Maximal symplectic packings in $\mathbb{P}^2$

Emmanuel Opshtein

Abstract

In this paper we describe the intersections between the balls of maximal symplectic packings of $\mathbb{P}^2$. This analysis shows the existence of singular points for maximal packings of $\mathbb{P}^2$ by more than three equal balls. It also yields a construction of a class of very regular examples of maximal packings by five balls.

1. Introduction

The symplectic packing problem is the question of identifying the conditions on the radii of k balls for being able to pack them symplectically in a given manifold. It was first considered by Gromov as a problem whose answer distinguishes symplectic geometry from the volume-preserving one: the restrictions are sometimes stronger than the volume obstruction alone [Gro85]. For instance, the non-squeezing theorem asserts that no ball of radius bigger than one can be packed into the infinite volume cylinder $B^2(1) \times \mathbb{R}^{2n}$. In the special case of $\mathbb{P}^2$, this problem has been completely solved. Apart from the volume obstruction, symplectic packings by less than eight balls are submitted to finitely many purely symplectic obstructions discovered by Gromov [Gro85] and by McDuff and Polterovich [MP94]. Biran proved that the symplectic obstructions disappear for more than nine balls [Bir99].

This paper is aimed at describing what the packings of $\mathbb{P}^2$ by balls of maximal radii look like. We are particularly interested in understanding their intersection properties. Let us first define our setting. Throughout this paper, the symplectic structure on $\mathbb{P}^2$ is given by the standard Fubini–Study form normalized so that $\int_{\mathbb{CP}^1} \omega = \pi$. With this normalization, $\mathbb{CP}^2 \backslash \mathbb{CP}^1$ is symplectomorphic to the standard unit *open* ball $B^4(1)$ in $\mathbb{C}^2$.

DEFINITION 1.1. A *maximal symplectic packing* of M by k balls is a symplectic embedding $\varphi : (B(r_1) \amalg \ldots \amalg B(r_k), \omega_{\mathrm{st}}) \hookrightarrow (M, \omega)$ where the radii are such that there exists no symplectic packing of M by balls of radii $(r_1, \ldots, r_i + \varepsilon, \ldots, r_k)$. It will be said to be *smooth* if each $\varphi_i := \varphi_{|B(r_i)}$ extends to a smooth embedding of the closed balls. It will be said to be *regular* if these maps have only a finite number of singular points on the boundary: each φ_i extends to a topological embedding of the closed ball which is a smooth embedding of $\overline{B(r_i)} \backslash \{p_i^1, \ldots, p_i^{n_i}\}$.

The space of regular maximal symplectic packings by a fixed number of balls is naturally endowed with the Hausdorff topology for compact sets. Throughout this paper, genericity is meant with respect to this topology, and should be understood in a strong sense: a property is generic if it is true for an open dense set. Our first theorem deals with smooth maximal symplectic packings of $\mathbb{P}^2$.

THEOREM 1. *The generic intersection patterns between the balls of smooth maximal symplectic packings of $\mathbb{P}^2$ are as follows.*

(a) *Generically, the closed balls of a smooth maximal symplectic packing of $\mathbb{P}^2$ by two balls intersect precisely along one common Hopf circle of their boundary.*

(b) *Generically, any two closed balls of a smooth maximal symplectic packing of $\mathbb{P}^2$ by three equal balls intersect precisely along one common Hopf circle of their boundary.*

(c) *Generically, the two smallest balls of a smooth maximal symplectic packing of $\mathbb{P}^2$ by three non-equal balls do not intersect, while the intersection of the biggest ball with any of the others is exactly one common Hopf circle of their boundaries.*

(d) *There exists no smooth maximal symplectic packing of $\mathbb{P}^2$ by more than three equal balls.*

The Hopf circles of a ball are the characteristic leafs of its boundary (see § 3.1). Theorem 1 is better understood in the light of Karshon's examples [MP94]: it says that generic smooth maximal symplectic packings look very much like those she produced (see § 2). The idea is to translate the maximality property to the existence of characteristic circles in the intersections of the boundary spheres. These characteristics give rise to (maybe singular) symplectic spheres, whose intersection properties lead to the desired uniqueness. The approach is based on a strong connection observed by Paternain, Polterovich and Siburg [PPS03] or Laudenbach and Sikorav [LS94] between symplectic non-removable intersection and closed characteristics.

The importance of the characteristic foliation in the present study is the reason for our definition of smooth or regular maximal symplectic packing to be so restrictive. Before going further, a discussion of the existence of the objects under consideration is needed. A non-constructive argument due to McDuff shows that there always exist symplectic packings by *open* balls of maximal radii, with no guaranteed boundary regularity. On the other hand, several explicit examples are available. We already mentioned Karshon's smooth packings by two or three balls. Generalizing her construction, Traynor [Tra95] and Schlenk [Sch05] produced examples of maximal symplectic packings of $\mathbb{P}^2$ by five and six balls, which unfortunately fail by far to be regular. This is not surprising in view of Theorem 1(d). As far as we are concerned, the achieved boundary regularity is nevertheless as difficult to handle as McDuff's abstract maximal packings: no convenient notion of characteristic foliation on the boundary of the balls can be defined. The second result of this paper exhibits the relevance of our definition of regularity. Allowing only finitely many singularities enables us to produce interesting maximal packings of $\mathbb{P}^2$, at least by five balls.

THEOREM 2. *There exist regular maximal symplectic packings of $\mathbb{P}^2$ by five equal balls.*

The construction relies on a decomposition theorem of Kähler manifolds due to Biran [Bir01].

Our third result generalizes Theorem 1 to the regular setting. The intersections between the balls of a regular maximal symplectic packing is a union of *Hopf circles* of the boundary spheres which provide a 'grid' of a topological 2-fold, symplectic away from its singularities.

THEOREM 3. *Given a regular maximal symplectic packing of a symplectic manifold, there exists at least one 'supporting surface': a closed topological surface covered by the balls of the packing and whose intersection with any ball is a union of smooth symplectic discs bounded by Hopf circles. Generically, the balls of the packing intersect exactly along the Hopf circles contained in the supporting surfaces.*

The existence result above can be sharpened when the singularities are simple enough (see Definition 5.3 of packings of simple type). For five balls, for instance, the intersection pattern must be the same as in the constructed examples (Theorem 2).

THEOREM 4. *Regular maximal symplectic packings of $\mathbb{P}^2$ by five equal balls which have simple type have exactly one supporting surface, of symplectic area 2π, intersecting each ball in exactly*

1559

one Hopf disc. Generically, these maximal packings thus intersect along exactly one Hopf circle of each of the boundary spheres.

The paper is organized as follows. In §2, we discuss previously known examples of maximal symplectic packings and we construct new ones (Theorem 2). These examples shed some light on Theorems 1, 3 and 4 by providing relevant illustrations. In the third section, we explain the link between non-removable intersections and characteristic foliations in the setting of smooth balls, and prove Theorem 1. The purpose of §4 is to adapt the tools of §3 to non-smooth objects. We prove Theorem 3 in §5 and conclude by a technical paragraph aimed at smoothening the supporting surfaces to prove Theorem 4.

2. Examples of maximal symplectic packings

The aim of this section is to provide examples of smooth or regular symplectic packings. We describe them in the light of the results stated in the Introduction.

Karshon's construction

Examples of smooth maximal symplectic packings of $\mathbb{P}^2$ were given by Karshon [MP94] and Traynor [Tra95]. They can be described in the following way. The momentum map (or the action–angle coordinates) presents $\mathbb{P}^2$ as a singular bundle over a closed triangle with two-dimensional tori as generic fibres. The balls forming the packing are fibred by these tori and project by the momentum map to closed triangles (see Figure 1).

The packing is smooth in the sense of Definition 1.1. Indeed, one of the balls is obtained by restricting the symplectic embedding

$$\Phi \;:\; \begin{aligned} B^4(1) &\longrightarrow \mathbb{P}^2, \\ (z_1, z_2) &\longmapsto [\sqrt{1 - |z|^2} : z_1 : z_2] \end{aligned}$$

to the ball of radius one-half. This ball is obviously a smooth one because it is the restriction of a bigger one. Now the two other balls are obtained from the first one by the global symplectomorphisms of $\mathbb{P}^2$ which consist of exchanging the coordinates $[z_0 : z_1 : z_2] \mapsto [z_1 : z_0 : z_2]$ or $[z_1 : z_2 : z_0]$. Thus all three balls are smooth, and the packing itself is a smooth one.

Note that the intersection between the closed balls in these examples is precisely one common circle of the Hopf fibration of the boundary spheres (point I). Moreover, the fibre over the line AC forms a two-dimensional sphere covered by the union of the two balls and whose intersection with each ball is a disc bounded by a Hopf circle. It is thus precisely one of the 'supporting surfaces' of the maximal packing. Theorem 1 says that every smooth maximal packing of $\mathbb{P}^2$ by two or three balls always has the same intersection patterns as those examples.

