J. Austral. Math. Soc. (Series A) 40 (1986), 234–252

ON VECTOR LATTICE-VALUED MEASURES II

T. V. PANCHAPAGESAN and SHIVAPPA VEERAPPA PALLED

(Received 15 July 1982; revised 10 February 1984)

Communicated by N. Moran

Abstract

For a weakly (σ, ∞)-distributive vector lattice V, it is proved that a $V \cup \{\infty\}$-valued Baire measure μ_0 on a locally compact Hausdorff space T admits uniquely regular Borel and weakly Borel extensions on T if and only if μ_0 is 'strongly regular at ∞'. Consequently, for such a vector lattice V every V-valued Baire measure on a locally compact Hausdorff space T has unique regular Borel and weakly Borel extensions. Finally some characterisations of a weakly (σ, ∞)-distributive vector lattice are given in terms of the existence of regular Borel (weakly Borel) extensions of certain $V \cup \{\infty\}$-valued Baire measures on locally compact Hausdorff spaces.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 28 B 15.

Introduction

The proof of Lemma 2.1 in [9] is incorrect, as it is tacitly assumed at the end of page 280 of [9] that the sequence $\{U_K\}$ is increasing. But this need not happen, though $\{B_n\}$ is an increasing sequence. However, the lemma is true due to the results of Matthes [11]. The lemma has been proved also in Hrachovina [10] and in Riečan [12].

The purpose of the present paper is to provide a proof for the part (1) $\Rightarrow$ (2) of the said theorem of Wright. Following a method different from that of Wright, we prove mainly that when a vector lattice V is weakly (σ, ∞)-distributive, a $V \cup \{\infty\}$-valued Baire measure μ_0 on a locally compact Hausdorff space T admits uniquely regular Borel and weakly Borel extensions if and only if 'μ_0 is strongly regular at ∞'. Further, it is shown that when a Baire measure μ_0 on T is just V-valued, then μ_0 is always strongly regular at ∞. Thus the proof of the part

(1) $\Rightarrow$ (2) of Theorem 3.3. of [9] is obtained as a very particular case of the present work.

In this connection we would like to draw the attention of the reader to Theorem N of Wright [8]. There is a flaw in the proof of this theorem, as $\bigvee_{n=1}^{\infty}\sum_{1}^{n} mO_r$ on page 79 of [8] can be ∞, since $\{O_r\}$ need not be pairwise disjoint. But, the flaw can be rectified by replacing $\{O_n\}$ by $\{G_n\}$ where $G_n = O_n \setminus \bigcup_{r=1}^{n-1} O_r$.

Section 1 gives some basic definitions and results from the literature, which are needed in the sequel. In Section 2 we show that the $\hat{\mu}$-outer (inner) regularity of some sets in a locally compact Hausdorff space T implies the $\hat{\mu}$-outer (inner) regularity of some other sets in T, where $\hat{\mu}$ is a $V \cup \{\infty\}$-valued Baire or Borel measure on T and V is a weakly (σ, ∞)-distributive vector lattice. The notion of 'being regular at ∞' is introduced and it is shown that when $\hat{\mu}$ is regular at ∞, $\hat{\mu}$ is regular if and only if every set in $\hat{\mathscr{C}}$ is outer regular or every bounded set in $\hat{\mathscr{U}}$ is inner regular where $\hat{\mathscr{C}} = \mathscr{C}$ (or $\mathscr{C}_0$ respectively) is the collection of all compact sets (or compact G_δ's resp.) in T, $\hat{\mathscr{U}} = \mathscr{U}$ (or $\mathscr{U}_0$ resp.) is the collection of all open Borel (or Baire resp.) sets in T and $\hat{\mu}$ is a $V \cup \{\infty\}$-valued Borel (Baire) measure on T with V weakly (σ, ∞)-distributive. Consequently, every $V \cup \{\infty\}$-valued Baire measure μ_0 on T is regular if μ_0 is 'regular at ∞' and V is weakly (σ, ∞)-distributive.

Section 3 deals with the regular weakly Borel extension of $V \cup \{\infty\}$-valued Baire measures on a locally compact Hausdorff space T. When V is weakly (σ, ∞)-distributive, it is shown that every regular $V \cup \{\infty\}$-valued Borel measure on T admits a unique regular weakly Borel extension. A more general form of Lemma 2.1 of Wright [9] is obtained here as Lemma 3.5. For such vector lattices V, we show that a $V \cup \{\infty\}$-valued Baire measure μ_0 on T admits uniquely regular $\hat{V} \cup \{\infty\}$-valued Borel and weakly Borel extensions if and only if μ_0 is 'strongly regular at ∞'. Also it is proved that when μ_0 is V-valued, μ_0 is always strongly regular at ∞. The last theorem of this section gives a number of characterizations of a weakly (σ, ∞)-distributive vector lattice and generalizes Theorem 3.3 of [9] to the set up of locally compact Hausdorff spaces.

1. Preliminaries

Throughout this paper V will denote a boundedly σ-complete vector lattice with $\hat{V}$ its Dedekind completion; $V^+ = \{x \in V\colon x \geqslant 0\}$ and the supremum of any unbounded collection of elements in V^+ is taken to be infinity and the infinity is denoted by "∞". The partial ordering and addition operation of V to $V \cup \{\infty\}$ are extended in the obvious way.

DEFINITION 1.1. A $V \cup \{\infty\}$-*valued measure* is a map $\mu\colon \mathscr{R} \to V \cup \{\infty\}$ where $\mathscr{R}$ is a ring of subsets of a set T such that

(i) $\mu(E) \geqslant 0$ for every E in $\mathscr{R}$;

(ii) $\mu(\phi) = 0$;

(iii) $\mu(\bigcup_1^\infty E_n) = \bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \mu(E_i)$, where $\{E_i\}$ is a sequence of pairwise disjoint sets in $\mathscr{R}$, with $\bigcup_1^\infty E_i \in \mathscr{R}$.

We say $\mu(E) < \infty$ or $\mu(E)$ is finite if $\mu(E) \in V$.

For each positive element h in V, let

$$V[h] = \{b \in V\colon -rh \leqslant b \leqslant rh \text{ for some positive } r \in \mathbb{R}\}$$

where $\mathbb{R}$ denotes the real line.

THEOREM 1.2 (Stone, Krein, Kakutani, Yosida). *There exists a compact Hausdorff space S such that $V[h]$ is vector lattice isomorphic to $C(S)$, the algebra of all real valued continuous functions on S, where $h > 0$, $h \in V$, a vector lattice. Further, when V is boundedly complete (σ-complete) then so is $V[h]$, $V[h]$ is a Banach space in the order unit norm, the isomorphism is also isometric and $C(S)$ is a Stone algebra (σ-Stone algebra) in the sense that S is extremally disconnected (S is totally disconnected with the property that the closure of every countable union of clopen subsets of S is open).*

For details one may refer to Kadison [2] and Vulikh [4]. We shall use the terms Stone algebra and σ-Stone algebra in the above sense.

From the result of Wright [8, 9] one can define a weakly (σ, ∞)-distributive vector lattice as below.

DEFINITION 1.3. A σ-Stone algebra $C(S)$ is said to be *weakly (σ, ∞)-distributive* (*weakly σ-distributive*) if and only if each meagre subset (σ-meagre subset) of S is nowhere dense. Consequently, a boundedly σ-complete vector lattice V is said to be weakly (σ, ∞)-distributive if and only if for $h > 0$ in V, $V[h]$ is weakly (σ, ∞)-distributive.

