Canad. Math. Bull. Vol. **42** (3), 1999 pp. 307–320

On the Moduli Space of a Spherical Polygonal Linkage

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Abstract. We give a "wall-crossing" formula for computing the topology of the moduli space of a closed n-gon linkage on $\mathbb{S}^2$. We do this by determining the Morse theory of the function ρ_n on the moduli space of n-gon linkages which is given by the length of the last side—the length of the last side is allowed to vary, the first $(n-1)$ side-lengths are fixed. We obtain a Morse function on the $(n-2)$-torus with level sets moduli spaces of n-gon linkages. The critical points of ρ_n are the linkages which are contained in a great circle. We give a formula for the signature of the Hessian of ρ_n at such a linkage in terms of the number of back-tracks and the winding number. We use our formula to determine the moduli spaces of all regular pentagonal spherical linkages.

1 Introduction

Our goal in this paper is to give a "wall-crossing" formula for determining the topology of the moduli space of a closed n-gon linkage on $\mathbb{S}^2$. We will give definitions in Section 2. The definitions of the configuration space and the moduli space $M(\Lambda, X)$ of a general linkage Λ in a constant curvature space X are given in [KM3].

Let $r = (r_1, r_2, \ldots, r_n)$ be an n-tuple of real numbers satisfying $0 < r_i < \pi$. Let $N_{r'}$ be the moduli space of the free $(n-1)$-gon spherical linkage with side-lengths $r' := (r_1, \ldots, r_{n-1})$, so $N_{r'}$ is the quotient by SO(3) of the subspace $\tilde{N}_{r'} \subset (\mathbb{S}^2)^n$ defined by

$$\tilde{N}_{r'} = \{u = (u_1, \ldots, u_n) \in (\mathbb{S}^2)^n : d(u_i, u_{i+1}) = r_i, 1 \leq i \leq n-1\}.$$

Here d is the spherical distance. The points $u_1, u_2, \ldots, u_n$ are called the vertices of the linkage $T \in \tilde{N}_{r'}$. Clearly $N_{r'} \cong (\mathbb{S}^1)^{n-2}$. We will study the Morse theory of the function $\rho_n \colon N_{r'} \to \mathbb{R}$ given by

$$\rho_n(u) = d(u_1, u_n).$$

We will restrict to u's such that $0 < \rho_n(u) < \pi$ so that ρ_n is differentiable. Notice that

$$M_r := \rho_n^{-1}(r_n) \subset N_{r'}$$

is the *moduli space* of *closed polygonal linkages* in $\mathbb{S}^2$ with the side-lengths $(r_1, \ldots, r_n)$.

Received by the editors October 14, 1997.

The first author was partially supported by NSF grant DS-96-26633 at the University of Utah. The second author was partially supported by NSF grant DMS-95-04134 at the University of Maryland.

AMS subject classification: Primary: 14D20; secondary: 14P05.

Definition We define the closed n-gon linkage $P = P(T)$ associated to a free $(n-1)$-gon linkage T to be the linkage obtained by adding the length-minimizing geodesic segment[1] $(u_n, u_1) = e_n \subset \mathbb{S}^2$ joining u_n to u_1.

Thus r_n is the length of the new edge e_n. Hence, in terms of deformations of the closed n-gon P in $\mathbb{S}^2$, we can obtain $N_{r'}$ by fixing the lengths of the first $n-1$ sides and *letting the length of the last side vary*.

In order to state the Main Theorem we will need some definitions.

Definition A linkage in $\mathbb{S}^2$ is degenerate if it lies in a great circle γ of $\mathbb{S}^2$.

Suppose now that P is a degenerate closed n-gon linkage contained in a great circle γ. We orient γ and define $\epsilon_i \in \{\pm 1\}$ to be 1 if the orientation of the i-th edge of P agrees with that of γ and -1 otherwise. We say that the i-th edge of P is a *forward-track* if $\epsilon_i = 1$ and a *back-track* otherwise. We let $f = f(P)$ be the number of forward-tracks and $b = b(P)$ be the number of back-tracks so $f + b = n$. Define the winding number $w = w(P)$ by

$$\sum_{i=1}^{n} \epsilon_i r_i = 2\pi w.$$

The numbers b, f and w depend on the orientation of γ. We will deal with this below.

We will see that the critical points of ρ_n on $N_{r'}$ are the degenerate linkages. If T is a degenerate free $(n-1)$-gon linkage our goal is to give a formula for the signature of the Hessian $D^2\rho_n|_T$ in terms of $b(P)$, $f(P)$ and $w(P)$ where $P = P(T)$ is the associated closed n-gon linkage (see above). Clearly we must give a rule for orienting the great circle $\gamma \supset T$.

***Definition* (orienting γ)** Suppose $u = (u_1, u_2, \dots, u_n)$ is a closed degenerate linkage contained in a great circle γ. Orient γ so that the arc joining u_1 to u_n is positively directed. Thus an edge e_i is a back-track if it has the same direction as $e_n = (u_n, u_1)$.

We will prove the following theorem (with b, f and w defined using the above orientation of γ).

Main Theorem *Let $T \in N_{r'}$ be a degenerate free $(n-1)$-gon linkage and P be the associated degenerate closed n-gon linkage. Then the signature of $D^2\rho_n|_T$ is*

$$\big(b(P) + 2w(P) - 1,\ f(P) - 2w(P) - 1\big).$$

Remark The analogue of the Main Theorem for polygonal linkages in the Euclidean plane was proved in Lemma 11 of [KM1].

The Main Theorem reduces the description of the moduli spaces of spherical polygonal linkages to the combinatorics of the chambers of the polyhedron $D_n(\mathbb{S}^2)$ (see Section 2). These computations are manageable for $n = 4, 5, 6$ but become formidable for $n \geq 7$. In [G] the moduli spaces of all spherical n-gons for $n = 4, 5, 6$ are determined. In this paper we illustrate the wall-crossing formula by describing the moduli spaces of regular spherical pentagons.

[1] In what follows (a, b) will always denote the shortest geodesic segment connecting non-antipodal points a, b in $\mathbb{S}^2$.