FIGURE 1. Maximal symplectic packings of $\mathbb{P}^2$ by three balls.

1560

From this example, it is also easy to see that the assumption concerning the equality of the balls in Theorem 1(d) cannot be dropped. Consider actually a symplectic packing $\{B_1, B_2\}$ of $\mathbb{P}^2$ by smooth balls of radii $r_2 \ll 1$ and $r_1 = \sqrt{1 - r_2^2}$, centred at the points A and C with the notation of Figure 1. Looking at them in $B^4(1)$ rather than $\mathbb{P}^2$ (their interiors do not intersect the $\mathbb{P}^1$ at infinity), we see one big ball B_1' centred at the origin and one thin compact set B_2' (not a closed ball any more) very close to the Hopf circle of $\partial B^4(1)$ corresponding to A. There clearly exist approximately $1/r_2^2$ unitary transformations of $\mathbb{C}^2$ (leaving B_1' invariant) which take B_2' to disjoint compact sets. After collapsing the Hopf circles of $\partial B^4(1)$ to get $\mathbb{P}^2$, these compact sets transform to disjoint symplectic balls of the same radius r. Together with B_1, they provide a smooth maximal symplectic packing of $\mathbb{P}^2$ by approximately $1/r_2^2$ balls.

Biran's decomposition theorem (see [Bir01])

The examples of regular packings to come are based on a decomposition result due to Biran which we describe now briefly. Given an integral Kähler manifold (M, ω, J) (i.e. with $[\omega] \in H^2(M, \mathbb{Z})$), there always exists a complex hypersurface X which is Poincaré dual to $k\omega$ for some big enough integer k. Biran showed that there exists a CW-complex $\Delta_X \subset M$ (of empty interior) associated to this hypersurface, whose complement in M is a *standard symplectic disc bundle over X*. It turns out that some maximal packings appear very clearly in these special coordinates. In order to explain this, let us discuss briefly the structure of these standard bundles. Their symplectic type is that of the unit disc bundle associated to a Hermitian line bundle of first Chern class $c_1 = [k\tau]$, where $\tau = \omega_{|X}$. The symplectic structure itself is explicitly given by

$$\omega := \pi^*\tau + d(r^2\alpha),$$

where r is the radial coordinate in the fibre and α is the 1-form whose restriction to the fibres is $\alpha_{|\pi^{-1}(x)} = 1/k \, d\theta$ on $E\backslash\{r = 0\}$ and $d\alpha = -\pi^*\tau$. Notice that, although θ is not defined globally on E because of non-vanishing Chern class, the differential form '$d\theta$' is perfectly defined away from the zero section.

The connection between standard symplectic bundles and balls comes from the following simple observation. The restriction of these bundles to a symplectic ellipsoid in X is symplectomorphic to an ellipsoid. In the following lemma, $\mathcal{E}_a = \mathcal{E}_{(a_1,\ldots,a_n)}$ denotes the standard ellipsoid in $\mathbb{C}^n$:

$$\mathcal{E}_a := \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \;\middle|\; \frac{|z_1|^2}{a_1} + \cdots + \frac{|z_n|^2}{a_n} < 1 \right\}.$$

LEMMA 2.1. *Consider the trivial disc bundle* $\pi : \mathcal{E}_a \times \mathbb{D} \longrightarrow \mathcal{E}_a$ *over an ellipsoid of $\mathbb{C}^n$, equipped with the symplectic structure defined by* $\omega := \pi^*\omega_{\mathrm{st}} + d(r^2\alpha)$, $\alpha_{|\{x\}\times\mathbb{D}} = 1/k \, d\theta$ *and* $d\alpha = -\pi^*\omega_{\mathrm{st}}$, *where (r, θ) are the polar coordinates on $\mathbb{D}$. Then there exists a smooth function $h : \mathcal{E}_a \longrightarrow \mathbb{R}$ such that the map*

$$
\begin{array}{rcl}
\Phi \;:\; (\mathcal{E}_a \times \mathbb{D}, \omega) & \longrightarrow & (\mathcal{E}_{a,1/k}, \omega_{\mathrm{st}}), \\
(z, w) & \longmapsto & \left(\sqrt{1 - |w|^2}\, z, \dfrac{1}{\sqrt{k}} e^{i\, h(z)} w \right)
\end{array}
$$

is a symplectomorphism.

Proof. Consider the coordinates $(z, w) = (r_1, \theta_1, \ldots, r_n, \theta_n, r, \theta)$ on $\mathcal{E}_a \times \mathbb{D}$. The symplectic form ω is given in these coordinates by

$$\omega = \sum_{i=1}^{n} dr_i^2 \wedge d\theta_i + d(r^2\alpha).$$

1561

Taking into account the identities $\alpha_{|\{x\}\times\mathbb{D}} = 1/k\,d\theta$ and $d\alpha = -\sum dr_i^2 \wedge d\theta_i$, we get

$$\omega = (1 - r^2)\sum_{i=1}^{n} dr_i^2 \wedge d\theta_i + dr^2 \wedge \alpha$$

$$= \sum_{i=1}^{n} d[(1 - r^2)r_i^2] \wedge d\theta_i + dr^2 \wedge \left[\alpha + \sum_{i=1}^{n} r_i^2\,d\theta_i\right]$$

$$= \sum_{i=1}^{n} d[(1 - r^2)r_i^2] \wedge d\theta_i + \frac{1}{k}dr^2 \wedge [d\theta + \beta],$$

where $\beta := k[\alpha - 1/k\,d\theta + \sum_{i=1}^{n} r_i^2\,d\theta_i]$. The form β is defined on $\mathcal{E}_a \times (\mathbb{D}\backslash\{0\})$ and is closed. Moreover, its action on the fundamental group of $\mathcal{E}_a \times \mathbb{D}\backslash\{0\}$ is trivial because $\beta_{|\{x\}\times\mathbb{D}} = 0$. Hence there is a smooth function $h : \mathcal{E}_a \times \mathbb{D}\backslash\{0\} \longrightarrow \mathbb{R}$ such that $\beta = dh$. Notice now that h does not depend on w because β vanishes on the vertical discs. This function thus extends to $\mathcal{E}_a \times \mathbb{D}$ and depends only on z. We finally get

$$\omega = \sum_{i=1}^{n} d[(1 - r^2)r_i^2] \wedge d\theta_i + \frac{1}{k}dr^2 \wedge d(\theta + h)$$

$$= \Phi^*\left[\sum_{i=1}^{n} dr_i^2 \wedge d\theta_i + dr^2 \wedge d\theta\right],$$

where Φ is the announced map. It clearly sends $\mathcal{E}_a \times \mathbb{D}$ to the ellipsoid $\mathcal{E}_{a,1/k}$. $\square$

Since a complex hypersurface of a Kähler manifold is Kähler, an obvious iteration leads to the following corollary. It seems to hold true also in general compact symplectic manifolds due to Donaldson's results on the existence of symplectic hypersurfaces [Don96].

COROLLARY 2.2. *Every Kähler manifold has a full packing by one ellipsoid.*

Maximal full packing of $\mathbb{P}^n$ by k^n balls

We make a digression at this point to explain how to construct a full symplectic packing of $\mathbb{P}^2$ by four *open* balls of radius $1/\sqrt{2}$. The generalization to k^n balls of maximal radius $1/\sqrt[n]{k}$ in $\mathbb{P}^n$ is straightforward. As far as I know, although their existence is well known from McDuff and Polterovich's [MP94] or Traynor's work [Tra95] via McDuff's argument [McD98], no example of such packings has been available up to now.

Consider the quadric $Q := \{z_0^2 + z_1^2 + z_2^2 = 0\}$ in $\mathbb{P}^2$ with homogeneous coordinates $[z_0 : z_1 : z_2]$ – recall that it is diffeomorphic to a 2-sphere. Then $\pi : \mathbb{P}^2\backslash\mathbb{RP}^2 \longrightarrow Q$ is a standard disc bundle with fibres of area $\pi/2$. Choose four disjoint open discs D_i in Q of area $\pi/2$, $i = 1,\ldots,4$. Then the sets $B_i := \pi^{-1}(D_i) \subset \mathbb{P}^2$ are obviously disjoint. Moreover, the previous lemma shows that they are symplectic balls of radius $1/\sqrt{2}$. Notice also that the boundary singularities of these balls (inavoidable by Theorem 1(d)) are easily describable in terms of the singularities of the discs D_i thanks to the explicit formula for the symplectomorphism Φ of Lemma 2.1.