PROPOSITION 1.4. *A boundedly σ-complete vector lattice V is weakly (σ, ∞)-distributive if and only if $\hat{V}$ is so.*

2. Regular $V \cup \{\infty\}$-valued measures on locally compact Hausdorff spaces

Throughout this paper T will denote a locally compact Hausdorff space; $\mathscr{B}_0$ and $\mathscr{B}$ will denote respectively the class of all Baire sets and all Borel sets in T, i.e. $\mathscr{B}_0$ is the σ-ring generated by all compact G_δ's in T and $\mathscr{B}$ is the σ-ring generated by all compact sets in T.

A $V \cup \{\infty\}$-valued measure μ on $\mathscr{B}_0$ ($\mathscr{B}$) is called a *Baire* (*Borel*) *measure* on T if $\mu(K) < \infty$ for each compact G_δ (compact) K in T. A subset A of T is said to be *bounded* (*σ-bounded*) if A is contained in some compact set (in the union of a sequence of compact sets) in T.

A V-valued measure μ on a ring $\mathscr{R}$ of sets is said to be bounded if there exists an $h \in V^+$ such that $\mu(E) \leqslant h$ for every $E \in \mathscr{R}$. Accordingly, a bounded V-valued measure μ on $\mathscr{B}_0$ ($\mathscr{B}$) is a bounded V-valued Baire (Borel) measure on T.

We use the following notations in the sequel: $\mathscr{C}$ and $\mathscr{C}_0$ denote respectively the class of all compact sets and all compact G_δ sets in T; $\mathscr{U}$ and $\mathscr{U}_0$ denote respectively the class of all open Borel sets and all open Baire sets in T.

In this section, following Halmos [1, Section 52], we use the symbols $\hat{\mathscr{B}}$, $\hat{\mathscr{C}}$, $\hat{\mathscr{U}}$, $\hat{\mu}$ to denote either $\mathscr{B}$, $\mathscr{C}$ $\mathscr{U}$, μ ($V \cup \{\infty\}$-valued Borel measure) or $\mathscr{B}_0$, $\mathscr{C}_0$, $\mathscr{U}_0$, μ_0 ($V \cup \{\infty\}$-valued Baire measure).

DEFINITION 2.1. A set E in $\hat{\mathscr{B}}$ is said to be *outer regular* (with respect to $\hat{\mu}$) if $\hat{\mu}(E) = \bigwedge_{\hat{V}}\{\hat{\mu}(U): E \subseteq U \in \hat{\mathscr{C}}\}$ and *inner regular* (with respect to $\hat{\mu}$) if $\hat{\mu}(E) = \bigvee_{\hat{V}}\{\hat{\mu}(C): E \supseteq C \in \hat{\mathscr{C}}\}$. We say that E is *regular* (with respect to $\hat{\mu}$) if E is both inner and outer regular (with respect to $\hat{\mu}$); $\hat{\mu}$ is said to be *regular* if every $E \in \hat{\mathscr{B}}$ is regular (with respect to $\hat{\mu}$).

Hereafter we shall assume that V is a weakly (σ, ∞)-distributive vector lattice and as in the numerical case (see Halmos [1, Section 52]) we show in this section that the outer (inner) regularity of some sets implies the outer (inner) regularity of some other sets.

Let $C(S)$ be a weakly (σ, ∞)-distributive Stone algebra. Let $\{f_\alpha^i\}_{\alpha \in A_i}$ be an increasing net in $C(S)$, bounded above with $\bigvee_{\alpha \in A_i} f_\alpha^i = f^i$; let $\{g_\beta^i\}_{\beta \in B_i}$ be a decreasing net in $C(S)$ bounded below with $\bigwedge_{\beta \in B_i} g_\beta^i = g^i$ ($i = 1, 2, \dots$). Since a countable union of meagre sets is meagre, by Lemma 1.1 of Wright [6], there exists a meagre set $M \subseteq S$, such that

$$f^i(s) = \sup_{\alpha \in A_i} f_\alpha^i(s), \qquad i = 1, 2 \dots,$$

$$g^i(s) = \inf_{\beta \in B_i} g_\beta^i(s), \qquad i = 1, 2 \dots,$$

for $s \in S \setminus M$. Since $C(S)$ is weakly (σ, ∞)-distributive, by Definition 1.3 M is nowhere dense and hence so is $\overline{M}$. So $S \setminus \overline{M}$ is dense and open in S. Let $s_0 \in S \setminus \overline{M}$. There exists a clopen neighbourhood K of s_0 contained in $S \setminus \overline{M}$. Let χ_K be the characteristic function of K. Then $\{f_\alpha^i \chi_K\}_{\alpha \in A_i}$ ($\{g_\beta^i \chi_K\}_{\beta \in B_i}$) ascends (descends) pointwise on the compact set S to $f^i(g^i)$ and hence by Dini's theorem the convergence is uniform in each case, for $i = 1, 2, \dots$. Let $\varepsilon > 0$ be given. Then for each i, there exist $\alpha_i \in A_i$, $\beta_i \in B_i$ such that

$$f^i\chi_K - \varepsilon/2^i < f_{\alpha_i}^i \chi_K, \; g^i\chi_K + \varepsilon/2^i > g_{\beta_i}^i \chi_K \quad \text{for } i = 1, 2, \dots.$$

NOTATION I. The situation and the arguments given above occur often in the sequel, sometimes with $i = 1, 2, \ldots, n$. Hence to avoid repetition we shall refer to the above situation together with the above arguments as 'Step A' and we shall use the notations M, K, χ_K, s_0, etc, in the same sense as in the above, without any explicit description.

LEMMA 2.2. *Let $\hat{\mu}$ be a $V \cup \{\infty\}$-valued Baire (Borel) measure on T where V is weakly (σ, ∞)-distributive.*

(i) *If every set in $\hat{\mathscr{C}}$ is outer regular so is every proper difference of two sets in $\hat{\mathscr{C}}$.*

(ii) *If every bounded set in $\hat{\mathscr{U}}$ is inner regular, then so is every proper difference of two sets in $\hat{\mathscr{C}}$.*

PROOF. Let C and D be two sets in $\hat{\mathscr{C}}$ such that $C \supseteq D$.

(i) Since C is outer regular by hypothesis

$$\hat{\mu}(C) = \bigwedge_{\hat{V}} \{\hat{\mu}(U) \colon C \subseteq U \in \hat{\mathscr{U}}\}. \tag{1}$$

Again by hypothesis on $\hat{\mu}$, $\hat{\mu}(C) < \infty$. Hence there exists a $U_0 \in \hat{\mathscr{U}}$ with $C \subseteq U_0$ and $\hat{\mu}(U_0) < \infty$. From (1) and the monotoneity of $\hat{\mu}$ it follows that

$$\hat{\mu}(C) = \bigwedge_{\hat{V}} \{\hat{\mu}(U \cap U_0) \colon C \subseteq U \in \hat{\mathscr{U}}\}.$$

Let $\hat{\mu}(U_0) = h \in V$. Since V is weakly (σ, ∞)-distributive, $\hat{V}$ and $\hat{V}[h]$ are weakly (σ, ∞)-distributive by Proposition 1.4. Hence by Theorem 1.2, $\hat{V}[h] \cong C(S)$, a weakly (σ, ∞)-distributive Stone algebra. Let us identify $\hat{V}[h]$ with $C(S)$. Now, given $\varepsilon > 0$, by Step A, there exists a $U_1 \in \hat{\mathscr{U}}$ such that $C \subseteq U_1$ and

$$\hat{\mu}(C)\chi_K + \varepsilon > \hat{\mu}(U_1 \cap U_0)\chi_K. \tag{2}$$

Since $C \setminus D \subseteq (U_1 \cap U_0) \setminus D \in \hat{\mathscr{U}}$, by (2) we have

$$\{\hat{\mu}((U_1 \cap U_0) \setminus D) - \hat{\mu}(C \setminus D)\}\chi_K = \hat{\mu}\{(U_1 \cap U_0) \setminus C\}\chi_K < \varepsilon. \tag{3}$$