This paper depends on the result of [KM2] *that* ρ_n *is a Morse function*. This result is what underlies the deformation arguments in Lemma 5.4 and Lemma 5.6. This paper completes the computation of the signature of $D^2\rho_n$ in Theorem 8.10 of that paper. In the appendix to this paper we patch up an error in [KM2] which allows us to apply the results of that paper that we need here.

Acknowledgements We would especially like to thank Amy Galitzer for allowing us to use the results of her thesis here and for many helpful conversations. We would also like to thank Robert Bryant who suggested the wall-crossing approach to the moduli spaces of polygonal linkages when we were working on [KM1].

2 Preliminaries

Definition 2.1 A closed spherical n-gon $P = (e_1, \dots, e_n)$ is an n-tuple of oriented geodesic arcs e_j (in $\mathbb{S}^2$) of lengths between 0 and π (inclusive) such that the end-point of e_{i-1} is equal to the initial point of e_i, $0 \leq i \leq n$ (the indices are taken modulo n).

Definition 2.2 Let $\mathcal{P}_n(\mathbb{S}^2)$ be the space of closed n-gons on $\mathbb{S}^2$ with geodesic edges.

We let r_i be the length of e_i in the spherical metric. The arcs $e_1, \dots, e_n$ will be called the edges of P. We will use $u = (u_1, \dots, u_n)$ to denote the set of vertices of P, that is, the set of initial points of the edges e_i. We will soon restrict ourselves to n-gons P with the property that $0 < r_i < \pi$, $1 \leq i \leq n$. In this case P is determined by its vertices $u_1, \dots, u_n$ and we may write $P = u = (u_1, \dots, u_n)$.

Definition 2.3 Let $\rho\colon \mathcal{P}_n(\mathbb{S}^2) \to (\mathbb{R}_+)^n$ defined by $\rho(u) = r = (r_1, \dots, r_n)$ be the side length map. That is, the distances, $d(u_i, u_{i+1})$ in the spherical metric satisfy $d(u_i, u_{i+1}) = r_i$ for $1 \leq i \leq n$ where we consider $u_{n+1} = u_1$.

Definition 2.4 $D_n(\mathbb{S}^2) = \rho\bigl(\mathcal{P}_n(\mathbb{S}^2)\bigr)$ is the space of possible side lengths. We let $\tilde{M}_r := \rho^{-1}(r)$ be the configuration space of closed n-gon linkages in $\mathbb{S}^2$ with the side-lengths r.

It is immediate that $\tilde{M}_r$ is the set of real points of the affine variety over $\mathbb{R}$ (*i.e.*, $\tilde{M}_r$ is a real algebraic set) defined by

$$u_i \cdot u_{i+1} = \cos r_i, \quad 1 \leq i \leq n,$$

where $\vec{x} \cdot \vec{y}$ denotes the scalar product in $\mathbb{R}^3$. The group $\mathrm{SO}(3)$ acts on $\tilde{M}_r$ according to

$$g(u) = (gu_1, \dots, gu_n), \quad u \in \tilde{M}_r, \quad g \in \mathrm{SO}(3).$$

Definition 2.5 The moduli space M_r of n-gon linkages on $\mathbb{S}^2$ with side lengths $r = (r_1, \dots, r_n)$ is defined to be the quotient space of $\tilde{M}_r$ by $\mathrm{SO}(3)$.

We now prove that M_r has the structure of a real algebraic set—here we assume $0 < r_i < \pi$, $1 \leq i \leq n$. Let $\vec{\epsilon}_1, \vec{\epsilon}_2, \vec{\epsilon}_3$ denote the standard basis of $\mathbb{R}^3$.

Lemma 2.6 *Define $\Sigma_r \subset \tilde{M}_r$ by $\Sigma_r = \{u \in \tilde{M}_r : u_1 = \vec{\epsilon}_1,\ u_n = \cos r_n \vec{\epsilon}_1 + \sin r_n \vec{\epsilon}_2\}$. Then Σ_r is a cross-section to the orbits of $\mathrm{SO}(3)$ on $\tilde{M}_r$.*

Proof Obvious. ■

Since the quotient map $\tilde{M}_r \to M_r$ induces a homeomorphism from Σ_r to M_r and Σ_r is a real algebraic set, M_r is a real algebraic set by transport of structure. In what follows we identify M_r and Σ_r. Notice that

$$M_r = \rho_n^{-1}(r_n), \quad \rho_n \colon N_{r'} \to \mathbb{R}, \quad \rho_n(P) = r_n, \quad \text{where } r = (r_1, \dots, r_n).$$

We let $\mathcal{Q}_n(\mathbb{S}^2)$ be the quotient space of $\mathcal{P}_n(\mathbb{S}^2)$ by $\mathrm{SO}(3)$ and let $\pi \colon \mathcal{Q}_n(\mathbb{S}^2) \to (\mathbb{R}_+)^n$ be the map induced by ρ. Hence for $r \in (\mathbb{R}_+)^n$

$$M_r = \pi^{-1}(r).$$

Our strategy is to study how the fibers of π vary as r varies in $D_n(\mathbb{S}^2)$.

We have

Lemma 2.7

(i) The Zariski tangent space $T_u(\tilde{M}_r)$ is given by

$$T_u(\tilde{M}_r) = \ker d\rho|_u.$$

(ii) The Zariski tangent space $T_u(M_r)$ is given by

$$T_u(M_r) = \ker d\pi|_u.$$

Corollary 2.8 *The variety $\tilde{M}_r$ (resp. M_r) is smooth if and only if r is a regular value of ρ (resp. π).*

From [KM2], Theorem 1.1 we deduce

Theorem 2.9 *Let $P \in \mathcal{P}_n(\mathbb{S}^2)$ (resp. $\mathcal{Q}_n(\mathbb{S}^2)$). Then P is a critical point of ρ (resp. π) if and only if P is degenerate.*

3 The Results of A. Galitzer

In [G], A. Galitzer has described $D_n(\mathbb{S}^2)$. We will need some notation to describe her results. If $I \subset \{1, 2, \dots, n\}$ we let $\bar{I}$ denote the complement of I, $|I|$ be the cardinality of I and $r_I = \sum_{i \in I} r_i$. Define a polyhedron $K_n \subset \mathbb{R}^n$ by the system of inequalities

$$0 \le r_i \le \pi, \quad 1 \le i \le n, \quad \text{and}$$
$$r_I \le r_{\bar{I}} + (|I| - 1)\pi, \quad I \subset \{1, 2, \dots, n\}, \quad \text{with } |I| \text{ odd}.$$

Then Galitzer proves

https://doi.org/10.4153/CMB-1999-037-x Published online by Cambridge University Press

Theorem 3.1 $K_n = D_n(\mathbb{S}^2)$.