Regular maximal packing of $\mathbb{P}^2$ by five balls (see Figure 2)

As above, consider the quadric $Q := \{z_0^2 + z_1^2 + z_2^2 = 0\}$, and the projection $\pi : \mathbb{P}^2\backslash\mathbb{RP}^2 \longrightarrow Q$ which gives $\mathbb{P}^2\backslash\mathbb{RP}^2$ the structure of a standard disc bundle with fibres of area $\pi/2$. Cover Q by five closed discs of area $2\pi/5$ with finite number of singularities on their boundaries. We claim that we can find the desired balls of the packing inside the sets fibred upon these discs. To see this, we construct a regular symplectic embedding of a ball of radius R into $\pi^{-1}(D)$, where D is any closed disc in Q of area $\pi R^2 < \pi/2$, with a finite number of singularities $p_1,\ldots,p_k \in \partial D$. First identify $\pi^{-1}(D)$
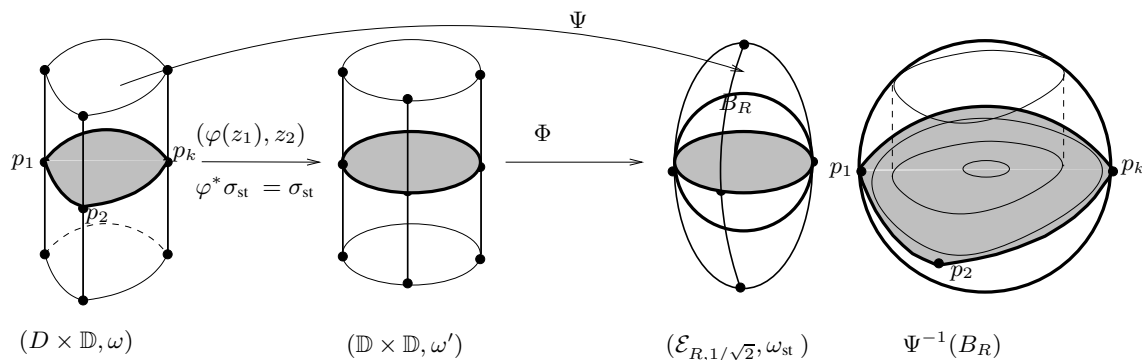
FIGURE 2. Regular packing of a standard disc bundle over a disc by a ball.

with $D \times \mathbb{D}$. Then $(\pi^{-1}(D), \omega_{FS})$ is symplectomorphic to a standard ellipsoid $\mathcal{E} := \mathcal{E}_{R, 1/\sqrt{2}}$ via a map Ψ which is the composition of a fibred map $\varphi \times \mathrm{Id}$ sending $D \times \mathbb{D}$ to the standard bidisc $\mathbb{D}_R \times \mathbb{D}$ with the map Φ of Lemma 2.1 where now $k = 2$. The boundary regularity of the symplectomorphism $\Psi : \pi^{-1}(D) \longrightarrow \mathcal{E}$ can be described as follows. First it extends to a homeomorphism between $\overline{D} \times \mathbb{D}$ (i.e. $\overline{\pi^{-1}(D)}$ minus the section at infinity) and $\overline{\mathcal{E}} \backslash C_\infty$ where $C_\infty := \{|z_2|^2 = 1/\sqrt{2}, z_1 = 0\}$. Moreover, this extension is a local diffeomorphism except at the singular points $\bigcup \{p_i\} \times \mathbb{D}$ of $\pi^{-1}(D)$. Now the ellipsoid $\overline{\mathcal{E}}$ contains a Euclidean closed ball B of radius R whose boundary intersects $\partial \mathcal{E}$ only along the 'zero section' $\{z_2 = 0, |z_1| = R\}$. The map $\Psi^{-1} : B \longrightarrow \pi^{-1}(D)$ is therefore a regular symplectic embedding of a ball of radius R in $\pi^{-1}(D)$.

3. Non-removable intersections in smooth maximal packings

The aim of this section is to introduce the main tool of the paper – namely the link between non-removable intersections in symplectic geometry and characteristic foliations – following [PPS03, LS94]. We also prove Theorem 1. Dealing only with smooth objects first permits us to avoid the technical difficulties arising in the context of regular packings.

Let us explain first how the genericity is achieved in Theorem 1. We define a *Hamiltonian perturbation of a maximal packing* $\{\varphi_i\}_{i=1,\dots,k}$ as a family $\{\varphi_i^t\} := \{\Phi_{X_{H_i}}^t \circ \varphi_i\}$, where the time-dependent Hamiltonian functions H_i on M are such that the open balls $\varphi_i^t(B(r_i))$ remain disjoint for all $t \in [0, \varepsilon]$. Note that these Hamiltonian perturbations of packings enable us to break intersections. The properties stated in Theorem 1 are generic precisely because they are always true after a possible Hamiltonian perturbation of the packing.

3.1 Pushing into a ball by a Hamiltonian: Sullivan's lemma

As explained in [PPS03] and [LS94], the characteristic foliation plays a central role in the phenomena of 'symplectic non-removable intersection'. Loosely speaking, the reason for a compact set inside the smooth boundary of an open set U not to be displaceable inside $\overline{U}$ by a Hamiltonian vector field is that it contains a closed invariant set of the characteristic foliation of ∂U.

DEFINITION 3.1. The *characteristic distribution* $\mathcal{L}$ of a hypersurface S in a symplectic manifold (M, ω) is the kernel of the restriction of ω to S (i.e. for all $x \in S$, $\mathcal{L}_x := \ker \omega_{x|T_x S}$). The *characteristic foliation* of S is the integral foliation associated to this one-dimensional characteristic distribution.

The characteristic foliation is obviously preserved by any map $\varphi : S \subset (M, \omega) \longrightarrow S' \subset (M', \omega')$ with $\varphi^* \omega'_{|TS'} = \omega_{|TS}$.

1563

Example. On the Euclidean sphere $S(r) \subset \mathbb{C}^n$, the characteristic distribution is given by $\mathcal{L}_x = \mathrm{Span}_{\mathbb{R}}(i\vec{N}(x))$ where $\vec{N}(x) := x/\|x\|$. The characteristic foliation is thus the classical foliation of $S(r)$ by Hopf circles. Any symplectic smooth (or regular) embedding of a Euclidean closed ball of $\mathbb{C}^n$ into M is called a *smooth (or regular) symplectic closed ball of* (M, ω). If B is a symplectic closed ball in M corresponding to a regular embedding $\varphi : \overline{B(r)} \longrightarrow M$, the characteristic leaves of $\partial B = \varphi(S(r))$ are the images by φ of the Hopf circles of $S(r)$ and will be called the Hopf circles of B. Finally, the Hopf discs of a symplectic ball B are the images of the intersections of $B(r)$ with the complex lines of $\mathbb{C}^n$.

The vector field $i\vec{N}(x)$ defines and orients the characteristic foliation of the Euclidean sphere S in $\mathbb{C}^n$. The Hamiltonian vector field $\vec{X}_H$ associated to a smooth function $H : \mathbb{C}^n \longrightarrow \mathbb{R}$ points inside B at a point x of S if and only if $i\vec{N} \cdot H(x) < 0$. Indeed, we have

$$\vec{X}_H \cdot \vec{N}(x) < 0 \quad \Longleftrightarrow \quad \omega(\vec{X}_H(x), i\vec{N}(x)) < 0 \quad \Longleftrightarrow \quad dH_x(i\vec{N}(x)) < 0.$$

In other words, the Hamiltonian vector field $\vec{X}_H$ pushes into B at a point x of S if H *is decreasing along the characteristic foliation at this point.* The following lemma explains along which compact sets of S one can push into B in a Hamiltonian way. It is a particular case of a more general result due to Sullivan [Sul76] (see also [LS94, LM95]).

LEMMA 3.2. *Let K be a compact set of $S^{2n-1} \subset \mathbb{C}^n$ and $\mathcal{C}$ the (compact) set of Hopf circles belonging to K. There exists a smooth function $H : S^{2n-1} \longrightarrow \mathbb{R}$ such that $i\vec{N} \cdot H < 0$ on $K \backslash \mathcal{C}$ and $dH = 0$ on $\mathcal{C}$.*

