, if this were so, then $12q+10 \geqq |X_r \cup X_s \cup X_t| \geqq 15q+12$, which is not true. If $12q+10-f$ is 0 or 1, then there is no μ_{rs} to consider. If $12q+10-f$ is 2, and $|X_{12q+9-f} \cap X_{12q+10-f}| = q+1$, then replacing $X_{12q+10-f}$ by $\sim X_{12q+10-f}$ gives $|X_{12q+9-f} \cap \sim X_{12q+10-f}| = 5q+4$. Suppose $12q+10-f$ is 3, and suppose there is a pair of subsets X_r, X_s such that $|X_r \cap X_s| = q+1$. If X_t is the third subset, then $|X_r \cap X_t| \neq |X_s \cap X_t|$; for $|X_r \cap X_t| = |X_s \cap X_t| = q+1$ is impossible, and, if $|X_r \cap X_t| = |X_s \cap X_t| = 5q+4$, then $|X_r \cap \sim X_t| = |X_s \cap \sim X_t| = q+1$, which is again impossible. Now, it may be assumed that $|X_r \cap X_t| = q+1$ and $|X_s \cap X_t| = 5q+4$. Thus, replacing X_r by $\sim X_r$ gives $|\sim X_r \cap X_s| = |\sim X_r \cap X_t| = |X_s \cap X_t| = 5q+4$, as desired. Now let $12q+10-f \geqq 4$, and suppose there are 2 subsets X_r, X_s such that $|X_r \cap X_s| = q+1$. For a third subset, say X_t, it may be assumed that $|X_s \cap X_t| = 5q+4$ and $|X_r \cap X_t| = q+1$. Now consider a fourth subset X_u. It is necessary that $|X_r \cap X_u| \neq |X_s \cap X_u|$; say $|X_r \cap X_u| = q+1$ and $|X_s \cap X_u| = 5q+4$. Then it follows that $|X_t \cap X_u| = 5q+4$. Now note that $|\sim X_s \cap X_r| = |\sim X_s \cap \sim X_t| = |\sim X_s \cap \sim X_u| = |X_r \cap \sim X_t| = |X_r \cap \sim X_u| = |\sim X_t \cap \sim X_u| = 5q+4$. Therefore, replacing X_s, X_t, and X_u by their complements gives a design in which all $\mu_{rs} = 5q+4$. This proves:

THEOREM 3.4. *The incidence matrix of the combinatorial design associated with an* $H(4q+3, 12q+10)$ *matrix yields the incidence matrix of an* $(m, v, k_1, \lambda_1, k_2, \lambda_2, f-1, \lambda_3)$*-design, where* $m = 12q+9$, $v = 12q+10$, $k_1 = 2q+2$, $\lambda_1 = q+1$, $k_2 = 6q+5$, $\lambda_2 = 5q+4$, $\lambda_3 = q+1$, *and* f *is some fixed integer satisfying* $1 \leqq f \leqq 12q+10$. *The converse is also valid.*

4. A particular class of $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-designs. The particular class of $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-designs which is considered in this section is that obtained from $H(4q + 3, 12q + 10)$ matrices as stated in Theorem 3.4. The existence of these designs will be determined for $q = 0$, $q = 1$, and $q = 2$ (and as a consequence of Theorem 3.4, it follows that the existence of $H(4q + 3, 12q + 10)$ matrices will be determined for these three values of q). It is simple to show that these designs exist when $q = 0$ or $q = 1$, since it is not too difficult to exhibit the corresponding $H(4q + 3, 12q + 10)$ matrices. One observes that $J_{10} - 2I_{10}$ is an $H(3, 10)$ matrix; and, since $J_{11} - 2I_{11}$ is an $H(7, 11)$ matrix,

$$(J_{11} - 2I_{11}) \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

is an $H(7, 22)$ matrix. However, as will be seen in the sequel, the known proof that these designs do not exist when $q = 2$ is not as simple.