In addition she proves that the codimension 1 faces of $D_n(\mathbb{S}^2)$ are given by the intersections of the hyperplanes corresponding to the above inequalities with K_n, *i.e.*, the above representation of K_n is irredundant.

The space $\mathcal{Q}_n$ is difficult to work with since the mapping π is not differentiable. To remedy this we let $\mathcal{P}_n^0$ denote the open subset of $\mathcal{P}_n$ corresponding to those n-gons such that successive vertices u_i, u_{i+1} $(i \in \mathbb{Z}/n)$ do not coincide and are not antipodal. We let $\mathcal{Q}_n^0$ denote the quotient of $\mathcal{P}_n^0$ by $\mathrm{SO}(3)$. Then $\mathcal{Q}_n^0$ is naturally a smooth manifold of dimension $2n-3$. Indeed, $\mathcal{Q}_n^0$ is naturally diffeomorphic to the submanifold $\Sigma \subset \mathcal{P}_n^0$ consisting of those n-gons with the vertex set $u = (u_1, \ldots, u_n)$ satisfying

$$u_1 = \vec{\epsilon}_1, \quad u_n \cdot \vec{\epsilon}_3 = 0, \quad u_n \cdot \vec{\epsilon}_2 > 0 \quad \text{and} \quad 0 < d(u_i, u_{i+1}) < \pi, \quad 1 \le i \le n.$$

Recall $\vec{\epsilon}_1, \vec{\epsilon}_2, \vec{\epsilon}_3$ is the standard basis of $\mathbb{R}^3$.

Note that $\Sigma_r = M_r \cap S$ (see Lemma 2.6) and that $K_n^0 \subset \pi(\mathcal{Q}_n^0)$, where K_n^0 is the interior of K_n. We will henceforth replace π by its restriction to $\mathcal{Q}_n^0$.

We shall see shortly (Theorem 3.3) that the set of critical values of π inside K_n^0 is the union of certain hyperplane sections of K_n^0. We call these hyperplane sections *walls* of K_n. Connected components in K_n^0 of the complement of the union of the walls are called *chambers*. In [G], Galitzer determines the walls of K_n. We again summarize her results.

Let $I \subset \{1, \ldots, n\}$ be any non-empty subset. For each nonnegative integer w let $H_{I,w}$ denote the hyperplane in $\mathbb{R}^n$ defined by the equation

$$r_I - r_{\bar{I}} = 2\pi w.$$

The intersection of such a hyperplane with K_n^0 is called a *wall*.

We then have the following lemma of Galitzer

Lemma 3.2 $H_{I,w} \cap K_n^0 \neq \varnothing \Leftrightarrow |I| \ge 2w + 2$.

Proof Suppose $r^* \in H_{I,w} \cap K_n^0$. Since $r^* \in H_{I,w}$ we have

$$r_I^* - r_{\bar{I}}^* = 2\pi w.$$

Assume first that $|I|$ is odd. Since $r^* \in K_n^0$ we also have

$$r_I^* - r_{\bar{I}}^* < (|I| - 1)\pi.$$

Hence $2\pi w < (|I| - 1)\pi$ and

$$|I| > 2w + 1.$$

Now assume that $|I|$ is even. We have the trivial inequality

$$r_I^* - r_{\bar{I}}^* < |I|\pi.$$

Since $r_I^* - r_{\bar{I}}^* = 2\pi w$ we obtain $2\pi w < |I|\pi$ and $|I| > 2w$. Hence $|I| \ge 2w + 1$, but $|I|$ is even, so we obtain $|I| \ge 2w + 2$.

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To prove the converse we first note that there exists a cross-section $s_{I,w}\colon H_{I,w}\cap(0,\pi)^n \to \mathcal{Q}_n^0$ to the restriction of π to $\pi^{-1}(H_{I,w})$ defined inductively as follows. Let $r^* \in H_{I,w}\cap(0,\pi)^n$. The vertices u_1 and u_n are determined by the condition that the image of $s_{I,w}$ belongs to Σ_{r^*} (see Lemma 2.6). Place the vertex u_{n-1} on the equator so that e_{n-1} is a forward track (and $d(u_{n-1},u_n) = r^*_{n-1}$) if $n-1 \in I$ and on the other side of u_n if $n-1 \in \bar{I}$. Continue inductively. The resulting degenerate linkage closes up because $r^*_I - r^*_{\bar{I}} = 2\pi w$.

We claim that $H_{I,w}\cap(0,\pi)^n \neq \varnothing$ if and only if $|I| \geq 2w+1$. Necessity is easy, if r^* is in the intersection then

$$r^*_I - r^*_{\bar{I}} = 2\pi w \Rightarrow 2\pi w < r^*_I < \pi|I|.$$

We prove sufficiency by constructing r^* in the intersection so that $r^*_i = \rho, i \in I$ and $r^*_i = \delta, i \in \bar{I}$. Hence ρ and δ must satisfy $|I|\rho - |\bar{I}|\delta = 2\pi w$. Suppose first that $\delta = 0$. Then $\rho := 2\pi w/|I| < \pi$. Now choose $\epsilon > 0$ such that $\epsilon/|I| < \pi - \rho$ and $\epsilon/|\bar{I}| < \pi$. Change ρ to $\rho + \epsilon/|I|$ and δ to $\epsilon/|\bar{I}|$. Then r^* is in the intersection and the claim follows.