Proof. We first reduce Lemma 3.2 to finding convenient functions on solid tori by a partition of unity argument. Denote the Hopf projection by $\pi : S^{2n-1} \longrightarrow \mathbb{P}^{n-1}$. Given a locally finite open cover by balls $\mathcal{U} := \{U_\alpha\}$ of $\pi(K \backslash \mathcal{C})$, the open set $U := \mathbb{P}^{n-1} \backslash K$ completes $\mathcal{U}$ to a locally finite cover of $\mathbb{P}^{n-1} \backslash \pi(\mathcal{C})$. Consider a smooth partition of unity $\{\Phi_\alpha, \Phi\}$ associated to (U_α, U). Also choose a smooth non-negative function θ on $\mathbb{P}^{n-1}$ which vanishes on $\pi(\mathcal{C})$ at order two and is positive away from $\pi(\mathcal{C})$. We claim that if we can find functions $h_\alpha : \pi^{-1}(U_\alpha) \longrightarrow \mathbb{R}$ such that $i\vec{N} \cdot h_\alpha < 0$ on $K \backslash \mathcal{C} \cap \pi^{-1}(U_\alpha)$ then the function $H = \theta \circ \pi \sum \lambda_\alpha \Phi_\alpha \circ \pi \cdot h_\alpha$ is of the required type, provided that the positive constants λ_α are small enough to guarantee the smoothness of H (the choice $\lambda_\alpha := 2^{-\alpha} \|\Phi_\alpha \circ \pi \cdot h_\alpha\|_{\mathcal{C}^1}^{-1}$ is convenient). Indeed, we have

$$i\vec{N} \cdot H = \theta \circ \pi \sum \lambda_\alpha \Phi_\alpha \circ \pi \cdot i\vec{N} \cdot h_\alpha < 0 \quad \text{on } K \backslash \mathcal{C}, \tag{1}$$

and

$$dH = \left(\sum \lambda_\alpha \Phi_\alpha \circ \pi \cdot h_\alpha \right) d(\theta \circ \pi) + (\theta \circ \pi) d\left(\sum \lambda_\alpha \Phi_\alpha \circ \pi \cdot h_\alpha \right) = 0 \quad \text{on } \mathcal{C}. \tag{2}$$

The first equality holds because $\theta \circ \pi$ and $\Phi_\alpha \circ \pi$ are constant along $i\vec{N}$. The second holds because both $\theta \circ \pi$ and $d\theta \circ \pi$ vanish on $\mathcal{C}$.

Thus it only remains to build the cover of $\pi(K \backslash \mathcal{C}) \subset \mathbb{P}^{n-1}$ by balls and the suitable functions on the solid tori which project to these balls by π. In a neighbourhood of $x \in \pi(K \backslash \mathcal{C})$, the compactness of K allows us to find a local section $s : B_{\delta(x)} \longrightarrow S^{2n-1}$ of π and constants $\varepsilon(x) > 0$ such that the intervals

$$I(\widetilde{x}) := \{\Phi_{i\vec{N}}^t(s(\widetilde{x})), t \in \,]-\varepsilon(x), \varepsilon(x)[\}, \quad \widetilde{x} \in B_{\delta(x)}$$

contain no points of K. These balls $B_{\delta(x)}$ provide a cover of $\pi(K \backslash \mathcal{C})$, for which we now construct the functions h_x. For this, first identify $\pi^{-1}(B_{\delta(x)})$ with $B_{\delta(x)} \times S^1$ by the map $\varphi : B_{\delta(x)} \times S^1 \longrightarrow \pi^{-1}(B_{\delta(x)})$ defined by $\varphi(\widetilde{x}, t) := \Phi_{i\vec{N}}^{t+\varepsilon(x)}(s(\widetilde{x}))$. By φ^{-1}, $I(\widetilde{x})$ is taken to $]-2\varepsilon(x), 0[$ and $i\vec{N}$ to $\partial/\partial t$.

1564

In these coordinates, we can define the function h_x on $B_{\delta(x)} \times S^1$ by

$$\begin{cases} h_x(\widetilde{x}, t) = -t & \text{for } t \in [0, 1 - 2\varepsilon(x)], \\ h_x \text{ is smooth on } B_{\delta(x)} \times S^1. \end{cases}$$

Such a function can obviously be defined, and it fits with the requirement $i\vec{N} \cdot h_\alpha < 0$ on $K \backslash \mathcal{C} \cap \pi^{-1}(U_\alpha)$. $\quad\square$

COROLLARY 3.3 (Perturbation procedure for smooth balls). *Let M be a symplectic manifold, $B \subset M$ a smooth symplectic closed ball and U an open set of M. Assume that $U \cap B = \emptyset$, denote $K := \partial U \cap B$ and $\mathcal{C}$ the Hopf circles of ∂B contained in K. Then there is a Hamiltonian function H on M such that $\Phi^\varepsilon_{X_H}(B) \cap \overline{U} = \Phi^\varepsilon_{X_H}(B) \cap \partial U = \mathcal{C}$ for all small enough $\varepsilon > 0$.*

Proof. It is well known that there exists a symplectomorphism Φ between a neighbourhood $N(\partial B)$ of ∂B in M and a neighbourhood $N(S)$ of $S(r)$ in $\mathbb{C}^n$. This map sends K to $K' = \Phi(K)$, and $\mathcal{C}$ to $\mathcal{C}' = \Phi(\mathcal{C})$, where $\mathcal{C}'$ is exactly the set of Hopf circles of $S(r)$ contained in K'. Consider a function h on $S(r)$ associated to $(K', \mathcal{C}')$ as in Lemma 3.2. Extend it to a smooth function of $\mathbb{C}^n$ with compact support in $N(S)$. The function $H := h \circ \Phi$, *a priori* defined on $N(\partial B)$, can be extended to M by setting $H = 0$ outside $N(\partial B)$. As explained above, the vector field X_H points inside B on $K \backslash \mathcal{C}$ and vanishes on $\mathcal{C}$. $\quad\square$

This corollary is unfortunately only useful when dealing with smooth balls. In our context, B will usually be one of the (only regular) balls B_i and U the union of the other balls $\bigcup_{j \neq i} B_j$. We will prove the counterpart of this perturbation procedure for regular symplectic closed balls in the next section (see Proposition 4.3).

3.2 Symplectic spheres in maximal packings

We show here that any Hopf circle in the intersection of two symplectic balls along their boundaries gives rise to its own supporting sphere.

LEMMA 3.4. *Let B_1, B_2 be regular symplectic closed balls of a symplectic manifold (M, ω) with disjoint interior, of radii r_1, r_2 and centres O_1, O_2. Given any common Hopf circle C of ∂B_1 and ∂B_2, one can construct a topological 2-sphere S_C of M passing through C, O_1, O_2, smooth except along C and with symplectic area $\pi(r_1^2 + r_2^2)$. In dimension four, these spheres intersect precisely in O_1 and O_2 with intersection number 1 at each point.*

Proof. Let $\varphi_i : \overline{B(r_i)} \longrightarrow B_i$ be the corresponding symplectic embeddings and $C_i := \varphi_i^{-1}(C)$. Then C_i is a Hopf circle of $\partial B(r_i)$ and bounds a holomorphic disc $\mathbb{D}_{C_i} \subset B(r_i)$. Its image by φ_i is a symplectic disc D_i in B_i of area πr_i^2 passing through O_i and bounded by C (a Hopf disc). Since the interiors of B_1, B_2 are disjoint, when gluing D_1 with D_2 along their common boundary we get a topological 2-sphere S_C which is smooth except on C and has area $\mathcal{A}(S_C) = \pi r_1^2 + \pi r_2^2$. By construction, two such spheres S_C, $S_{C'}$ intersect only at O_1, O_2, and in dimension four their intersection numbers at O_i is 1 because they are the same as the intersection number of $\mathbb{D}_{C_i}$, $\mathbb{D}_{C_i'}$ in $B^4(r_i)$. $\quad\square$

3.3 Smooth maximal packings: proof of Theorem 1

Proof of Theorem 1(a). Fix a smooth maximal symplectic packing of $\mathbb{P}^2$ by two balls $\{B_1, B_2\}$ of radii r_1, r_2 verifying the maximality condition $r_1^2 + r_2^2 = 1$. Set $K := \partial B_1 \cap \partial B_2$ and let $\mathcal{C}$ be the set of Hopf circles of ∂B_1 contained in K. Note that such a circle is also a Hopf circle of ∂B_2 because ∂B_1 and ∂B_2 are tangent along K. The perturbation procedure of Corollary 3.3 shows that, after a possible Hamiltonian perturbation of the packing, we can assume that $K = \mathcal{C}$.