Specialising (3) at s_0, we obtain

$$\begin{aligned}\hat{\mu}((U_1 \cap U_0) \setminus D)(s_0) - \varepsilon &\leqslant \hat{\mu}(C \setminus D)(s_0) \\ &\leqslant \inf\{\hat{\mu}(U)(s_0) \colon C \setminus D \subseteq U \in \hat{\mathscr{U}}, U \subseteq U_0\}.\end{aligned}$$

Thus

$$\hat{\mu}(C \setminus D)(s_0) = \inf\{\hat{\mu}(U)(s_0) \colon C \setminus D \subseteq U \in \hat{\mathscr{U}}, U \in U_0\}.$$

Since s_0 is arbitrary in the dense set $S \setminus \overline{M}$,

$$\hat{\mu}(C \setminus D)(s) = \inf\{\hat{\mu}(U)(s) \colon C \setminus D \subseteq U \in \hat{\mathscr{U}}, U \subseteq U_0\} \tag{4}$$

https://doi.org/10.1017/S144678870002721X Published online by Cambridge University Press

for $s \in S \setminus \overline{M}$. But by Lemma 1.1 of Wright [6]

(5) $\left(\bigwedge\{\hat{\mu}(U): C \setminus D \subseteq U \in \hat{\mathscr{U}}, U \subseteq U_0\}\right)(s)$
$$= \inf\{\hat{\mu}(U)(s): C \setminus D \subseteq U \in \hat{\mathscr{U}}, U \subseteq U_0\}$$

for all $s \in S \setminus M_1$ (say), where M_1 is meagre. $M_2 = M \cup M_1$ is a meagre set and $\overline{M}_2$ is therefore nowhere dense in S as $C(S)$ is weakly (σ, ∞)-distributive. Hence by (4) and (5)

$$\hat{\mu}(C \setminus D)(s) = \left(\bigwedge\{\hat{\mu}(U): C \setminus D \subseteq U \in \hat{\mathscr{U}}, U \subseteq U_0\}\right)(s)$$

for all s in the dense set $S \setminus \overline{M}_2$. Hence as elemenet in $C(S)$,

$$\begin{aligned} \hat{\mu}(C \setminus D) &= \bigwedge_{C(S) \cong \hat{V}[h]} \{\hat{\mu}(U): C \setminus D \subseteq U \in \hat{\mathscr{U}}, U \subseteq U_0\} \\ &= \bigwedge_{\hat{V}}\{\hat{\mu}(U): C \setminus D \subseteq U \in \hat{\mathscr{U}}, U \subseteq U_0\} \\ &\geqslant \bigwedge_{\hat{V}}\{\hat{\mu}(U): C \setminus D \subseteq U \in \hat{\mathscr{U}}\} \\ &\geqslant \hat{\mu}(C \setminus D). \end{aligned}$$

Thus

$$\hat{\mu}(C \setminus D) = \bigwedge_{\hat{V}}\{\hat{\mu}(U): C \setminus D \subseteq U \in \hat{\mathscr{U}}\},$$

i.e. $C \setminus D$ is outer regular.

(ii) Let U be a bounded set in $\hat{\mathscr{U}}$ such that $C \subseteq U$. Then $\hat{\mu}(U) \in V$, from Theorem D, Section 50 of Halmos [1] and the hypothesis on $\hat{\mu}$.

By hypothesis the bounded set $U \setminus D$ (in $\hat{\mathscr{U}}$) is inner regular. Let $\hat{\mu}(U) = h$. Then $\hat{V}[h] \cong C(S)$, a weakly (σ, ∞)-distributive Stone algebra. Hereafter in the proof, let us identify $\hat{V}[h]$ with $C(S)$.

As $U \setminus D$ is inner regular, we have

$$\hat{\mu}(U \setminus D) = \bigvee_{C(S)}\{\hat{\mu}(F): U \setminus D \supseteq F \in \hat{\mathscr{C}}\}.$$

Now, given $\varepsilon > 0$, by Step A, there exists an $F_1 \in \hat{\mathscr{C}}$ such that $U \setminus D \supseteq F_1$ and

(6) $$\hat{\mu}(U \setminus D)\chi_K - \varepsilon < \hat{\mu}(F_1)\chi_K.$$

Since $C \setminus D = C \cap (U \setminus D) \supseteq C \cap F_1 \in \hat{\mathscr{C}}$, by (6) we have

(7) $\{\hat{\mu}(C \setminus D) - \hat{\mu}(C \cap F_1)\}\chi_K$
$$= \{\hat{\mu}((C \setminus D) \setminus F_1)\}\chi_K \leqslant \{\hat{\mu}((U \setminus D) \setminus F_1)\}\chi_K < \varepsilon.$$

Specialising at s_0, we obtain

$$\hat{\mu}(C \cap F_1)(s_0) + \varepsilon \geqslant \hat{\mu}(C \setminus D)(s_0) \geqslant \sup\{\hat{\mu}(F)(s_0): C \setminus D \supseteq F \in \hat{\mathscr{C}}\}$$

so that

$$\hat{\mu}(C \setminus D)(s_0) = \sup\{\hat{\mu}(F)(s_0): C \setminus D \supseteq F \in \hat{\mathscr{C}}\}.$$

Since s_0 is arbitrary in $S \setminus \overline{M}$, we obtain that

$$\hat{\mu}(C \setminus D)(s) = \sup\{\hat{\mu}(F)(s) : C \setminus D \supseteq F \in \hat{\mathscr{C}}\}$$

for all $s \in S \setminus \overline{M}$. By a dual argument of that following equation (5) in the proof of (i) of the lemma, we obtain that

$$\begin{aligned}\hat{\mu}(C \setminus D) &= \bigvee_{C(s) \cong V[h]} \{\hat{\mu}(F) : (C \setminus D) \supseteq F \in \hat{\mathscr{C}}\} \\ &= \bigvee_{\hat{V}} \{\hat{\mu}(F) : (C \setminus D) \supseteq F \in \hat{\mathscr{C}}\},\end{aligned}$$

i.e. $C \setminus D$ is inner regular.

NOTATION II. We shall denote by 'Step B' ('Step C') the arguments in the proof of (i) ((ii)) of the above lemma subsequent to inequality (3) ((7))

LEMMA 2.3. *Let $\hat{\mu}$ be a $V \cup \{\infty\}$-valued Baire (Borel) measure on T where V is a weakly (σ, ∞)-distributive vector lattice. A finite union of inner regular sets of finite measure is inner regular.*

PROOF. Let $\{E_1, E_2, \ldots, E_n\}$ be a finite disjoint family of inner regular sets of finite measure in $\hat{\mathscr{B}}$. Let $\hat{\mu}(E_i) < h \in V$ for $i = 1, 2, \ldots, n$. Since V is weakly (σ, ∞)-distributive, $\hat{V}[h] \cong C(S)$, a weakly (σ, ∞)-distributive Stone algebra. Hereafter let us identify $\hat{V}[h]$ with $C(S)$. Since E_i is inner regular of finite measure,

$$\hat{\mu}(E_i) = \bigvee_{C(S)} \{\hat{\mu}(C) : E_i \supseteq C \in \hat{\mathscr{C}}\}$$

for $i = 1, 2, \ldots, n$. Now, given $\varepsilon > 0$, by Step A, there exist $C_i \in \hat{\mathscr{C}}$ such that $C_i \subseteq E_i$ and

$$\hat{\mu}(E_i)\chi_K - \varepsilon/n < \hat{\mu}(C_i)\chi_K \quad \text{for } i = 1, 2, \ldots, n. \tag{8}$$