We now observe that the existence of the cross-section $s_{I,w}$ constructed above implies

$$H_{I,w}\cap(0,\pi)^n = H_{I,w}\cap K_n.$$

Put $\Delta := H_{I,w}\cap(0,\pi)^n$. Then Δ is the interior of a polyhedron of dimension $n-1$. Hence Δ cannot be contained in the $(n-2)$-skeleton of K_n. Thus Δ is either a face of dimension $n-1$ of K_n or else $H_{I,w}\cap K_n^0$ is nonempty. But if $H_{I,w}\cap K_n$ is a face of dimension $n-1$ it must be the face given by

$$r_I - r_{\bar{I}} = (|I|-1)\pi.$$

Consequently $2w = |I|-1$ and $|I| = 2w+1$. Thus $|I| \geq 2w+2$ implies that $H_{I,w}\cap K_n^0$ is nonempty. ■

The set of critical values of π is then determined by

Theorem 3.3 *Let $r \in K_n^0$. Then r is a critical value of π if and only if $r \in H_{I,w}$ for some $I, w \geq 0$ with $|I| \geq 2w+2$.*

Proof Clearly there exists a degenerate $u \in \pi^{-1}(r)$ if and only if r satisfies an equation of the form $r_I - r_{\bar{I}} = 2\pi w$. Now apply Theorem 2.9. ■

Remark 3.4 Since π is proper it is a fibration over each chamber and the topology of the fibers does not change within a chamber.

4 Recuttings and Flips of Spherical n-Gons

In this section we construct two groups acting on the space of spherical n-gons.

We first construct the group $\mathcal{R}$ of *recuttings*. Let $D'_n(\mathbb{S}^2) = \{r \in D_n(\mathbb{S}^2)\colon$ all components of r are distinct$\}$. Let $\mathcal{P}'_n(\mathbb{S}^2) = \rho^{-1}\big(D'_n(\mathbb{S}^2)\big)\cap\mathcal{P}^0_n(\mathbb{S}^2)$. The permutation group S_n operates naturally on $D'_n(\mathbb{S}^2)$. We will construct a group $\mathcal{R}$ acting on $\mathcal{P}'_n(\mathbb{S}^2)$ and an epimorphism $\phi\colon \mathcal{R} \to S_n$ so that the projection ρ is ϕ-equivariant:

$$\rho(gP) = \phi(g)\rho(P) \quad P \in \mathcal{P}'_n, \quad g \in \mathcal{R}.$$

We will call elements $g \in \mathcal{R}$ recuttings. Adler [A] defined recuttings for the Euclidean plane. Here we define the recuttings for the spherical case.

We define the *basic recuttings* $R_i \colon \mathcal{P}'_n(\mathbb{S}^2) \to \mathcal{P}'_n(\mathbb{S}^2)$, $1 \le i \le n$ as follows. Let $u \in \mathcal{P}'_n(\mathbb{S}^2)$ with $u = (u_1, u_2, \dots, u_n)$. Take any geodesic arc connecting the points u_{i-1} and u_{i+1}, and look at its perpendicular bisector. The bisector is unique because $r_{i-1} \neq r_i$. Reflect the point u_i through this perpendicular line to exchange r_{i-1} and r_i. Leave all other vertices fixed. This is what we will call the *basic recutting* R_i at the i-th vertex.

The equation for the basic recutting at the i-th vertex is as follows. Set $R_i(u) = (w_1, w_2, \dots, w_n)$. Then we have

$$w_i = u_i - 2\frac{u_i \cdot (u_{i+1} - u_{i-1})}{\|u_{i+1} - u_{i-1}\|^2}(u_{i+1} - u_{i-1})$$

and

$$w_j = u_j, \quad j \neq i.$$

Then the basic recuttings are well defined on the space $\mathcal{P}'_n(\mathbb{S}^2)$. We let $\mathcal{R}$ be the group generated by the basic recuttings. Since the generators act on $\mathcal{P}'_n(\mathbb{S}^2)$, so does $\mathcal{R}$. Notice that the action of $\mathcal{R}$ preserves the subset of degenerate polygons and their winding numbers and the orientation of their edges.

We next define the *basic flips* F_i, $1 \le i \le n$. We define $F_i \colon \mathcal{P}^0_n(\mathbb{S}^2) \to \mathcal{P}^0_n(\mathbb{S}^2)$, $1 \le i \le n$, by

$$F_i(u_1, \dots, u_n) = (u_1, \dots, -u_i, \dots, u_n).$$

We note that F_i induces the map $\bar{F}_i \colon D_n(\mathbb{S}^2) \to D_n(\mathbb{S}^2)$ given by

$$\bar{F}_i(r_1, \dots, r_n) = (r_1, \dots, \pi - r_{i-1}, \pi - r_i, \dots, r_n).$$

Note that flips F_i preserve the set of degenerate n-gons but change b and w by ± 1.

5 The Morse Theory of ρ_n

In this section we will prove the Main Theorem. We begin by discussing what we proved along these lines in [KM2]. Suppose $r^* \in K^0_n$ lies on the intersection of the walls

$$H_{I_1,w_1}, H_{H_2,w_2}, \dots, H_{I_p,w_p}.$$

Choose a degenerate linkage u^* with $\pi(u^*) = r^*$. Let γ be the great circle containing u^*.

Definition 5.1 The vertical line segment L through r^* will be the line segment defined by

$$r_i = r^*_i, \quad 1 \le i \le n-1 \quad \text{and} \quad r^*_n - \delta \le r_n \le r^*_n + \delta.$$

We assume that δ is chosen so that L does not intersect any wall except at r^*. Let $X_L = \pi^{-1}(L)$.