1565

Gromov's work [Gro85] shows that the balls of a smooth maximal packing of $\mathbb{P}^2$ by two balls cannot be disjoint (we also prove this in Proposition 4.1 in the more general setting of regular mappings). This implies that no Hamiltonian perturbation of our packing can lead to disjoint balls. In view of the perturbation procedure of Corollary 3.3, K must contain at least one Hopf circle of ∂B_1, so $\mathcal{C}$ is not empty. To prove that $\mathcal{C}$ is exactly one Hopf circle, we argue by contradiction and assume that $\mathcal{C}$ contains two circles C and C'. The topological 2-spheres S_C and $S_{C'}$ given by Lemma 3.4 intersect precisely at O_1, O_2 with intersection numbers 1 at each point. On the other hand, they both have symplectic area $\pi(r_1^2 + r_2^2) = \pi$ so they are in the homology class of a line L in $\mathbb{P}^2$. We thus have $2 = S_{C_1} \cdot S_{C_2} = L \cdot L = 1$, which is a contradiction. $\qquad\square$

Proof of Theorem 1(b,c). Let B_1, B_2, B_3 be the closed balls of our smooth maximal packing. We can assume that their radii satisfy $r_1 \geqslant r_2 \geqslant r_3$. Then the maximality condition is [Gro85]

$$r_1^2 + r_2^2 = r_1^2 + r_3^2 = 1.$$

As above, set $K_{ij} := B_i \cap B_j = \partial B_i \cap \partial B_j$ and let $\mathcal{C}_{ij}$ be the set consisting of the Hopf circles of K_{ij}. The first step is to prove that we can get rid of $K_{ij} \backslash \mathcal{C}_{ij}$ by a Hamiltonian perturbation. Unlike in the two-balls situation, this is not obvious at first glance because a Hopf circle of ∂B_1 could *a priori* be covered by $\partial B_2 \cup \partial B_3$ without being in any K_{ij}. But notice that the K_{ij} are pairwise disjoint compact sets (and remain so after perturbation) because the intersection of two of the K_{ij} is precisely $B_1 \cap B_2 \cap B_3$. This intersection must be empty because any point of it would be a singular point of at least one of the balls. Since Hopf circles are connected, we deduce that $K_{12} \cup K_{13}$ contains no Hopf circles besides those in $\mathcal{C}_{12}$ and $\mathcal{C}_{13}$. After a Hamiltonian perturbation of B_1 according to Corollary 3.3, we can assume that $K_{1i} = \mathcal{C}_{1i}$. After a second Hamiltonian perturbation, this time of B_2, we can also assume that $K_{23} = \mathcal{C}_{23}$.

Case 1: Theorem 1(b) ($r_1 = r_2 = r_3 = 1/\sqrt{2}$). In this case, any two of the balls B_1, B_2, B_3 form a maximal symplectic packing of $\mathbb{P}^2$ by two balls. As such, and in view of Theorem 1(a), $K_{ij} = \mathcal{C}_{ij}$ contains exactly one circle.

Case 2: Theorem 1(c) ($r_1 > r_2 = r_3$). In this case, (B_1, B_2) and (B_1, B_3) form maximal symplectic packings of $\mathbb{P}^2$ by two balls. As in the proof of Theorem 1(a), we conclude that $\mathcal{C}_{12}$ and $\mathcal{C}_{13}$ contain exactly one circle. Finally, if $\mathcal{C}_{23}$ would contain a circle, Lemma 3.4 would associate to it a toplogical 2-sphere of symplectic area $\pi(r_2^2 + r_3^2) \in\,]0, \pi[$, which is impossible, so $\mathcal{C}_{23}$ is empty. $\qquad\square$

Remark 3.5. This last argument shows that any two closed symplectic balls of radii r_1, r_2 in $\mathbb{P}^2$ with disjoint interiors can intersect along a full Hopf circle of the boundary of one of them (therefore of both) only if $r_1^2 + r_2^2 \in \mathbb{N}$.

Proof of Theorem 1(d). Suppose by contradiction that n closed balls $B_1, \ldots, B_n$ of the same radius r constitute a maximal packing of $\mathbb{P}^2$ ($n \geqslant 4$).

We know from [MP94] and [Bir99] that, for $n = 4$ and $n \geqslant 9$, such a packing fills the space. Since $B_1 \cup B_2$ does not fill the whole of $\mathbb{P}^2$, there exists a point $p \in \overline{\partial B_1 \cap \partial B_i} \backslash \mathrm{Int}_{\partial B_1} \partial B_1 \cap \partial B_i$ of the boundary of $\partial B_1 \cap \partial B_i$ in ∂B_1. Such a point belongs to a third ball B_j so our packing by $B_1, \ldots, B_n$ is not smooth, as already noted in the last paragraph. Moreover, the radius of the balls is not greater than $\sqrt{2/5} < \sqrt{1/2}$ for $n = 5, 6, 7, 8$. As before, we are in a non-removable intersection situation: the ball B_1 cannot be disjoint from the union of the other ones $\bigcup_{j \neq 1} B_j$ (see Proposition 4.1). In view of Corollary 3.3, this means that there is a Hopf circle C of ∂B_1 which is also covered by $\bigcup_{j \neq 1} \partial B_j$. Assume for instance that $C \cap \partial B_2 \neq \emptyset$. Since $r_1^2 + r_2^2 \leqslant 4/5 < 1$, Remark 3.5 shows that C cannot be entirely contained in ∂B_2. So there is a point p in the boundary of ∂B_2 in C. Such a point also belongs to another ball, so it is an intersection point of $\partial B_1, \partial B_2$ and ∂B_i for $i \neq 1, 2$. Our packing cannot be smooth. $\qquad\square$

3.4 A digression on Gromov's two-packing theorem

We end this section with a remark. The previous analysis on non-removable intersections provides an interesting interpretation of Gromov's theorem on two-packings. Consider a packing of $\mathbb{P}^2$ by two smooth balls $\{B_1, B_2\}$ of radii r_1, r_2. Assume that we know the following assertion to hold true.

$$\text{Every non-maximal two packing of } \mathbb{P}^2 \text{ extends to a smooth maximal packing.} \qquad (*)$$

Then from the previous analysis, we conclude that the maximal extensions $\{B_1', B_2'\}$ of $\{B_1, B_2\}$ intersect along a common Hopf circle of their boundaries, so their radii verify ${r_1'}^2 + {r_2'}^2 \in \mathbb{N}$ by Remark 3.5. Taking into consideration the volume restriction ${r_1'}^4 + {r_2'}^4 \leqslant 1$, we immediately get that ${r_1'}^2 + {r_2'}^2 = 1$, so $r_1^2 + r_2^2 \leqslant 1$. In this approach, the whole difficulty of the two-packing theorem thus lies on the assertion $(*)$. This assertion is *a posteriori* true, when knowing the two-packing theorem, Karshon's examples and McDuff's result on the connectedness of the space of symplectic packings in $\mathbb{P}^2$.

4. Regularizations and extensions of regular embeddings

This purely technical section is aimed at extending the perturbation procedure described in Corollary 3.3 to non-smooth balls. We first explain how to extend slightly a symplectic open ball in an open manifold M. It should be noticed that the following proposition applies to regular embeddings of a ball.

PROPOSITION 4.1. *Let $\varphi : B(r) \hookrightarrow M$ be a symplectic embedding of a Euclidean ball. Assume that φ extends smoothly to an embedding of $\overline{B(r)} \backslash K$ into M where K is a closed subset of $\partial B(r)$. If K contains no Hopf circle of $\partial B(r)$ then, for any open neighbourhood V of $\overline{\mathrm{Im}\,\varphi}$, there is a symplectic embedding $\widetilde{\varphi}_\varepsilon : B(r + \varepsilon) \hookrightarrow V$ for ε small enough.*

If $K = \emptyset$ then $\widetilde{\varphi}_\varepsilon$ can be chosen to be an extension of φ, i.e., $\widetilde{\varphi}_{\varepsilon \,|B(r)} = \varphi$.

The non-squeezing theorem shows that this proposition is sharp in the sense that K indeed has to be assumed not to contain any Hopf circle.

Proof. When the singular locus is empty $(K = \emptyset)$, this proposition is very classical. We nevertheless give its brief proof since it serves as a basis for the non-smooth case. In this situation, the image $\varphi(\partial B(r))$ is a smooth hypersurface of M. With $S := \partial B(r)$, the regular neighbourhood theorem gives a symplectomorphism $\psi : N_\varepsilon(S) \longrightarrow N(\varphi(S))$ from an ε-neighbourhood of the standard sphere of radius r in $\mathbb{C}^n$ to a neighbourhood of $\varphi(S)$ in M (see [Wei71] or [MS98, p. 101]). Moreover, this map can be chosen in such a way that $\psi_{|S} = \varphi_{|S}$ and $\psi'(x)\vec{N}(x) = \varphi'(x)\vec{N}(x)$ for any point $x \in S$ ($\vec{N}(x)$ again denotes $x/\|x\|$). The map $\varphi_\varepsilon : B(r + \varepsilon) \longrightarrow M$ defined by

$$\varphi_\varepsilon(x) := \begin{cases} \varphi(x) & \text{for } x \in B, \\ \psi(x) & \text{for } x \in N_\varepsilon(S) \backslash B \end{cases}$$

is $\mathcal{C}^1$-smooth, and hence a symplectic embedding of the ball of radius $r + \varepsilon$ into M.

For singular hypersurfaces, the standard neighbourhood theorem is not valid any more, hence the above proof has no straightforward generalization. However, it can be easily adapted by using a regularization trick explained in Lemma 4.2 below. It states that under the extension condition on φ in Proposition 4.1, there exists a 'regularization' $\widetilde{\varphi} : \overline{B(r)} \hookrightarrow M$ which is a symplectic embedding of the *closed* ball into the prescribed neighbourhood V of $\varphi(\overline{B(r)} \backslash K)$. Now the extension $\widetilde{\varphi}_\varepsilon$ of $\widetilde{\varphi}$ constructed in the case $K = \emptyset$ gives the desired map. $\square$

The result we have used in this proof is a particular case of the following lemma for $\mathcal{C} = \emptyset$. This more general version will nevertheless be useful in order to generalize Corollary 3.3.