Clearly we have that

$$\hat{\mu}\left(\bigcup_{i=1}^{n} E_i\right) \leqslant \hat{\mu}\left(\bigcup_{i=1}^{n} E_i \setminus \bigcup_{j=1}^{n} C_j\right) + \hat{\mu}\left(\bigcup_{j=1}^{n} C_j\right) \leqslant \sum_{i=1}^{n} \hat{\mu}(E_i \setminus C_i) + \hat{\mu}\left(\bigcup_{i=1}^{n} C_i\right),$$

and hence by (8)

$$\hat{\mu}\left(\bigcup_{i=1}^{n} E_i\right)(s_0) \leqslant \varepsilon + \hat{\mu}\left(\bigcup_{i=1}^{n} C_i\right)(s_0).$$

Thus

$$\hat{\mu}\left(\bigcup_1^n E_i\right)(s) = \sup\left\{\hat{\mu}(C)(s) : \bigcup_1^n E_i \supseteq C \in \hat{\mathscr{C}}\right\}$$

for all $s \in S \setminus \overline{M}$. Now by Step C of Notation II, $\bigcup_1^n E_i$ is inner regular.

LEMMA 2.4. *Let $\hat{\mu}$ be a $V \cup \{\infty\}$-valued Borel (Baire) measure on T, where V is weakly (σ, ∞)-distributive. Then the intersection of a decreasing sequence of $\hat{\mu}$-inner regular sets of finite measure in $\hat{\mathscr{B}}$ is $\hat{\mu}$-inner regular.*

PROOF. Let $\{E_i\}_1^\infty$ be a decreasing sequence of inner regular sets in $\hat{\mathscr{B}}$ of finite measure. Let $\hat{\mu}(E_1) = h$. Then $\hat{V}[h] \cong C(S)$ is weakly (σ, ∞)-distributive. Hereafter let us identify $\hat{V}[h]$ with $C(S)$. Since $\hat{\mu}(E_i) \leqslant \hat{\mu}(E_1)$, $\hat{\mu}(E_i) \in C(S)$ for $i = 1, 2, \ldots$. Since E_i is inner regular,

$$\hat{\mu}(E_i) = \bigvee_{C(S)} \{\hat{\mu}(C) \colon E_i \supseteq C \in \hat{\mathscr{C}}\}$$

for all $i = 1, 2, \ldots$. Now given $\varepsilon > 0$, by Step A, there exist $C_i \in \hat{\mathscr{C}}$ such that $C_i \subseteq E_i$ and

$$\hat{\mu}(E_i)\chi_K - \varepsilon/2^i < \hat{\mu}(C_i)\chi_K \tag{9}$$

for $i = 1, 2, \ldots$. If $C = \bigcap_{i=1}^\infty C_i$ and $E = \bigcap_{i=1}^\infty E_i$, then

$$\{\hat{\mu}(E) - \hat{\mu}(C)\}\chi_K = \hat{\mu}(E \setminus C)\chi_K \leqslant \hat{\mu}\left(\bigcup_{i=1}^\infty (E_i \setminus C_i)\right)\chi_K. \tag{10}$$

By (9) and the σ-subadditivity it follows that

$$\hat{\mu}\left(\bigcup_{i=1}^\infty (E_i \setminus C_i)\right)\chi_K \leqslant \sum_{i=1}^\infty \hat{\mu}(E_i \setminus C_i)\chi_K < \varepsilon. \tag{11}$$

Specialising (11) at s_0 and then varying s_0 in the dense set $S \setminus \overline{M}$ we have

$$\hat{\mu}(E)(s) = \sup\{\hat{\mu}(F)(s) \colon E \supseteq F \in \hat{\mathscr{C}}\}$$

for $s \in S \setminus \overline{M}$. By Step C, this implies that E is $\hat{\mu}$-inner regular.

DEFINITION 2.5. Let $\hat{\mu}$ be a $V \cup \{\infty\}$-valued measure on $\hat{\mathscr{B}}$ which is Borel or Baire according to $\hat{\mathscr{B}} = \mathscr{B}$ or $\hat{\mathscr{B}} = \mathscr{B}_0$. Then μ is said to be *regular at* ∞ if for each $E \in \hat{\mathscr{B}}$ with $\hat{\mu}(E) < \infty$ (i.e. $\hat{\mu}(E) \in V$) there exists a $U \in \hat{\mathscr{U}}$ such that $E \subseteq U$ and $\hat{\mu}(U) < \infty$.

When $\hat{\mu}$ is V-valued, trivially $\hat{\mu}$ is regular at ∞.

LEMMA 2.6. *Let $\hat{\mu}$ be a $V \cup \{\infty\}$-valued Baire (Borel) measure on T which is regular at ∞. If V is weakly (σ, ∞)-distributive then the union of a sequence of $\hat{\mu}$-outer regular sets in $\hat{\mathscr{B}}$ is $\hat{\mu}$-outer regular.*

PROOF. Let $\{E_i\}$ be a sequence of $\hat{\mu}$-outer regular sets in $\hat{\mathscr{B}}$, with $E = \bigcup_{i=1}^\infty E_i$. If $\hat{\mu}(E) = \infty$, clearly E is $\hat{\mu}$-outer regular. Hence let $\hat{\mu}(E) < \infty$. Then by hypothesis, there exists a U_0 in $\hat{\mathscr{U}}$ such that $E \subseteq U_0$ and $\hat{\mu}(U_0) < \infty$. Let $\hat{\mu}(U_0) = h$. Then $\hat{V}[h] \cong C(S)$ is a weakly (σ, ∞)-distributive Stone algebra and hereafter we shall identify $\hat{V}[h]$ with $C(S)$.

By monotoneity of $\hat{\mu}$ and by the $\hat{\mu}$-outer regularity of E_i, we have

$$(12) \qquad \hat{\mu}(E_i) = \bigwedge_{C(S)} \{\hat{\mu}(U_0 \cap U) : E_i \subseteq U \in \hat{\mathscr{U}}\}$$

$i = 1, 2, \ldots$. Then by Step A, given $\varepsilon > 0$, there exist $U_i \in \hat{\mathscr{U}}$ with $E_i \subseteq U_i$ such that

$$(13) \qquad \hat{\mu}(E_i)\chi_K + \varepsilon/2^i > \hat{\mu}(U_0 \cap U_i)\chi_K.$$

Let $O_i = U_i \cap U_0$ and $O = \bigcup_{i=1}^{\infty} O_i$. Now $\bigcup_{i=1}^{\infty}(O_i \setminus E_i) \subseteq U_0$ and

$$(14) \qquad (\hat{\mu}(O) - \hat{\mu}(E))\chi_K = \hat{\mu}(O \setminus E)\chi_K \leqslant \hat{\mu}\left(\bigcup_{i=1}^{\infty}(O_i \setminus E_i)\right)\chi_K.$$

By (13) and the σ-additivity we have that

$$(15) \qquad \hat{\mu}\left(\bigcup_{i=1}^{\infty}(O_i \setminus E_i)\right)\chi_K \leqslant \sum_{i=1}^{\infty} \hat{\mu}(O_i \setminus E_i)\chi_K < \varepsilon.$$

By the usual argument, (15) and Step B give

$$(16) \qquad \hat{\mu}(E) = \bigwedge_{\hat{V}} \{\hat{\mu}(U) : E \subseteq U \in \hat{\mathscr{U}}, U \subseteq U_0\}.$$

Now from monotoneity of $\hat{\mu}$ it follows that

$$\hat{\mu}(E) = \bigwedge_{\hat{V}} \{\hat{\mu}(U) : E \subseteq U \in \hat{\mathscr{U}}\}$$

so E is $\hat{\mu}$-outer regular.