Lemma 5.2 *X_L is a smooth submanifold of $\mathcal{Q}_n$ diffeomorphic to the $(n-2)$-torus. Moreover $X_L \cong N_{r'}$, where $r' := (r^*_1, \dots, r^*_{n-1})$ (see Section 1).*

Proof We first observe that $\rho^{-1}(L)$ is diffeomorphic to $\mathbb{S}^2 \times (\mathbb{S}^1)^{n-1}$. Indeed a point in $\rho^{-1}(L)$ is a closed n-gon where the lengths of the first $(n-1)$-sides are prescribed to be $r_1^*, r_2^*, \ldots, r_{n-1}^*$ but the length of the n-th side is not determined. The operation of forgetting the n-th side gives an isomorphism to the moduli space of the free linkage with $(n-1)$-edges. The $\mathbb{S}^2$ factor comes from the position of the first vertex u_1, the circle factors come from the angles between successive edges. The quotient $\pi^{-1}(L) = \rho^{-1}(L)/\operatorname{SO}(3)$ can be obtained by fixing the position of the first edge. Clearly $X_L \cong N_{r'}$. ■

In [KM2], Theorem 8.10, we proved

Theorem 5.3 *$\rho_n|X_L$ is a Morse function with a finite collection of critical points $u^*_{(1)} \cup \cdots \cup u^*_{(p)}$, all located on the critical fiber M_{r^*}. Each critical point $u^*_{(i)}$ corresponds to a degenerate n-gon linkage in M_{r^*} with f_i forward-tracks, b_i back-tracks and the winding number w_i contained in a great circle γ_i. Then the signature of the Hessian of $\rho_n|X_L$ at $u^*_{(i)}$ is either $(f_i - 2w_i - 1, b_i + 2w_i - 1)$ or $(b_i + 2w_i - 1, f_i - 2w_i - 1)$ depending on the orientations of γ_i, $1 \le i \le p$.*

We now concentrate on a single critical point $u^* = T^*$ of ρ_n contained in a great circle γ with the associated closed polygon P^* which has f forward-tracks and winding number w. We orient γ as described in Section 2 (*i.e.*, in the direction of rotation from u_1 to u_n). Let L^* be a vertical segment through $\rho(u^*)$.

We begin the proof of the Main Theorem with

Lemma 5.4 *There exists a vertical line segment $L^\# \subset D_n(\mathbb{S}^2)$ and a degenerate free $(n-1)$-gon linkage $T^\#$ with $\pi(T^\#) = r^\# \in L^\#$ such that*

(i) The forward-tracks of the associated closed linkage $P(T^\#)$ are the first f edges of $T^\#$.
(ii) $w(T^\#) = w(T^*)$, $f\big(P(T^\#)\big) = f$.
(iii) signature $D^2(\rho_n|X_{L^\#})|_{T^\#}$ = signature $D^2(\rho_n|X_{L^*})|_{T^*}$.
(iv) $r^\#$ belongs to exactly one wall in $D_n(\mathbb{S}^2)$ and does not belong to any minor wall.

Proof The hyperplanes $r_i = r_j$ intersect the hyperplane $r_I - r_{\bar{I}} = 2\pi w$ transversally. Hence $H_{I,w} \cap D_n'(\mathbb{S}^2)$ is the complement of a union of hyperplane sections of $H_{I,w}$ and hence is dense. Thus there exists $\bar{r} \in H_{I,w}$ close to r^* such that components of $\bar{r}$ are distinct. We let $\bar{L}$ be the vertical segment passing through $\bar{r}$, $X_{\bar{L}} = \pi^{-1}(\bar{L})$ and $\bar{u} = s_{I,w}(\bar{r})$ (see Lemma 3.3). We claim

$$\text{signature } D^2(\rho_n|X_{\bar{L}})|_{\bar{u}} = \text{signature } D^2(\rho_n|X_{L^*})|_{u*}.$$

To see this let B be the line segment in $H_{I,w}$ joining $\bar{r}$ to r^*. For $b \in B$, let L_b be the vertical segment through b and $u_b = s_{I,w^{(b)}}$. We obtain the curve $D^2(\rho_n|X_{L_b})|_{u_b}$ which joins the two Hessians above. By Theorem 5.3 these quadratic forms are nondegenerate and the claim follows. The same argument proves that we can choose $\bar{r}$ which belongs to exactly one wall.

We now choose a permutation σ of the set $\{1, 2, \ldots, n\}$ which fixes n and sends $I := \{i_1, \ldots, i_f\}$ to $\{1, 2, \ldots, f\}$. Choose a recutting R in the subgroup of $\mathcal{R}$ generated by $\{R_2, \ldots, R_{n-2}\}$ such that $\phi(R) = \sigma$. Put $r^\# = \sigma(\bar{r})$ and $u^\# = R(\bar{u})$. The line segment $\bar{L}$ through $\bar{r}$ is carried by σ to the line segment $L^\#$ through $r^\#$. Hence the corresponding manifold $X_{\bar{L}}$ is carried to $X_{L^\#}$ by R. We claim

$$\text{signature } D^2(\rho_n|X_{L^\#})|_{u^\#} = \text{signature } D^2(\rho_n|X_{L'})|_{\bar{u}}.$$

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Indeed since $\rho_n|X_{L^\#} = \rho_n \circ R|_{X_{\tilde L}}$ we find that

$$dR_{\tilde u} \colon T_{\tilde u}(X_{\tilde L}) \longrightarrow T_{u^\#}(X_{L^\#})$$

is an isometry of the quadratic form on the right-hand side to that on the left-hand side. ∎

We can now reduce to the case $w = 0$.

Lemma 5.5 *There exists a flip F such that $\tilde T = F(T^\#)$ satisfies*

(i) $b(\tilde T) = b(T^\#) + 2w(T^\#)$
(ii) $w(\tilde T) = 0$
(iii) signature $D^2(\rho_n|X_{\tilde L})|_{\tilde T}$ = *signature* $D^2(\rho_n|X_{L^\#})|_{T^\#}$.

Here $\tilde L = \bar F(L^\#)$.

Proof We consider the case $w > 0$ (the case when $w < 0$ is treated similarly, just instead of flipping forward-tracks we flip back-tracks). We let F be the product of flips given by

$$F = F_2 \circ F_4 \circ \cdots \circ F_{2w}.$$

We note that since $f \geq 2w + 2 > 2w$ all the edges that are flipped are forward-tracks (and they become back-tracks after flipping). Thus (i) and (ii) are clear. The statement (iii) is proved in the same fashion as (iii) in the previous lemma. ∎

We let K be the set of forward tracks of $\tilde T$ (or the associated closed n-gon linkage $\tilde P$). Hence $\tilde r = \pi(\tilde P)$ is on the wall $H_{K,0}$.