LEMMA 4.2. *Let* $\varphi : \overline{B(r)}\backslash K \hookrightarrow M$ *be a symplectic embedding of a closed ball minus a singular compact set* $K \subset \partial B(r)$ *inside an open manifold* M. *Denote by* $\mathcal{C}$ *the set of Hopf circles of* $\partial B(r)$ *contained in* K. *There exists a symplectic embedding* $\widetilde{\varphi}$ *of* $\overline{B(r)}\backslash \mathcal{C}$ *into any given neighbourhood* V *of* $\operatorname{Im} \varphi = \varphi(\overline{B(r)}\backslash K)$.

Proof. Consider an open set $U \subset \partial B(r)\backslash K$ which contains at least one point of each Hopf circle of $\partial B(r)\backslash \mathcal{C}$. As in the previous proof, φ can be extended to a symplectic embedding of a shell $U_\varepsilon := \{[x, (1 + \varepsilon(x))x[, \ x \in U\}$, where ε is a small positive function on U. For ε sufficiently small, φ sends $B'_\varepsilon := B(r)\backslash K \cup U_\varepsilon$ into V. Lemma 3.2 shows that there exists a smooth function $h : \partial B(r) \longrightarrow \mathbb{R}$ such that $i\vec{N} \cdot h < 0$ on $\partial B(r)\backslash(U \cup \mathcal{C})$ with $dh = 0$ on $\mathcal{C}$. When extending this function to $\mathbb{C}^n$, we get a Hamiltonian function whose flow has the property that, for t small enough, the set $\Phi^t_{\vec{X}_h}(\overline{B(r)}\backslash \mathcal{C})$ is contained in B'_ε. The map

$$\widetilde{\varphi} := \varphi \circ \Phi^t_{\vec{X}_h} : \overline{B(r)}\backslash \mathcal{C} \hookrightarrow V$$

gives the desired symplectic embedding of $\overline{B(r)}\backslash \mathcal{C}$ into V. $\qquad\square$

Before stating the announced generalization of the perturbation procedure of Corollary 3.3 to regular balls, we need to broaden slightly the notion of Hamiltonian perturbation. We say that a path $\{\varphi_t, B_t\}$ of regular symplectic balls in M is a *regular Hamiltonian deformation* if there exist smooth functions H_t defined on the interior of B_t such that $\varphi_t = \Phi^t_{X_{H_t}} \circ \varphi$. The point is that we do not impose these functions to be defined in all of M, so they may have singularities at the boundary. However, we demand the whole path to be made of regular symplectic balls. Now, a regular Hamiltonian perturbation of a packing is a regular Hamiltonian perturbation of each of its balls such that the configuration of the balls at any time t remains a packing. To illustrate this definition, consider the special case of Lemma 4.2 when M is an open manifold with boundary ∂M and $\varphi : (B(r), K) \hookrightarrow (M, \partial M)$ is a regular symplectic ball with a part K of its boundary sent into ∂M. Then the map $\widetilde{\varphi}$ is in fact a regular Hamiltonian deformation of φ associated to the function $h \circ \varphi^{-1}$. Moreover, $\widetilde{\varphi}_{|\mathcal{C}} = \varphi_{|\mathcal{C}}$. This obvious remark is exactly the content of the next proposition.

PROPOSITION 4.3 (Perturbation procedure for regular balls). *Let* U *be an open set of a symplectic manifold* M *and* B *a regular symplectic ball in* $M\backslash U$. *Set* $K := \partial B\backslash\partial U$ *and let* $\mathcal{C}$ *be the set of Hopf circles of* ∂B *contained in* K. *Then there is a regular Hamiltonian perturbation* $(B_t)_{t<\varepsilon}$ *of* B *with* $B_t \cap \partial U = \mathcal{C}$ *for any positive* t.

As a corollary of Proposition 4.1, we get a non-removable intersection property for regular maximal symplectic packings. No ball of such a packing is completely disjoint from all other balls.

COROLLARY 4.4 (Non-removable intersection). *Let* $(B_i)_{i=1,\dots,k}$ *be a regular maximal symplectic packing of* M. *Then*

$$\text{for all } i = 1, \dots, k, \quad B_i \cap \left(\bigcup_{j \neq i} B_j \right) \neq \emptyset.$$

Moreover, this intersection contains at least one Hopf circle of ∂B_i.

Proof. Call φ_i the symplectic embeddings of $B(r_i)$ into M corresponding to B_i. Suppose by contradiction that $K := B_i \cap (\bigcup_{j \neq i} B_j)$ contains no Hopf circle of ∂B_i. The map φ_i is a regular symplectic embedding of $\overline{B(r_i)}\backslash K$ into the open symplectic manifold $M' = M\backslash \bigcup_{j \neq i} B_j$. By Proposition 4.1, φ_i can be 'extended' to a symplectic embedding $\widetilde{\varphi}_i : B(r_i + \varepsilon) \hookrightarrow M'$. The maps $\varphi_1, \dots, \widetilde{\varphi}_i, \dots, \varphi_k$ thus provide a symplectic packing of M by k balls of radii $(r_1, \dots, r_i + \varepsilon, \dots, r_k)$. This contradicts the definition of a maximal symplectic packing. $\qquad\square$

5. Non-smooth symplectic packings

We turn to the problem of identifying non-smooth (but regular) maximal symplectic packings. We first prove the existence of supporting surfaces of such packings (Theorem 3). We then remark that passing from Theorem 3 to the precise statement of Theorem 4 is a matter of being able to perturb the topological supporting surface to a smooth symplectic surface (Lemma 5.1). We finally show that the smoothening is possible under the hypothesis of Theorem 4, thus proving it.

5.1 Existence of a supporting surface: proof of Theorem 3

The idea of the proof is very simple. The non-removable intersection property of the packing implies the existence of a Hopf circle C_1 of ∂B_1 contained in the union of the other balls. Let us denote by $\mathcal{C}_{1i}$ the set of circles of B_i intersecting C_1 along an open set ($\mathcal{C}_{1i}$ may be empty or contain several circles). Denote by S_0 the Hopf disc in B_1 bounded by C_1 and by S_1 the surface obtained by gluing the Hopf discs corresponding to $\mathcal{C}_{1i}$ to S_0. Obviously no point of C_1 is a boundary point of S_1. Now there are two possibilities. Either the $\mathcal{C}_{1i}$ are covered by the balls B_j for $j \neq i$ or not. In the first case, we can glue to S_1 the Hopf discs corresponding to the circles of $\mathcal{C}_{1ij}$ ($\mathcal{C}_{1ij}$ is the set of circles of ∂B_j intersecting a circle of $\mathcal{C}_{1i}$ in an open set). We obtain a surface S_2 with boundary points neither in C_1 nor in C_{1i}, which we can use in order to iterate the construction. In the latter case, we can get rid of one of the circles of $\mathcal{C}_{1i}$ from the intersections between the balls by a small Hamiltonian perturbation. The effect of this transformation is that C_1 is not covered by the other balls any more. It actually means that C_1 was not relevant for our purpose, but there clearly exists another circle of ∂B_1 to which we can apply the previous procedure. The reason why this iteration process stops and produces a closed surface is the finiteness condition on the singularities. In order to make the preceding iterative process rigorous and clear, we encode the situation into a graph.

Let $B_1, \ldots, B_k$ be the balls of a regular maximal symplectic packing of M. We already know how to associate a supporting topological sphere to any common Hopf circle of two balls of the packing (see Lemma 3.4). We thus consider in the following only those Hopf circles not concerned by this basic construction. Define

$$\mathcal{S}_i := \{x \in \partial B_i \mid \exists j \neq i, C_x \subset \partial B_j\},$$

$$\mathcal{C}_i := \left\{ x \in \partial B_i \mid C_x \subset \bigcup_{j \neq i} \partial B_j, C_x \not\subset \partial B_j \ \forall j \neq i \right\},$$

where C_x denotes the possibly singular Hopf circle of ∂B_i passing through x. Clearly, if one of the $\mathcal{C}_i$ is empty, then Corollaries 4.4 and 3.4 show that there must be a supporting 2-sphere of the packing passing through B_i. As before, our aim is to explain to what $\mathcal{C}_i$ can be reduced after Hamiltonian perturbation of the packing. Notice that, by definition, each circle of $\mathcal{C}_i$ contains at least one triple intersection point between the balls of the packing, and there are only finitely many such points because of the regularity condition on the packing. Since any two Hopf circles of a given ball are disjoint, each $\mathcal{C}_i$ thus contains only a finite number of Hopf circles of ∂B_i. Consider henceforth the finite graph G whose vertices are the Hopf circles contained in one of the $\mathcal{C}_i$, and the edges are the pairs of such circles which share an open arc. Also colour the vertices black when they represent a circle $C \in \mathcal{C}_i$ which is also contained in $\bigcup_{j \neq i} \mathcal{C}_j$ and red otherwise.