LEMMA 4.1. *The parameter f in Theorem 3.4 must satisfy the following: if $x = (4q + 3)(f - (6q + 5))$, then x must be a solution of the Diophantine equation $x^2 + y^2 = ((4q + 3)(6q + 5) + 2(q + 1))^2$.*

Let A be the incidence matrix of the design associated with an $H(4q + 3, 12q + 10)$ matrix, as given in Theorem 3.4. Let B be the $(0, 1)$-matrix obtained from A by adjoining an initial row of $+1$'s. With some work, it may be determined that

$$(\det B)^2 = \det(BB^T) = (q + 1)^{12q+8}a, \tag{4.1}$$

where

$$a = -f^2(4q + 3)^2 + 2f(4q + 3)^2(6q + 5) + 4(q + 1)(24q^2 + 39q + 16). \tag{4.2}$$

It follows from Equation (4.1) that a must be a square, say $a = y^2$. Now, from Equation (4.2), a may be re-written in the form

$$a = -(4q + 3)^2(f - (6q + 5))^2 + ((4q + 3)(6q + 5) + 2(q + 1))^2,$$

which easily gives the desired result.

When $q = 2$, the parameter f can be determined by using Lemma 4.1; indeed, f must equal 17. Thus, the following may be stated.

LEMMA 4.2. *If an $(m = 33, v = 34, k_1 = 6, \lambda_1 = 3, k_2 = 17, \lambda_2 = 14, f - 1 = 16, \lambda_3 = 3)$-design exists, then there exists a $(0, 1)$-matrix $M_{16,34}$ satisfying $MM^T = 3I_{16} + 3J_{16}$.*

The proof that an $(33, 34, 6, 3, 17, 14, 16, 3)$-design cannot exist will be accomplished by showing that such a matrix M cannot exist. This proof will be given in several lemmas. If P_{16} and Q_{34} are permutation matrices, then

$$(PMQ)(PMQ)^T = PMM^TP^T = P(3I_{16} + 3J_{16})P^T = 3I_{16} + 3J_{16}.$$

Consequently, the rows of M may be permuted and the columns of M may be permuted without changing the significant properties of M. This fact will be used throughout the subsequent proof, and a particular usage will not necessarily be noted explicitly. Let η_r denote the rth row vector of $M = [m_{ij}]$; and let $\eta_r^{(j)} = (m_{r1}, \ldots, m_{rj})$, the j dimensional vector whose entries are the first j entries of η_r. Throughout the proof it will be assumed that

$$\eta_1 = 111\ 111\ 000\ 000 \ldots 000 \text{ and}$$
$$\eta_2 = 111\ 000\ 111\ 000 \ldots 000.$$

LEMMA 4.3. *The matrix M can have at most three rows which have $+1$'s in the same three columns. If M has three such rows, then every remaining row has exactly two $+1$'s in those three columns.*

It will be sufficient to show that either $\eta_1^{(3)} \circ \eta_j^{(3)} < 3$ for $j \geqq 3$, or $\eta_3^{(3)} = (111)$ and $\eta_1^{(3)} \circ \eta_j^{(3)} = 2$ for $j \geqq 4$. Let

$$\begin{aligned} \eta_3 &= 111\ 000\ 000\ 111\ 000\ 000\ 000 \ldots 000, \\ \eta_4 &= 111\ 000\ 000\ 000\ 111\ 000\ 000 \ldots 000, \text{ and} \\ \eta_5 &= 111\ 000\ 000\ 000\ 000\ 111\ 000 \ldots 000. \end{aligned} \tag{4.3}$$

For $j > 5$, suppose first that $\eta_1^{(3)} \circ \eta_j^{(3)} = 0$. Then $\eta_1 \circ \eta_j = \eta_2 \circ \eta_j = 3$ requires that $\eta_j^{(9)} = (000\ 111\ 111)$. However, $\eta_3 \circ \eta_j = 0$, instead of 3. Now suppose that $\eta_1^{(3)} \circ \eta_j^{(3)} = 1$. In order that $\eta_1 \circ \eta_j = \eta_2 \circ \eta_j = \eta_3 \circ \eta_j = 3$, $\eta_j^{(12)}$ must be *essentially* (100 110 110 110), which is impossible. By "essentially" will be meant to within a permutation of the columns. In this particular case, the permutation could interchange the ith, the $(i + 1)$st, and the $(i + 2)$nd columns, where $i = 1, 4, 7, 10$. Next, supposing that $\eta_1^{(3)} \circ \eta_j^{(3)} = 2$, a similar argument shows that $\eta_j^{(18)}$ must be essentially (110 100 100 100 100 100), which is impossible. Consequently, $\eta_j^{(3)} = (111)$ for all j. In this case, M could have at most 10 rows. At this stage, it has been shown that M can have at most 4 rows with $+1$'s in the same three columns.