We next deform $\tilde r$ along the wall $H_{K,0}$ to $\hat r$ such that $\hat r_1 + \hat r_2 + \cdots + \hat r_n < 2\pi$. The corresponding degenerate closed n-gon linkage $s_{K,0}(\hat r) = \hat u$ will have perimeter less than 2π. To accomplish this let $A \subset D_n(\mathbb{S}^2) \cap H_{K,0}$ be the line segment

$$A = \{\lambda \tilde r : \epsilon < \lambda < 1 + \epsilon\}.$$

Choose λ_0 such that $\sum_{i=1}^n \lambda_0 \tilde r_i < 2\pi$. Let $\hat r = \lambda_0 \tilde r$ and $\hat L$ be the vertical segment through $\hat r$. Put $\hat u = s_{K,0}(\hat r)$.

Lemma 5.6 *The signature of* $D^2(\rho_n|X_{\hat L})|_{\hat u}$ *is equal to the signature of* $D^2(\rho_n|X_{\tilde L})|_{\tilde u}$.

Proof For $a \in A$ define L_a and u_a as in the proof of Lemma 5.4. We obtain the curve $D^2(\rho_n|X_{L_a})|_{u_a}$ and the proof goes as in Lemma 5.4. ∎

Let $\hat f$ (resp. $\hat b$) be the number of forward-tracks (resp. back-tracks) of $\hat u$. By Lemma 5.5, $\hat f = f(P) - 2w(P)$ and $\hat b = b(P) + 2w(P)$.

We complete the proof of the Main Theorem by

Proposition 5.7 *The signature of* $D^2(\rho_n|X_{\hat L})|_{\hat u}$ *is* $(\hat b - 1, \hat f - 1)$.

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The proposition will be a consequence of the next three lemmas. In what follows let $\hat{P} = \hat{u} = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n)$ be a degenerate closed n-gon linkage of perimeter less that 2π. We assume that $\pi(\hat{P})$ belongs to exactly one wall. Then any vertex u_i is connected to u_1 by a unique geodesic segment (u_1, u_i) which does not degenerate to a point.

Following [KK] we introduce local coordinates $\psi_2, \psi_3, \ldots, \psi_{n-1}$ on $X_{\hat{L}}$ by defining ψ_i to be the signed angle at u_i between the oriented segment (u_1, u_i) and the oriented edge e_i. For instance if $u_i = \vec{\epsilon}_2$, $u_{i+1} = -\vec{\epsilon}_1$ then $\psi_i = 0$. If $u_{i+1} = (\vec{\epsilon}_1 + \vec{\epsilon}_2)/\sqrt{2}$ then $\psi_i = \pi$. We then have

Lemma 5.8 *$\psi_2, \psi_3, \ldots, \psi_{n-1}$ are local coordinates near $\hat{u}$.*

Proof See [KK, Section 3]. ■

Remark 5.9 In [KK] the authors study free linkages in $\mathbb{S}^3$. Our coordinates are obtained from theirs by dropping their vector field Y. Thus we use an orthonormal frame (X, Z) where Z is the radial field.

We now have the clever observation of [KK], the reason for choosing the above coordinates.

Lemma 5.10

$$\frac{\partial^2 \rho_n}{\partial \psi_i \partial \psi_j}\Big|_{\hat{u}} = 0, \ i \neq j$$

Proof Assume $i < j$. Then by [KK, p. 84] we find that the restriction

$$\frac{\partial \rho_n}{\partial \psi_j}\Big|_{\psi_k = \hat{\psi}_k, \quad k \neq i}$$

of the partial derivative to the curve

$$\Gamma_k := \{\psi_k = \hat{\psi}_k, k \neq i\}$$

is identically zero as a function of ψ_i, this implies the lemma. Below we sketch a proof of vanishing of this derivative. We give the picture (Figure 1) in the Euclidean case with $\psi_j = 0$. We draw only the vertices u_1, u_i, u_j and u_n.

Pick a point u on the curve Γ_k. Then the points u_1, u_j, u_n belong to a common geodesic circle in $\mathbb{S}^2$. As ψ_j varies the line segment (u_j, u_n) rotates around u_j. Clearly the vertex u_n moves along a (small) circle tangent at $\psi_j = 0$ to the bigger circle which is the level set of ρ_n for the fixed values of ψ_i and $\psi_k = \hat{\psi}_k$, $k \neq i$. Hence $\frac{\partial \rho_n}{\partial \psi_j}|_{\Gamma_k}$ is identically zero as a function of ψ_i. ■

Lemma 5.11

(i) If $\hat{e}_i$ is a back-track then $\frac{\partial^2 \rho_n}{\partial \psi_i^2}|_{\hat{u}} > 0$.

(ii) If $\hat{e}_i$ is a forward-track then $\frac{\partial^2 \rho_n}{\partial \psi_i^2}|_{\hat{u}} < 0$.

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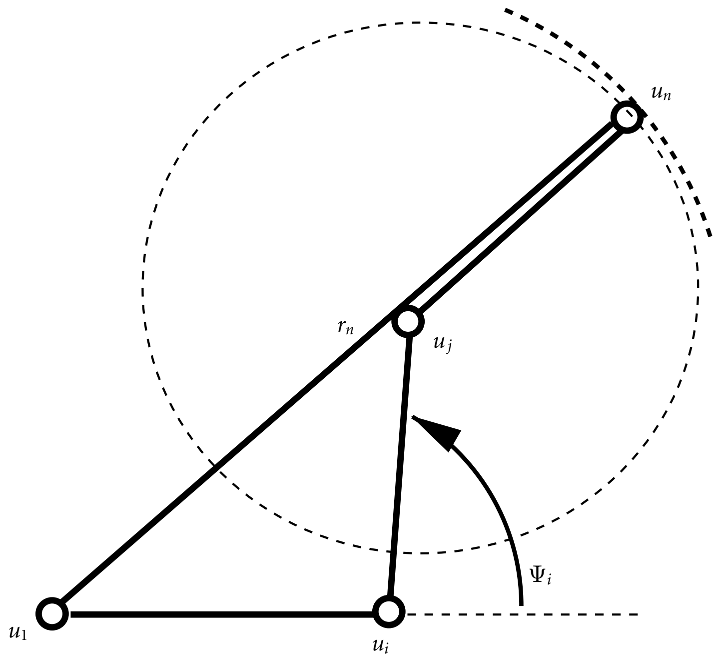

Figure 1: Vanishing of the derivative.