A red vertex is a Hopf circle C of ∂B_i which is covered by the other balls but not by the union of the $\mathcal{C}_j$ for $j \neq i$. Since each $\mathcal{C}_j$ is compact, there must be an open arc $I \subset C$ which is a piece of a Hopf circle of ∂B_j, not covered itself by the other balls of the packing. The perturbation procedure then allows us to produce a new packing $(\widetilde{B}_1, \ldots, \widetilde{B}_k)$ very close to the original one, with intersection between the balls unchanged except that $\widetilde{B}_i \cap \widetilde{B}_j = B_i \cap B_j \backslash I$. The graph $\widetilde{G}$ associated to the perturbed maximal packing is thus a subgraph of G obtained by erasing the vertex C together with

1569

all its adjacent edges, and turning all its neighbours in G to red. In particular, $\widetilde{G}$ has one vertex less than G. This process can be iterated as long as there is a red vertex in the graph. Because the initial graph is finite, there must be a stabilization after a finite number of steps. We conclude that some Hamiltonian perturbation of the packing leads to a graph which is only black. We will now suppose that G itself has only black vertices. Applying the perturbation procedure once more, we can arrange that

$$B_i \cap \bigcup_{j \neq i} B_j = \mathcal{C}_i \cup \mathcal{S}_i.$$

We claim that each connected component $\hat{G}$ of G corresponds to a supporting surface. Actually, let S be the union of the Hopf discs corresponding to the Hopf circles of $\hat{G}$. It is obviously a connected space, covered by the closed balls of the packing. We need however a brief discussion of the regularity of this space before we can assure that it is a topological surface. Inside the balls, S is an immersed symplectic surface, whose only self-intersection points are positive and located at the origin of the balls. Consider now a point x of $S \cap \partial B_i$ which is not a singular point of any ball. In particular it is not a triple intersection point of the packing, so S cannot be made of more than two discs in a neighbourhood. Moreover x belongs to a Hopf circle of $\mathcal{C}_i$. The fact that this Hopf circle is black coloured implies that x is in the boundary of at least two Hopf discs. It follows that S is locally made of exactly two smooth Hopf discs glued along an interval around x. It is easily checked that S is even locally diffeomorphic to a cylinder $\{y = |x|\} \times \mathbb{R}$ at these points. We thus conclude that S is indeed a closed topological surface with finitely many possible singular points located at the singular points of the packing.

5.2 Refinement of Theorem 3 when the supporting surfaces are smooth

In the concrete case of five balls in $\mathbb{P}^2$, we explain now how to sharpen Theorem 3 and get Theorem 4. We consider for the remainder of this paragraph a regular maximal symplectic packing of $\mathbb{P}^2$ by five equal balls $\varphi_1, \ldots, \varphi_5$ (or $B_1, \ldots, B_5$), of radii $\sqrt{2/5}$, and a supporting surface S of the packing. Then S is made of several Hopf discs, each of which is of area $2/5\pi$ because its boundary is a Hopf circle of the boundary of a ball. Since the Fubini–Study form is integral and S is a closed surface, the number of discs is a multiple $5k$ of 5, the symplectic area of S is $2k\pi$, and its homology class is $2k[L]$ (L is a line in $\mathbb{P}^2$).

LEMMA 5.1. *Assume that there is a smooth symplectic immersion $\mathcal{C}^0$-close to S, whose only self-intersections are at the origins of the balls. Then S is of area 2π, made of one Hopf disc in each ball of the packing, and it is the unique supporting surface of the packing.*

Proof. Denote by k_i the number of Hopf discs of $S \cap B_i$, so that $k_1 + k_2 + \cdots + k_5 = 5k$. The self-intersection of S near the origin O_i of B_i (namely the number of double points of generic perturbations of S near O_i) is then given by

$$\delta_i = \frac{k_i(k_i - 1)}{2}.$$

The assumption on the smoothening of S means that one can find a symplectically immersed surface $\widetilde{S}$ homologous to S (hence in the homology class $2k[L]$) with positive self-intersections. They are located in small neighbourhoods of the O_i and are the same as those of $S \cap B_i$. The total self-intersection number of $\widetilde{S}$ is thus

$$\delta := \sum_{i=1}^{5} \delta_i = \sum_{i=1}^{5} \frac{k_i(k_i - 1)}{2}.$$

Taking into account that $\sum k_i = 5k$, we easily get that

$$\delta \geqslant \frac{5k(k-1)}{2}. \tag{3}$$

The positivity of the self-intersections together with the fact that $\widetilde{S}$ is symplectic imply that $\widetilde{S}$ is actually a J-holomorphic curve for an almost complex structure on $\mathbb{P}^2$ (see [AL94]). It must therefore satisfy the adjunction inequality, which gives in our present situation [MS98]:

$$\delta \leqslant \frac{(2k-1)(2k-2)}{2} = (2k-1)(k-1). \tag{4}$$

It follows from (3) and (4) that $k = 1$ and $\delta = 0$. We thus conclude that S was made of five Hopf discs, one in each ball. Finally, we argue by contradiction to prove that S is the only supporting surface of the packing. Assume that there is a supporting surface S' distinct from S, made of k_i' Hopf discs in each ball, and of total symplectic area $2k'\pi$ (so that $\sum k_i = 5k'$). Since the only intersection points between S and S' are the centres of the balls, we get

$$4k' = 2L \cdot 2k'L = S \cdot S' = \sum_{i=1}^{5} S \cdot_{O_i} S' = \sum_{i=1}^{5} k_i \cdot 1 = 5k'.$$

This is the desired contradiction. $\qquad\square$

Note that the previous computation can be made for seven (respectively eight) balls. When they are symplectically smoothable in the previous sense, the supporting surfaces are at most seven (respectively eight), all of area 3π (respectively 6π). Each one intersects six of the balls through one Hopf disc and the last one through two Hopf discs (respectively intersects seven of the balls through two Hopf discs and the last one through three).

5.3 Smoothening of the supporting surfaces for packings of simple type

In this paragraph, we show that the smoothening required by Lemma 5.1 can be achieved when precise conditions on the singularities of the packings are given. Recall that the singularities of S are the union of a finite set of singularities of the packing, segments of the characteristic foliations joining precisely these points, and more exceptionally full Hopf circles. The first step is to deal with the generic singular points of S.

LEMMA 5.2. *Let S be a symplectic surface singular along a segment or a circle Γ. Assume that a neighbourhood of Γ in S is*

(i) *either globally diffeomorphic to*

$$\{y = \alpha(z)|x|\} \subset \mathbb{R}^2(x,y) \times]0,1[(z), \tag{5}$$

   where α is a continuous function which vanishes at 0 and 1 if Γ is a segment,

(ii) *or locally diffeomorphic to $\{y = \alpha(z)|x|\}$, where α is continuous if Γ is a circle. Then S can be smoothened to a symplectic surface by a $\mathcal{C}^0$-perturbation.*

Proof. Consider first the case of a segment Γ of singularities. Using Moser's argument, a neighbourhood of Γ can be presented symplectically as the cylinder

$$V_\varepsilon := \{|x_1| < 1 + \varepsilon, |y_1| < \varepsilon, |z_2| < \varepsilon\} \subset \mathbb{C}^2(z_1 = x_1 + iy_1, z_2)$$

in such a way that Γ corresponds to $[-1,1] \times \{0\}$ and S corresponds to a union of two symplectic surfaces S_1, S_2 with common boundary Γ. Since S_1 and S_2 are symplectic, the identification can be done so that also $T_q S_1 = \mathrm{Span}_{\mathbb{R}}(\partial/\partial x_1, \partial/\partial y_1)$ and $T_q S_2 = \mathrm{Span}_{\mathbb{R}}(\partial/\partial x_1, -\partial/\partial y_1 - u(x_1)\, \partial/\partial z_2)$ ($q \in \Gamma$) where $u(x_1)$ is a continuous complex-valued function along Γ which vanishes for $x_1 \notin]-1,1[$.

1571

If the neighbourhood of Γ is small enough, S_1 can be straightened to the strip $A = \{|x_1| < 1 + \varepsilon,$ $0 \leqslant y_1 < \varepsilon, z_2 = 0\}$ by a map h which is $\mathcal{C}^1$-close to the identity (and even tangent to the identity along Γ). This map may distort ω but by no more than an ε-factor. We produce our symplectic smoothening of S by cutting S_1 and replacing it by a 'very symplectic' surface $\Sigma_u \subset V_\varepsilon$ which interpolates smoothly between $(\Gamma, \mathrm{Span}_\mathbb{R}(\partial/\partial x_1, \partial/\partial y_1 + u(x_1)\,\partial/\partial z_2))$ and A.