Suppose that M has 4 such rows and let η_3 and η_4 be as in (4.3). In the above argument η_5 was used only in the proof that $\eta_1^{(3)} \circ \eta_j^{(3)} \neq 2$. Consequently, in this case, $\eta_1^{(3)} \circ \eta_j^{(3)} = 2$ for $j \geqq 5$. In order that $\eta_1 \circ \eta_j = \eta_2 \circ \eta_j = \eta_3 \circ \eta_j = \eta_4 \circ \eta_j = 3$, it is necessary that η_j be essentially (110 100 100 100 100 000 ... 000), for $j \geqq 5$. Thus, M must have the form

$$M = [S_{16,15} \quad \theta_{16,19}],$$

where $SS^T = 3I_{16} + 3J_{16}$ and every entry in θ is 0. But such a matrix S cannot exist. Therefore, M has at most 3 rows with $+1$'s in the same three columns. The preceding argument has also shown that if M has three such rows, then $\eta_1^{(3)} \circ \eta_j^{(3)} \neq 0, 1$ for $j \geqq 4$. Hence $\eta_1^{(3)} \circ \eta_j^{(3)} = 2$ for $j \geqq 4$, and the lemma is established.

LEMMA 4.4. *If the matrix M has three rows with $+1$'s in the same three columns, then M must have three rows η_r, η_s, and η_t such that $\eta_r^{(6)} = \eta_s^{(6)} = \eta_t^{(6)}$.*

Let η_3 be as described in (4.3). Since (by Lemma 4.3) $\eta_1^{(3)} \circ \eta_j^{(3)} = 2$ for $j \geqq 4$, and since $\eta_1 \circ \eta_j = 3$, it follows that $\eta_j^{(6)}$ is essentially (110 100). Here, the permutation can interchange the ith, $(i + 1)$st, and $(i + 2)$nd columns for $i = 1, 4$; and hence there are exactly nine possibilities for $\eta_j^{(6)}$. They are

$$\begin{array}{lll} \sigma_1 = (110\ 100), & \sigma_2 = (110\ 010), & \sigma_3 = (110\ 001), \\ \sigma_4 = (101\ 100), & \sigma_5 = (101\ 010), & \sigma_6 = (101\ 001), \\ \sigma_7 = (011\ 100), & \sigma_8 = (011\ 010), \text{ and} & \sigma_9 = (011\ 001). \end{array}$$

If the above permutation is not trivial for $i = 1$ and also for $i = 4$, then the resulting σ_j satisfies $\sigma_1 \circ \sigma_j = 1$. Clearly, for each σ_i there are exactly four σ_j such that $\sigma_i \circ \sigma_j = 1$. Since $\eta_j^{(6)}$ is essentially σ_1 for $j \geqq 4$, clearly M must contain two rows η_r and η_s such that $\eta_r^{(6)} = \eta_s^{(6)}$. Let $\eta_r^{(6)} = \eta_s^{(6)} = \sigma_t$, and suppose that $\eta_u^{(6)} = \sigma_w$ and that $\sigma_t \circ \sigma_w = 1$. Then, essentially,

$$\begin{aligned} \eta_r &= 110\ 100\ 100\ 110\ 000\ 000 \ldots 000, \\ \eta_s &= 110\ 100\ 010\ 001\ 100\ 000 \ldots 000, \text{ and} \\ \eta_u^{(6)} &= 101\ 010. \end{aligned}$$