LEMMA 2.7. *Let $\hat{\mu}$ be a $V \cup \{\infty\}$-valued Baire (Borel) measure in T where V is just a boundedly σ-complete vector lattice. Then*

(i) *the union of an increasing sequence of inner regular sets in $\hat{\mathscr{B}}$ is inner regular; and*

(ii) *the intersection of a decreasing sequence of outer regular sets of finite measure in $\hat{\mathscr{B}}$ is outer regular.*

PROOF. The proof is similar to the proofs of Lemmas 2.5 and 2.6.

LEMMA 2.8. *Let $\hat{\mu}$ be a $V \cup \{\infty\}$-valued Baire (Borel) measure on T where V is a weakly (σ, ∞)-distributive vector lattice. Then a necessary and sufficient condition that every set in $\hat{\mathscr{C}}$ be $\hat{\mu}$-outer regular is that every bounded set in $\hat{\mathscr{U}}$ be $\hat{\mu}$-inner regular.*

PROOF. Suppose that every set in $\hat{\mathscr{C}}$ is outer regular and let U be a bounded set in $\hat{\mathscr{U}}$. Let C be a set in $\hat{\mathscr{C}}$ such that $U \subseteq C$. By Theorem D, Section 51 of Halmos [1], $C \setminus U \in \hat{\mathscr{C}}$. By the finiteness of $\hat{\mu}(C)$ and by the outer regularity of C, there exists a U_0 in $\hat{\mathscr{U}}$ such that $C \subseteq U_0$ and $\hat{\mu}(U_0) < \infty$. Let $\hat{\mu}(U_0) = h \in V$. Then by monotoneity of $\hat{\mu}$ and by the outer regularity of $C \setminus U$ we have

$$\hat{\mu}(C \setminus U) = \bigwedge_{\hat{V}[h]} \{\hat{\mu}(W \cap U_0) : C \setminus U \subseteq W \in \hat{\mathscr{U}}\}. \tag{17}$$

By hypothesis, $\hat{V}[h]$ is a weakly (σ, ∞)-distributive Stone algebra. Applying Step A to (17) and modifying the argument in Theorem E, Section 52 of Halmos [1] suitably and finally making use of Step C we can easily show that U is inner regular.

Now, conversely suppose that every bounded set in $\hat{\mathscr{U}}$ is inner regular. Let C be in $\hat{\mathscr{C}}$. Let U be a bounded set in $\hat{\mathscr{U}}$ such that $C \subseteq U$. Since $U \setminus C$ is a bounded set in $\hat{\mathscr{U}}$, by hypothesis

$$\hat{\mu}(U \setminus C) = \bigvee_{\hat{V}} \{\hat{\mu}(F) : U \setminus C \supseteq F \in \hat{\mathscr{C}}\}.$$

Since U is a bounded set in $\hat{\mathscr{U}}$, by Theorem D, Section 50 of Halmos [1], $U \subseteq G$ for some $G \in \hat{\mathscr{C}}$, so that $\hat{\mu}(G)$ and hence $\hat{\mu}(U)$ are finite. Let $\hat{\mu}(U) = h$ in V. Then

$$\hat{\mu}(U \setminus C) = \bigvee_{\hat{V}[h]} \{\hat{\mu}(F) : U \setminus C \supseteq F \in \hat{\mathscr{C}}\}. \tag{18}$$

Applying Step A to (18) and modifying the argument in Theorem E of [1] suitably and finally making use of Step B, we can easily prove that C is outer regular.

Now we give the main theorem of this section.

THEOREM 2.9. *Let V be a weakly (σ, ∞)-distributive vector lattice and let $\hat{\mu}$ be a $V \cup \{\infty\}$-valued Baire (Borel) measure on T. If $\hat{\mu}$ is further regular at ∞ then the outer regularity of every set in $\hat{\mathscr{C}}$ or the inner regularity of every bounded set in $\hat{\mathscr{U}}$ is a necessary and sufficient condition for the regularity of $\hat{\mu}$. Thus if $\hat{\mu}$ is V-valued, the conclusion holds.*

PROOF. Due to the availability of Lemmas 2.2, 2.3, 2.4, 2.6, 2.7 and 2.8 in place of Theorems A, B, C, D and E of Section 52 of Halmos [1], the Theorem follows from an argument very similar to that in the proof of Theorem F of Section 52, Halmos [2].

COROLLARY 2.10. *Let V be a weakly (σ, ∞)-distributive vector lattice. Then every $V \cup \{\infty\}$-valued Baire measure on a locally compact Hausdorff space T is regular if and only if it is regular at ∞. Consequently, every V-valued Baire measure on T is regular.*

3. Regular weakly Borel and Borel extensions of $V \cup \{\infty\}$-valued Baire measures

$\mathscr{B}_W$ will denote the σ-algebra generated by all closed sets of the locally copact Hausdorff space T and the members of $\mathscr{B}_W$ are called weakly Borel sets. A $V \cup \{\infty\}$-valued measure μ on $\mathscr{B}_W$ with $\mu(C) < \infty$ for compact sets C in T will be called a weakly Borel measure. This section is devoted to the study of the regular weakly Borel extension of regular $V \cup \{\infty\}$-valued Borel and Baire measure on T. As discussed in the outset, the arguments in the proof of Lemma 2.1 of Wright [9], at the end of page 280 [9] are faulty, since the sequence $\{U_k\}$ there need not be increasing though $\{B_n\}$ there is an increasing sequence. However, we obtain in this section a stronger result generalising the said lemma of Wright [9] to locally compact Hausdorff spaces, and use it to give some characterizations of weakly (σ, ∞)-distributive vector lattices.

Definition 3.1. A weakly Borel measure μ: $\mathscr{B}_w \to V \cup \{\infty\}$ is said to be *regular* if the following conditions hold:

(i) for each $E \in \mathscr{B}_w$

$$\mu(E) = \bigvee \{\mu(K) \colon K \subseteq E \text{ and } K \text{ is compact}\};$$

(ii) for each E in $\mathscr{B}_w$

$$\mu(E) = \bigvee \{\mu(E \cap U) \colon U \text{ is open and } \mu(U) < \infty\}.$$

We say that the weakly Borel measure μ is *quasi-regular* if condition (ii) holds for each E in $\mathscr{B}_W$ and condition (i) holds for all open sets E.

Let $\tau = \{E \colon E \subseteq T,\ E \cap A \in \mathscr{B} \text{ for every } A \in \mathscr{B}\}$. Then τ is a σ-algebra containing $\mathscr{B}$.

Definition 3.2. Let μ be a $V \cup \{\infty\}$-valued Borel measure on T. Then the $\hat{V} \cup \{\infty\}$-valued set function $\hat{\mu}$ on τ is defined by

$$\hat{\mu}(E) = \bigvee_{\hat{V}} \{\mu(A) \colon A \in \mathscr{B},\ A \subseteq E\}$$

for every E in τ.