Proof We prove (i) and leave (ii) to the reader. We let ψ_i be a value close to $\hat{\psi}_i = \pi$ and consider the curve $\psi_j = \hat{\psi}_j$, $j \neq i$. We obtain the picture described on Figure 2 (again we have drawn the Euclidean case).
Here we have omitted all vertices except $u_1, u_i, u_{i+1}, u_{n-1}$ and u_n and assumed (in the Figure 2) that $\hat{\psi}_{i+1} = 0$ and $\hat{\psi}_{n-1} = \pi$.

We set $d(u_1, u_i) = a$, $d(u_{i+1}, u_n) = b$. From the spherical "law of cosines" (see [B, Proposition 18.6.8]) we have

$$\cos(r_n + b) = \cos a \cos r_i + \sin a \sin r_i \cos(\pi - \psi_i).$$

Differentiating implicitly we obtain

$$\frac{\partial^2 \rho_n}{\partial \psi_i^2}\Big|_{\hat{u}} = \frac{\sin a \sin \hat{r}_i}{\sin(\hat{r}_n + b)}.$$

Since the perimeter of $\hat{u}$ is less than 2π we have $a < \pi$, $\hat{r}_n + b < \pi$ and (i) follows. ∎

With this, Proposition 5.7 and the Main Theorem are proved.

6 The Wall-Crossing Formula and Regular Spherical Pentagons

In this section we explain how the Main Theorem can be used to describe how the moduli spaces M_r change as we cross a wall. As an illustration of our technique we describe the moduli spaces of regular spherical pentagons.

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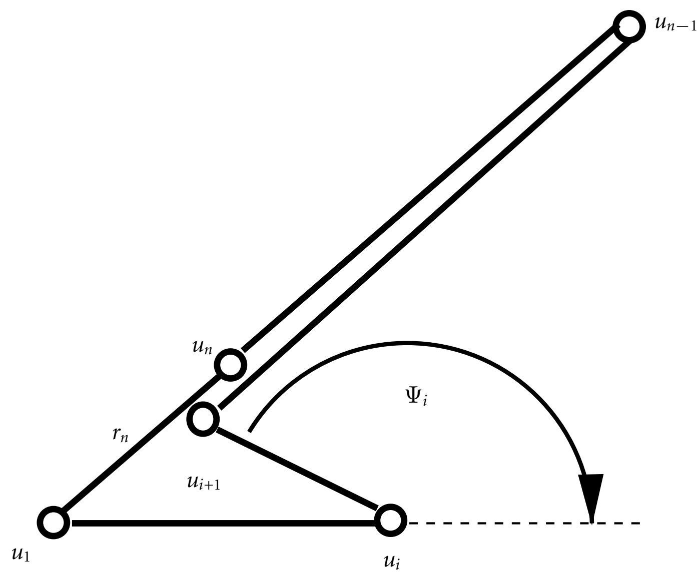

Figure 2: The sign of the second derivative.

We first claim that any wall-crossing can be effected by a vertical segment. Indeed as we have seen the walls are given by $r_I - r_{\bar{I}} = 2w\pi$ with $|I| \geq 2w + 2$. Let n_I be a normal vector to the above wall. Recall that the vector $\nu_n = (0, 0, \ldots, 0, 1)$ is parallel to a vertical segment through this wall. Since $\nu_n \cdot n_I \neq 0$ any vertical segment is transverse to a wall and the claim follows.

From the Main Theorem we obtain

Theorem 6.1 **(The wall-crossing formula)** *Suppose we cross the wall $H_{I,w}$ at $r_n = r_n^*$ along a vertical segment L with $r_n^* - \delta \leq r_n \leq r_n^* + \delta$. Then*

(i) $M_{r^*+\delta}$ is obtained from $M_{r^*-\delta}$ by attaching an $(f - 2w - 1)$-handle.
(ii) $M_{r^*-\delta}$ is obtained from $M_{r^*+\delta}$ by attaching some $(b + 2w - 1)$-handle.

We now apply our formula to describe the moduli spaces of regular spherical pentagons M_r with $r = (a, a, a, a, a)$. The description of the moduli space M_r for $\frac{2\pi}{5} < a < \frac{2\pi}{3}$ was first done in [G] by a different method. Assume first that $0 < a < \frac{2\pi}{5}$. Since the perimeter of P is less than 2π the moduli space $M_r = M_r(\mathbb{S}^2)$ is diffeomorphic to the corresponding Euclidean moduli space $M_r = M_r(\mathbb{R}^2)$ by [S]. Hence by [KM1, Theorem 2], M_r is the genus four surface, $0 < a < \frac{2\pi}{5}$.

Now as a goes from $\frac{2\pi}{5} - \delta$ to $\frac{2\pi}{5} + \delta$ we pass through the wall $r_1 + r_2 + r_3 + r_4 + r_5 = 2\pi$. We now describe what happens as we cross this wall using Theorem 6.1. Set $r_1 = r_2 = r_3 = r_4 = \frac{2\pi}{5}$ and let r_5 go from $\frac{2\pi}{5} - \delta$ to $\frac{2\pi}{5} + \delta$. The critical point $T \in N_r$ corresponding to the critical value $r_5 = \frac{2\pi}{5}$ is represented by the degenerate free 4-gon linkage with $P = P(T)$ obtained by dividing the equator γ into 5 equal parts proceeding anticlockwise around the equator and taking the first four segments. Our orientation rule requires us to orient the equator so that the positive direction is clockwise hence

$$b(P) = 5, \quad f(P) = 0, \quad w(P) = -1.$$

According to the main theorem the signature of $D^2\rho_5|_L$ is $(2, 1)$. Since ρ_5 increases as we cross the wall we obtain Theorem 6.1 of [G]:

$$M_r \text{ is the genus five surface}, \quad \text{if } \frac{2\pi}{5} < a < \frac{2\pi}{3}.$$

The point $r = (\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3})$ lies on the intersection of five walls of the form

$$r_i + r_j + r_k + r_l - r_m = 2\pi.$$

There are two cases to consider, $m = 5$ and $m \neq 5$. We will analyse the first case and leave the second to the reader.