Because $\eta_2 \circ \eta_u = 3$, η_u must have only one $+1$ in the 7th, 8th, and 9th columns. Assume that this $+1$ occurs in the 7th column. Then, since $\eta_u \circ \eta_r = 3$, there is only one $+1$ in the 9th and 10th columns. So far, five of the columns containing $+1$'s in η_u have been determined; and, at this stage, η_s and η_u have only one $+1$ in common, so that $\eta_s \circ \eta_u \leqq 2$. Similar arguments will resolve the cases when the $+1$ in η_u is assumed to occur first in the 8th column and then in the 9th column. Consequently, since M has two rows η_r and η_s satisfying $\eta_r^{(6)} = \eta_s^{(6)} = \sigma_t$, every other row η_u (excluding η_1 and η_2) must have $\eta_u^{(6)} = \sigma_w$, where $\sigma_t \circ \sigma_w \neq 1$. This means that at most five σ_j's can occur in the first six columns of M; and, since M has thirteen rows η_j with $\eta_j^{(6)} = \sigma_1$ essentially, the lemma has been proven.

LEMMA 4.5. *The matrix M cannot have three rows which have* $+1$'s *in the same three columns.*

If M has three such rows, by Lemma 4.4, M also has rows η_r, η_s and η_t such that $\eta_r^{(6)} = \eta_s^{(6)} = \eta_t^{(6)}$. Essentially, $\eta_r^{(6)} = \sigma_1$, and $\eta_1, \eta_r, \eta_s, \eta_t$ are four rows with $+1$'s in the 1st, 2nd, and 4th columns. This contradicts the result in Lemma 4.3.

If η_j, $j \geqq 3$, is a row with exactly one $+1$ in the first three columns, then, since $\eta_1 \circ \eta_j = 3$, $\eta_j^{(6)}$ must be essentially (100 110). Since the permutation here can interchange the ith, $(i + 1)$st, and $(i + 2)$nd columns for $i = 1, 4$, there are exactly nine possibilities. They are

$$\begin{array}{lll} \xi_1 = (001\ 011), & \xi_2 = (001\ 101), & \xi_3 = (001\ 110), \\ \xi_4 = (010\ 011), & \xi_5 = (010\ 101), & \xi_6 = (010\ 110), \\ \xi_7 = (100\ 011), & \xi_8 = (100\ 101), \text{ and} & \xi_9 = (100\ 110). \end{array}$$

https://doi.org/10.4153/CJM-1972-114-7 Published online by Cambridge University Press

LEMMA 4.6. *For each i, $1 \leqq i \leqq 9$, not both ξ_i and σ_i can be initial segments of rows in the matrix M.*

Let η_r be a row which has σ_i as its initial segment, and let η_s be a row which has ξ_i as its initial segment. Then, essentially, $\eta_r = (110\ 100\ 100\ 110\ 000 \ldots 000)$, and $\eta_s^{(6)} = (001\ 011)$. Because $\eta_2 \circ \eta_r = 3$, it follows that $\eta_r \circ \eta_s \leqq 2$, a contradiction.

It is now possible to show that such a matrix M cannot exist. From Lemma 4.5, $\eta_j^{(3)} \neq (111)$ for $j \geqq 3$. Clearly, there can be at most one row η_r in M such that $\eta_r^{(3)} = (000)$; and so, every remaining row η_i must have $\eta_1^{(3)} \circ \eta_i^{(3)} = 1$ or 2. There cannot be two rows η_a, η_b in M which have the same σ_s as their initial segment; for otherwise, rows η_1, η_a, and η_b are three rows having $+1$'s in the same three columns, which contradicts Lemma 4.5. Similarly, there cannot be two rows in M having the same ξ_u as their initial segment. Now, because of Lemma 4.6, there can be at most 12 rows in M. This concludes the proof of

THEOREM 4.1. *The combinatorial designs associated with $H(4q + 3, 12q + 10)$ matrices (as given in Theorem 3.4) exist for $q = 0$ and $q = 1$, and they do not exist for $q = 2$.*

REFERENCES

1. M. Hall, Jr., *Combinatorial theory* (Blaisdell, Waltham, Mass., 1967).
2. O. Marrero and A. T. Butson, *Modular Hadamard matrices and related designs* (to appear in J. Combinatorial Theory).
3. H. J. Ryser, *Combinatorial mathematics* (Wiley, New York, 1963).
4. D. A. Sprott, *Some series of partially balanced incomplete block designs*, Can. J. Math. *7* (1955), 369–381.

Francis Marion College,
Florence, South Carolina;
University of Miami,
Coral Gables, Florida