Lemma 3.3. *$\hat{\mu}$ of Definition 3.2 is a $\hat{V} \cup \{\infty\}$-valued measure on τ and is an extension of the $V \cup \{\infty\}$-valued Borel measure μ, when V is a weakly (σ, ∞)-distributive vector lattice.*

Proof. Clearly $\hat{\mu}$ is an extension of μ. Further, $\hat{\mu}$ is monotone and is also given by

$$\hat{\mu}(E) = \bigvee_{\hat{V}} \{\mu(E \cap A) \colon A \in \mathscr{B}\}, \quad \text{for } E \in \tau.$$

https://doi.org/10.1017/S144678870002721X Published online by Cambridge University Press

Clearly

$$\hat{\mu}(E) \geqslant 0 \quad \text{for every } E \text{ in } \tau.$$

(a) We claim that if $\{F_i\}$ is an increasing sequence of members of τ with $\bigcup_{i=1}^{\infty} F_i = F$, then $\hat{\mu}(F) = \bigvee_1^{\infty} \hat{\mu}(F_i)$. For, by monotoneity of $\hat{\mu}$, it suffices to show that $\hat{\mu}(F) \leqslant \bigvee_1^{\infty} \hat{\mu}(F_i)$. If $A \in \mathscr{B}$, as μ is a measure on $\mathscr{B}$, we have

$$\mu(F \cap A) = \bigvee_{i=1}^{\infty} \mu(F_i \cap A) \leqslant \bigvee_1^{\infty} \hat{\mu}(F_i)$$

so that

$$\hat{\mu}(F) = \bigvee\{\mu(F \cap A) \colon A \in \mathscr{B}\} \leqslant \bigvee_1^{\infty} \hat{\mu}(F_i).$$

(b) We claim that $\hat{\mu}$ is countably subadditive on τ. For, if $\{E_i\}_1^{\infty}$ is a sequence of members of τ, then by the countable subadditivity of μ on $\mathscr{B}$, we have

$$\begin{aligned}\sum_{i=1}^{n} \hat{\mu}(E_i) &\geqslant \bigvee\left\{\sum_{i=1}^{n} \mu(E_i \cap A) \colon A \in \mathscr{B}\right\}\\ &\geqslant \bigvee\left\{\mu\left(\left(\bigcup_1^n E_i\right) \cap A\right) \colon A \in \mathscr{B}\right\} = \hat{\mu}\left(\bigcup_1^n E_i\right),\end{aligned}$$

so that, by (a),

$$\bigvee_{n=1}^{\infty} \sum_{i=1}^{n} \hat{\mu}(E_i) \geqslant \bigvee_{n=1}^{\infty} \hat{\mu}\left(\bigcup_1^n E_i\right) = \hat{\mu}\left(\bigcup_1^{\infty} E_i\right).$$

(c) We claim that $\hat{\mu}$ is finitely additive on τ. For this we use the hypothesis that V is weakly (σ, ∞)-distributive. Let E_1, E_2 be in τ, with $E_1 \cap E_2 = \varnothing$. If one of $\hat{\mu}(E_i)$, $i = 1, 2$, is infinite, then by monotoneity of $\hat{\mu}$, $\hat{\mu}(E_1 \cup E_2) = \hat{\mu}(E_1) + \hat{\mu}(E_2)$. Hence let $\hat{\mu}(E_i) < \infty$, $i = 1, 2$. Let $h = \hat{\mu}(E_1) + \hat{\mu}(E_2)$. Then $\hat{V}[h] \cong C(S)$ is a weakly (σ, ∞)-distributive Stone algebra and

$$\hat{\mu}(E_i) = \bigvee_{C(S)} \{\mu(E_i \cap A) \colon A \in \mathscr{B}\}$$

for $i = 1, 2$. Given $\varepsilon > 0$, by Step A of Section 2, there exist A_1, A_2 in $\mathscr{B}$ such that

$$\hat{\mu}(E_1)\chi_K - \varepsilon/2 < \mu(E_1 \cap A_1)\chi_K$$

and

$$\hat{\mu}(E_2)\chi_K - \varepsilon/2 < \mu(E_2 \cap A_2)\chi_K.$$

Specialising at s_0, we have

$$\begin{aligned}(\hat{\mu}(E_1) + \hat{\mu}(E_2))(s_0) - \varepsilon \leqslant (\mu(E_1 \cap A_1) + \mu(E_2 \cap A_2))(s_0) \\ = \mu((E_1 \cap A_1) \cup (E_2 \cap A_2))(s_0) \\ \leqslant \hat{\mu}(E_1 \cup E_2)(s_0).\end{aligned}$$

Note that by (b), $\hat{\mu}(E_1 \cup E_2) \leqslant \hat{\mu}(E_1) + \hat{\mu}(E_2)$ and hence $\hat{\mu}(E_1 \cup E_2) \in C(S)$. Since ε is arbitrary and since s_0 can vary over a dense subset of S we have

$$\hat{\mu}(E_1) + \hat{\mu}(E_2) \leqslant \hat{\mu}(E_1 \cup E_2).$$

This inequality and (b) imply (c).

The countable additivity of $\hat{\mu}$ on τ now follows from (a) and (c).

THEOREM 3.4. *If μ is a $V \cup \{\infty\}$-valued regular Borel measure on a locally compact Hausdorff space T and if V is weakly (σ, ∞)-distributive, then the set function $\bar{\mu}$ on $\mathscr{B}_W$ defined by*

$$\bar{\mu}(E) = \bigvee_{\hat{V}} \{\mu(A) \colon A \subseteq E,\ A \in \mathscr{B}\}$$

is a $\hat{V} \cup \{\infty\}$-valued regular weakly Borel measure on T, which extends μ. Further, $\bar{\mu}$ is the unique $\hat{V} \cup \{\infty\}$-valued regular weakly Borel extension of μ.

PROOF. Let $E \in \mathscr{B}_W$ and E be closed in T. Then $E \cap C \in \mathscr{C}$ for each $C \in \mathscr{C}$ and hence by Theorem E, Section 5 of Halmos [1], $A \cap E \in \mathscr{B}$ for each $A \in \mathscr{B}$. Thus each closed set E in T belongs to τ. Since τ is a σ-algebra and $\mathscr{B}_W$ is the σ-algebra generated by closed sets in T, $\mathscr{B}_W \subseteq \tau$. Clearly, $\bar{\mu} = \hat{\mu}|\mathscr{B}_W$ and consequently $\bar{\mu}$ is a $\hat{V} \cup \{\infty\}$-valued measure on $\mathscr{B}_W$, by Lemma 3.3, extending μ. For $C \in \mathscr{C}$, $\bar{\mu}(C) = \mu(C) < \infty$ and hence $\bar{\mu}$ is weakly Borel.

For $E \in \mathscr{B}_W$,

$$(19) \quad \bar{\mu}(E) = \bigvee \{\mu(A) \colon A \subseteq E,\ A \in \mathscr{B}\} \geqslant \bigvee \{\mu(C) \colon C \subseteq E, C \in \mathscr{C}\}.$$

As μ is regular, for each $A \in \mathscr{B}$, $A \subseteq E$,

$$\mu(A) = \bigvee \{\mu(C) \colon C \subseteq A, C \in \mathscr{B}\} \leqslant \bigvee \{\mu(C) \colon C \subseteq E, C \in \mathscr{C}\}$$

so that

$$(20) \qquad \bar{\mu}(E) \leqslant \bigvee \{\mu(C) \colon C \subseteq E, C \in \mathscr{C}\}.$$

Now by (19) and (20), for each $E \in \mathscr{B}_W$,

$$(21) \qquad \bar{\mu}(E) = \bigvee \{\mu(C) \colon C \subseteq E, C \in \mathscr{C}\}.$$

For $E \in \mathscr{B}_W$ and for $C \in \mathscr{C}$ with $C \subseteq E$, by Theorem D, Section 50 of Halmos [1], there exist sets U_0, C_0 such that U_0 is a σ-compact open set and C_0 is a compact G_δ with $C \subseteq U_0 \subseteq C_0 \subseteq T$. Thus $\mu(C) = \mu(C \cap E) \leqslant \mu(U_0 \cap E)$, U_0 is open Borel and $\mu(U_0) < \infty$. Thus for each C in $\mathscr{C}$ with $C \subseteq E$, there exists a corresponding open Borel set U_C such that $C \subseteq U_C, \mu(U_C) < \infty$ and $\mu(C) \leqslant \mu(U_C \cap E)$. Thus

$$\begin{aligned}\bar{\mu}(E) &= \bigvee\{\mu(C): C \subseteq E, C \in \mathscr{C}\} \quad \text{(from (21))}\\ &\leqslant \bigvee\{\mu(U_C \cap E): C \subseteq E, C \in \mathscr{C}\}\\ &\leqslant \bigvee\{\mu(U \cap E): U \text{ is an open Borel set}, \mu(U) < \infty\}\\ &\leqslant \bigvee\{\bar{\mu}(U \cap E): U \text{ is open}, \bar{\mu}(U) < \infty\}\\ &\leqslant \bar{\mu}(E).\end{aligned}$$

This and (21) prove that $\bar{\mu}$ is a regular weakly Borel measure on T.