We will identify the equator of $\mathbb{S}^2$ with the unit circle on the complex plane. Let T be the degenerate free 4-gon linkage with vertices $(1, \omega, \omega^2, 1, \omega)$ where $\omega = \exp(2\pi i/3)$. By our orientation convention the unit circle has the usual (*i.e.*, counterclockwise) orientation and

$$b(P) = 1, \quad f(P) = 4, \quad w(P) = 1.$$

Hence $D^2\rho_5|_T$ has signature $(2, 1)$. The equation of the wall we are considering is $r_1 + r_2 + r_3 + r_4 - r_5 = 2\pi$. Let $\alpha(r_1, r_2, r_3, r_4, r_5) = r_1 + r_2 + r_3 + r_4 - r_5$. As a increases from $\frac{2\pi}{3} - \delta$ to $\frac{2\pi}{3} + \delta$ we pass from the half-space $\alpha < 2\pi$ to $\alpha > 2\pi$. Now to apply the Theorem we set $r_1 = r_2 = r_3 = r_4 = \frac{2\pi}{3}$. To cross from $\alpha < 2\pi$ to $\alpha > 2\pi$ we see that r_5 must *decrease* from $\frac{2\pi}{3} + \delta$ to $\frac{2\pi}{3} - \delta$. Thus we attach the "positive" or "ascending" disk of ρ_5 (*i.e.*, the unit disk in a maximal subspace of the tangent space at T on which the quadratic form $D^2\rho_5|_T$ is *positive-definite*) as we pass through the critical point $r_5 = \frac{2\pi}{3}$. Hence we attach a 2-handle. We attach 2-handles at the other 4 critical points of ρ_5 corresponding to the critical value $r_5 = \frac{2\pi}{3}$ and we obtain

$$M_r \approx \mathbb{S}^2, \quad \text{if } \frac{2\pi}{3} < a < \frac{4\pi}{5}.$$

We cross no more walls of $D_5(\mathbb{S}^2)$ until we reach the face given by $r_1+r_2+r_3+r_4+r_5 = 4\pi$ when $a = \frac{4\pi}{5}$. The critical value $r_5 = \frac{4\pi}{5}$ corresponds to the single critical point $u = (1, \zeta^2, \zeta^4, \zeta^6, \zeta^8)$ where $\zeta = \exp(2\pi i/5)$. We have $u_5 = \exp(-4\pi i/5)$. Hence γ is oriented n the clockwise direction. We obtain

$$b(P) = 5, \quad f(P) = 0, \quad w(P) = -2$$

and accordingly the signature of $D^2\rho_5|_T$ is $(0, 3)$. Hence P is locally rigid.

We can in fact determine the moduli space M_r as follows. Apply the flips F_1 and F_3 to change r to r^* with $r_1^* = r_2^* = r_3^* = r_4^* = \frac{\pi}{5}$, $r_5^* = \frac{4\pi}{5}$. This is a standard "Euclidean" rigid linkage and $M_{r^*} =$ a point, as was to be expected since r is on a face.

Of course for $a > \frac{4\pi}{5}$, M_r is empty since we are outside $D_5(\mathbb{S}^2)$.

7 Appendix

The statement in Section 6 of [KM2] that $A^{\bullet}_{(2)}(M, adP)$ is a differential graded Lie algebra is false since the L^2-condition is not closed under bracket. Hence our proof that $B^{\bullet}(M, U; adP)$ is formal as a *differential graded Lie algebra* is not correct. However we can salvage all the results of [KM2] except the result that $B^{\bullet}(M, U; adP)$ is formal by the following "quick fix". First we apply the results of Section 5 of our paper [KM3] to deduce that the germ $(M_r, [P_0])$ is given by a single quadratic equation corresponding to the cup product: $q\colon H^1\big(B^{\bullet}(M, U; adP)\big) \to H^2\big(B^{\bullet}(M, U; adP)\big) = \mathbb{R}$.

Now we claim that the results of Section 7 of [KM2] do in fact compute q above. To see this we note first that the inclusion $B^{\bullet}(M, U, adP) \to A^{\bullet}_{(2)}(M, adP)$ is a quasi-isomorphism of *complexes*. The bracket of two elements of $A^1_{(2)}(M, adP)$ is integrable (but not necessarily square integrable) whence the integration pairing (using the trace on adP) is well-defined on $A^1_{(2)}(M, adP)$. By [Ga] it descends to cohomology and consequently agrees with q.

Remark 7.1 Formality of $B^{\bullet}(M, U; adP)$ follows from the recent result of P. Foth [F].

References

[A] V. Adler, *Recuttings of polygons*. Functional Anal. Appl. **27**(1993), 141–143.

[B] M. Berger, *Geometry II*. Universitext. Springer, New York, 1980.

[F] P. Foth, *Deformations of representations of fundamental groups of open Kähler manifolds*. Preprint, September, 1997.

[Ga] M. Gaffney, *A special Stokes theorem for complete Riemannian manifolds*. Ann. of Math. **60**(1954), 140–145.

[G] A. Galitzer, *The moduli space of polygon linkages in the 2-sphere*. Ph.D. thesis, University of Maryland, 1997.

[KK] P. Kirk and E. Klassen, *Representation spaces of Seifert fibered homology spheres*. Topology **30**(1991), 77–95.

[KM1] M. Kapovich and J. J. Millson, *On the moduli space of polygons in the Euclidean plane*. J. Differential Geom. **42**(1995), 133–164.

[KM2] ———, *Hodge theory and the art of paper folding*. Publ. Res. Inst. Math. Sci. **33**(1997) 1–33.

[KM3] ———, *The relative deformation theory of representations and flat connections and deformations of linkages in constant curvature spaces*. Compositio Math. **103**(1996), 287–317.

[S] M. Sargent, *Diffeomorphism equivalence of configuration spaces of polygons in constant curvature spaces*. Ph.D. thesis, University of Maryland, 1995.

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