The uniqueness of such a regular weakly Borel extension $\bar{\mu}$ of μ follows trivially from the relation (21).

This completes the proof of the theorem.

As pointed out at the beginning of this section, the proof of Lemma 2.1 of Wright [9] is defective. Now we obtain a generalization of his lemma for locally compact Hausdorff spaces.

LEMMA 3.5. *Let $\hat{\mu}$ be a $V \cup \{\infty\}$-valued quasi-regular weakly Borel measure on a locally compact Hausdorff space T, where V is a weakly (σ, ∞)-distributive vector lattice. If the restriction μ of $\hat{\mu}$ to $\mathscr{B}$ is regular at ∞, then μ and $\hat{\mu}$ are regular Borel and weakly Borel measures respectively.*

PROOF. The quasi-regularity of $\hat{\mu}$ implies that its Borel restriction μ is inner regular on $\mathscr{U}$ and hence by Theorem 2.9, μ is a regular Borel measure on T.

Now $\hat{\mu}$ can be considered as a quasi-regular weakly Borel extension of the Borel measure μ and hence by Lemma 3 of Wright [7], $\hat{\mu}$ is a unique quasi-regular weakly Borel extension of μ. But as μ is regular, by Theorem 3.4, μ has a unique regular weakly Borel extension and so this extension coincides with $\hat{\mu}$. Therefore $\hat{\mu}$ is a regular weakly Borel measure on T.

Now we turn to the study of regular Borel extensions of certain $V \cup \{\infty\}$-valued Baire measures on a locally compact Hausdorff space T. We know from the results of Wright [7] that when V is a boundedly σ-complete vector lattice, for each positive linear V-valued map Λ on $C_{00}(T)$, the algebra of all real valued continuous functions on T of compact support, there exists a unique quasi-regular $\hat{V} \cup \{\infty\}$-valued weakly Borel measure $\tilde{\mu}$ on T such that

$$\Lambda f = \int_T f \, d\tilde{\mu}, \qquad f \in C_{00}(T). \tag{22}$$

This representation in this form will hereafter be referred to as Wright's form of the Riesz representation theorem for Λ.

DEFINITION 3.6. Let V be a boundedly σ-complete vector lattice and Λ be a positive linear V-valued map on $C_{00}(T)$ with $\tilde{\mu}$ the representing quasi-regular weakly Borel measure in Wright's form of the Riesz representation theorem for Λ. We say that Λ is regular at ∞ if the Borel restriction μ of $\tilde{\mu}$ is regular at ∞ as a $\hat{V} \cup \{\infty\}$-valued Borel measure on T.

THEOREM 3.7 (The Riesz representation theorem). *Let V be a weakly (σ, ∞)-distributive vector lattice and T a locally compact Hausdorff space. Let Λ be a positive linear V-valued map on $C_{00}(T)$ and let Λ be regular at ∞. Then there exists a unique regular $\hat{V} \cup \{\infty\}$-valued Borel measure μ on T such that*

$$\Lambda f = \int_T f\,d\mu, \qquad f \in C_{00}(T).$$

Consequently, there exists a unique regular $\hat{V} \cup \{\infty\}$-valued weakly Borel measure $\tilde{\mu}$ on T such that $\tilde{\mu}$ is an extension of μ and that

$$\int_T f\,d\tilde{\mu} = \Lambda f, \qquad f \in C_{00}(T).$$

PROOF. By the hypothesis that Λ is regular at ∞ the Borel restriction μ of the representing quasi-regular weakly Borel measure $\tilde{\mu}$ in Wright's form of the Riesz representation theorem for Λ is regular at ∞.

Hence by Lemma 3.5, μ is a regular $\hat{V} \cup \{\infty\}$-valued Borel measure and $\tilde{\mu}$ is a regular $\hat{V} \cup \{\infty\}$-valued weakly Borel measure. Since μ is the restriction of $\tilde{\mu}$, equation (22) implies, in particular, that

$$\int_T f\,d\mu = \int_T f\,d\tilde{\mu} = \Lambda f, \qquad f \in C_{00}(T). \tag{22'}$$

Uniqueness of $\tilde{\mu}$ follows as a consequence of Theorem 1 of Wright [7].

If possible, let μ' be another regular $\hat{V} \cup \{\infty\}$-valued Borel measure on T such that

$$\Lambda f = \int_T f\,d\mu', \qquad f \in C_{00}(T).$$

Then by Theorem 3.4, μ' has a unique regular weakly Borel extension μ'' on T so that

$$\Lambda f = \int_T f\,d\mu' = \int_T f\,d\mu'', \qquad f \in C_{00}(T). \tag{23}$$

Equations (22′), (23) and the uniqueness of $\tilde{\mu}$ together imply that $\mu'' = \tilde{\mu}$. Hence $\mu' = \mu''|_{\mathscr{B}} = \tilde{\mu}|_{\mathscr{B}} = \mu$. Thus μ is unique.

This completes the proof of the theorem.

DEFINITION 3.8. Let V be a boundedly σ-complete vector lattice and μ_0 a $V \cup \{\infty\}$-valued Baire measure on T. Let

$$\Lambda f = \int_T f \, d\mu_0, \qquad f \in C_{00}(T).$$

Then Λ is a V-valued positive linear map on $C_{00}(T)$. We say that the Baire measure μ_0 is *strongly regular at* ∞ if Λ above is regular at ∞ in the sense of Definition 3.6.

The following theorem gives some sufficient conditions for a $V \cup \{\infty\}$-valued Baire measure on T to admit unique regular Borel and weakly Borel extensions.

THEOREM 3.9. *Let μ_0 be a $V \cup \{\infty\}$-valued Baire measure on a locally compact Hausdorff space T where V is a weakly (σ, ∞)-distributive vector lattice. Then there exist uniquely regular Borel and regular weakly Borel extensions of μ_0 if and only if μ_0 is strongly regular at ∞.*

PROOF. Let $\Lambda f = \int_T f \, d\mu_0$, $f \in C_{00}(T)$ and μ_0 be strongly regular at ∞. Then by the hypothesis, Λ is a V-valued positive linear map which is regular at ∞ and hence by Theorem 3.7 there exists a unique regular $\hat{V} \cup \{\infty\}$-valued Borel measure μ on T such that

$$\int_T f \, d\mu_0 = \Lambda f = \int_T f \, d\mu, \qquad f \in C_{00}(T), \tag{24}$$

and there exists a unique regular weakly Borel extension $\tilde{\mu}$ of μ on T.

Let $C_0 \in \mathscr{C}_0$. Then by Theorem A, Section 55 of Halmos [1] there exists a decreasing sequence (f_n) in $C_{00}(T)^+$ such that

$$\chi_{C_0}(x) = \lim_n f_n(x), \qquad x \in T.$$

Hence by an easy generalization of Proposition 3.5 of Wright [6] to $V \cup \{\infty\}$-valued measures,

$$\mu_0(C_0) = \int_T \chi_{C_0} \, d\mu_0 = \bigwedge_{n=1}^{\infty} \int_T f_n \, d\mu_0$$

and

$$\mu(C_0) = \int_T \chi_{C_0} \, d\mu_0 = \bigwedge_{n=1}^{\infty} \int_T f_n \, d\mu.$$

By these equations and equation (24), $\mu_0(C_0) = \mu(C_0)$, $C_0 \in \mathscr{C}_0$. Hence $\mu|_{\mathscr{B}_0} = \mu_0$. Since $\tilde{\mu}|_{\mathscr{B}} = \mu$, $\tilde{\mu}|_{\mathscr{B}_0} = \mu_0$. Thus μ and $\tilde{\mu}$ are respectively the unique regular Borel and regular weakly Borel extensions of μ_0.

The converse part of the theorem is obvious and this completes the proof of the theorem.

COROLLARY 3.10. *If μ_0 is a V-valued Baire measure on a locally compact Hausdorff space T where V is weakly (σ, ∞)-distributive, then μ_0 is strongly regular at ∞ and hence admits uniquely regular Borel and regular weakly Borel extensions.*

PROOF. By Theorem A of Section 51 in [1], every Borel set E in T is σ-bounded. Thus, if $E \subseteq \bigcup_{n=1}^{\infty} C_n$, by Theorem D, Section 50 of Halmos [1], there exist U_{0n} and C_{0n} such that

$$C_n \subseteq U_{0n} \subseteq C_{0n} \subseteq T$$

where U_{0n} is σ-compact open and C_{0n} is a compact G_δ. Thus $E \subseteq \bigcup_{n=1}^{\infty} C_{0n}$ and $\bigcup_{n=1}^{\infty} C_{0n} \in \mathscr{B}_0$. If μ is the quasi-regular weakly Borel measure on T representing the V-valued positive linear map Λ, defined by

$$\Lambda f = \int_T f \, d\mu_0, \qquad f \in C_{00}(T)$$

(see (22)), then from the proof of Theorem 3.9 it follows that $\mu|\mathscr{B}_0 = \mu_0$. Thus, μ is an extension of μ_0 and

$$\mu(E) \leqslant \mu\left(\bigcup_{n=1}^{\infty} C_{0n}\right) = \mu_0\left(\bigcup_{n=1}^{\infty} C_{0n}\right) \in V.$$

Thus $\mu|\mathscr{B}$ is $\hat{V}$-valued and hence $\mu|\mathscr{B}$ is regular at ∞. Therefore, μ_0 is strongly regular at ∞ and thus the corollary follows.

REMARK. By using the theory of V-valued contents and Carathéodory extensions of $V \cup \{\infty\}$-valued measures it will be shown in [3] that a $V \cup \{\infty\}$-valued Baire measure on a locally compact Hausdorff space T is regular at ∞ if and only if it is strongly regular at ∞, where V is weakly (σ, ∞)-distributive.

We conclude this section with the following theorem which characterizes weakly (σ, ∞)-distributive vector lattices.

THEOREM 3.11. *Let V be a boundedly σ-complete vector lattice with $\hat{V}$ its Dedekind completion. Then the following conditions are equivalent.*

(1) *V is weakly (σ, ∞)-distributive.*

(2) *Each $V \cup \{\infty\}$-valued Baire measure, which is strongly regular at ∞, on each locally compact Hausdorff space can be extended to a regular $\hat{V} \cup \{\infty\}$-valued Borel ((3) weakly Borel) measure.*

(4) *Each V-valued Baire measure on each locally compact Hausdorff space can be extended to a regular $\hat{V}$-valued Borel ((5) weakly Borel) measure.*

(6) *Each V-valued Baire measure on each compact Hausdorff space ((7) totally disconnected compact Hausdorff space) can be extended to a regular $\hat{V}$-valued Borel measure.*

(8) *The representing* $V \cup \{\infty\}$*-valued weakly Borel measure in* (22) *for a* V*-valued positive linear map* Λ *on* $C_{00}(T)$, *where* T *is a locally compact Hausdorff space and* Λ *is regular at* ∞, *is regular.*

(9) *The Borel restriction of the representing* $\hat{V} \cup \{\infty\}$*-valued weakly Borel measure of the* V*-valued positive linear map* Λ *on* $C_{00}(T)$, *where* T *is a locally compact Hausdorff space and* Λ *is regular at* ∞, *is regular.*

PROOF. By Theorem 3.9, (1) $\Rightarrow$ (2) and (3). By Corollary 3.10, (1) $\Rightarrow$ (4) and (5). Obviously (4) $\Rightarrow$ (6) $\Rightarrow$ (7) and (5) $\Rightarrow$ (6) $\Rightarrow$ (7). By the final part of Theorem 3.3 of Wright [9], (7) $\Rightarrow$ (1). By Corollary 3.10, (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5). Thus conditions (1) to (7) are equivalent.

By Theorem 3.7, (1) $\Rightarrow$ (8) and (9). But, if $\Lambda f = \int_T f\,d\mu_0$ where μ_0 is a V-valued Baire measure on a locally compact Hausdoff space T, then Λ is regular at ∞ by Corollary 3.10 and hence by (8) (by (9)) μ_0 has a regular weakly Borel (Borel) extension. Thus (8) $\Rightarrow$ (5) ((9) $\Rightarrow$ (4)) and hence (8) $\Leftrightarrow$ (1) ((9) $\Leftrightarrow$ (1)).

This completes the proof of the theorem.

Acknowledgements

The authors are thankful to the referee for calling their attention to the papers [10] and [12] and for suggesting shorter proofs of Lemmas 2.3, 2.4 and 2.6. The work has been supported by the C.D.C.H. projects CS-80-149, 150 of the Universidad de los Andes, Mérida, Venezuela.

References

[1] P. R. Halmos, *Measure theory* (D. Van Nostrand, New York, 1950).

[2] R. V. Kadison, 'A representation theory for commutative topological algebras', *Mem. Amer. Math. Soc.* **7** (1951).

[3] T. V. Panchapagesan and Shivappa Veerappa Palled, 'On vector lattice-valued measures-III', submitted.

[4] B. Z. Vulikh, *Introduction to the theory of partially ordered spaces* (Wolters-Noordhoff Scientific Publications Ltd., Groningen, 1967).

[5] J. D. M. Wright, 'Extensions of infinite vector lattice measures', *Quart. J. Math. Oxford* **23** (1972), 259–265.

[6] J. D. M. Wright, 'Stone algebra-valued measures and integrals', *Proc. London Math. Soc.* **19** (1969), 107–122.

[7] J. D. M. Wright, 'Vector lattice measures on locally compact spaces', *Math. Z.* **120** (1971), 193–203.

[8] J. D. M. Wright, 'The measure extension problem for vector lattices', Ann. Inst. Fourier (Grenoble) **21** (1971), 65–85.

https://doi.org/10.1017/S144678870002721X Published online by Cambridge University Press

[9] J. D. M. Wright, 'An algebraic characterization of vector lattices with the Borel regularity property', *J. London Math. Soc.* **7** (1973), 277–285.
[10] E. Hrachovina, 'About regular measures with values in ordered spaces', *Acta Math. Univ. Comenian.*, to appear.
[11] K. Matthes, 'Über eine Schar von Regularitätsbedingungen für Verbände', *Math. Nachr.* **22** (1960), 93–128.
[12] B. Riečan, 'On regular measures in ordered spaces', Proc. Fifth Prague Top. Symp. (1981), to appear.

Departamento de Matematica
Facultad de Ciencias
Universidad de Los Andes
Mérida
Venezuela

Department of Mathematics
State University College at Potsdam
Potsdam, New York 13676
U.S.A.