To this purpose, we consider a smooth 'profile' of maps $\varphi_v : [0, \varepsilon[\longrightarrow \mathbb{C}$ parameterized by $\mathbb{C}$, such that $\varphi_v(0) = 0$, $\dot{\varphi}_v(0) = v$, $\varphi_v \equiv 0$ on $[\varepsilon/2, \varepsilon[$ and $\varphi_0 \equiv 0$. Up to shrinking these maps, we can also arrange that $\partial\varphi_v/\partial v$ and $\dot{\varphi}_v$ are very small on big compact sets (for v). Then the surface

$$\Sigma_u := \{(x_1, y_1, \varphi_{u(x_1)}(y_1)), |x_1| < 1 + \varepsilon, 0 \leqslant y_1 < \varepsilon\} \subset \mathbb{C}^2$$

obviously interpolates ($\mathcal{C}^1$-)smoothly between $(\Gamma, \partial/\partial y_1 + u(x_1)\,\partial/\partial z_2)$ and A. Observe moreover that the tangent vectors to Σ_u,

$$v_1 := \frac{\partial}{\partial x_1} + \left(\frac{\partial\varphi_v}{\partial v} \cdot \frac{\partial u}{\partial x_1}\right)\frac{\partial}{\partial z_2} \quad \text{and} \quad v_2 := \frac{\partial}{\partial y_1} + \left(\frac{\partial\varphi_v}{\partial y_1}\right)\frac{\partial}{\partial z_2},$$

are small perturbations of $\partial/\partial x_1$ and $\partial/\partial y_1$ respectively provided that φ is sufficiently small in the $\mathcal{C}^1$-norm. The tangent planes to Σ_u are thus far from being Lagrangian.

If Γ is now a circle of singularities, consider a subsegment $\Gamma_1 \subset \Gamma$. As above, a neighbourhood of Γ_1 in S is the union of two smooth strips S_1 and S_2 glued along $\Gamma_1 = [-1, 1] \times \{0\} \subset \mathbb{C}^2$. Their tangent planes along Γ are $T_q S_1 = \mathrm{Span}_\mathbb{R}(\partial/\partial x_1, \partial/\partial y_1)$ and $T_q S_2 = \mathrm{Span}_\mathbb{R}(\partial/\partial x_1, -\partial/\partial y_1 - u(x_1)\partial/\partial z_2)$, where $u(x_1)$ is a continuous complex-valued function. The only difference with the previous situation is that u may not vanish at the boundary of Γ_1. Consider any smooth function $v : [-1, 1] \longrightarrow \mathbb{C}$ vanishing at -1 and 1, which coincides with u on $\Gamma_2 = [-1/2, 1/2]$. Modifying S exactly as above by Σ_v (instead of Σ_u), we obtain a symplectic $\mathcal{C}^0$-perturbation $\widetilde{S}$ of S which is no longer singular on the circle Γ but on the segment $\Gamma\backslash\Gamma_2$. By construction, the singularity of $\widetilde{S}$ has the form (5) and a $\mathcal{C}^0$-perturbation of $\widetilde{S}$ is a smooth symplectic surface. $\square$

Hence all the difficulty of our desingularization problem concentrates at the (unavoidable) intersection of S with singular points of the packing. To understand the situation in the greatest generality, we need local models for the singularities that may arise in regular packings. We do not pretend to find them in this paper. Instead, and rather as an illustration, we focus on the very special type of singularities which appear in the examples we constructed in § 2.

DEFINITION 5.3. We say that a boundary singularity $p = \varphi(q)$ of a symplectic ball $\{\varphi, B\}$ is *simple* if φ is continuously differentiable at q, with non-vanishing derivatives in any directions but the characteristic one. A regular packing is said to be *of simple type* if all the singularities are simple, and if there exists no intersection point between any four balls (we only allow triple intersection points).

Before proving Theorem 4, let us discuss the meaning of this definition in our context. Consider a regular symplectic ball $\{\varphi, B\}$ of a symplectic manifold and a simple singularity p of B. Denote by C_p and D_p the Hopf circle and disc passing through p. For $q = \varphi(\eta) \in C_p$, note also that $T^c(q) := \varphi'(\eta) \cdot T_\eta^\mathbb{C} \partial B$ and $\pi(q) := T^c(q)^{\perp\omega}$. Then D_p is tangent to $\pi(q)$ along C_p and, from the definition of a simple singularity, both symplectic plane distributions $T^c(q)$ and $\pi(q)$ have a well-defined limit $T^c(p)$ and $\pi(p)$ when q goes to p. In particular, D_p has a tangent plane $\pi(p)$ at p. Although $T_q C_p$ may not have a limit at p, C_p is tangent to $\pi(p)$ at p. Observing that the tangent plane to ∂B at $q \in C_p$ is $T^c(q) \oplus T_q C_p$ we see that $\partial B \cap U$ is $\mathcal{C}^0$-close to the cylinder $\pi(C_p) \times T^c(p)$ (and $B \cap U$ is close to $D_p \times T^c(p)$). This remark immediately yields the following lemma.

LEMMA 5.4. *Let B be a symplectic ball with a simple singularity at $p \in \partial B$. With the notation above, there is a neighbourhood U of p such that the linear projection $\pi : \overline{D_p} \cap U \longrightarrow \pi_p$ along $T^c(p)$ is an injective map.*

1572

This lemma prevents D_p from spiralling too much above its tangent plane, creating problematic cone singularities. The proof of Theorem 4 is achieved in two steps. First, we smoothen the supporting surface at each (simple type) singularity of the packing. Being cautious enough in the first step allows us to use Lemma 5.2 to get rid of the remaining circles or segments of singularities.

Proof of Theorem 4. Consider a maximal symplectic packing $\{B_1, \ldots, B_k\} \subset \mathbb{P}^2$ of simple type, and one of its supporting surfaces S. Let $\{p_1, \ldots, p_n\}$ be the singularities of the packing which belong to S. Then the singularities of S consist of the points p_i themselves, together with open segments of the characteristic foliation linking the p_i. The differentiable model of the singularities along these segments is locally $\{y = |x|\} \times \mathbb{R} \subset \mathbb{R}^3$. As announced, we are going to smoothen S near the p_i in such a way that the remaining segments of singularities have the global form (5). Consider henceforth one of the singular points $p := p_i$. Recall that p can only be an intersection between two or three balls. Assume first that p lies in the intersection between two balls only, say B_1 and B_2. Let us distinguish between two cases.

Case (a): $\pi_1(p) = \pi_2(p)$, $T_1^c(p) = T_2^c(p)$. All indices refer to the balls of the packing (for instance, $\pi_1(p)$ is the plane defined above for the ball B_1). Consider local symplectic coordinates taking p to the origin in $\mathbb{C}^2$ and $\pi(p)$ to $\{z_2 = 0\}$. Inside a small bidisc $Q_\varepsilon := \{|z_1| < \varepsilon\} \times \{|z_2| < \varepsilon\}$, the surface S together with its tangent planes are very close to $\{z_2 = 0\}$. The projection $\pi : S \cap Q_\varepsilon \longrightarrow \{z_2 = 0\}$ is therefore a covering map, which must be injective by Lemma 5.4. The intersection of S with ∂Q_ε is therefore the graph over $\partial \mathbb{D}_\varepsilon$ of a piecewise $\mathcal{C}^1$-smooth complex-valued function with small $\mathcal{C}^1$-norm. This function can obviously be extended to $\mathbb{D}_\varepsilon$ in such a way that its graph $\Sigma \subset Q_\varepsilon$ is tangent to S on ∂Q_ε (except at the singularities of $S \cap \partial Q_\varepsilon$), and that its singularities are located on segments inside $\{\varepsilon/2 < |z_1| < \varepsilon\}$ and have the form of Lemma 5.2. Moreover, this extension can be chosen $\mathcal{C}^1$-small, so that Σ remains a symplectic surface. Cutting $S \cap Q_\varepsilon$ and replacing it by Σ thus gives the desired smoothening of S at p.

Case (b): $\pi_1(p) \neq \pi_2(p)$. Consider a small neighbourhood U of p such that the common Hopf circle $C_p := C_{p1} = C_{p2}$ of ∂B_1, ∂B_2 passing through p is the union of two smooth (open) arcs ℓ_l and ℓ_r meeting at p. Assume also that $\pi_1(q) \neq \pi_2(q)$ for all $q \in \ell_l \cup \ell_r$. Inside U we have $T_q C_p = \pi_1(q) \cap \pi_2(q)$ so that ℓ_l and ℓ_r have common tangency $\pi_1(p) \cap \pi_2(p)$ at p. Since $D_i(p)$ has limit tangent plane $\pi_i(p)$ at p, if $\ell_l \cup \ell_r$ is $\mathcal{C}^1$-smooth then the model of the singularity of S at p is the same as at any generic point of C_p, and we can forget it. Otherwise $\ell_l \cup \ell_r$ is a $\mathcal{C}^1$-cusp, meaning that its projection to $\pi_1(p) \cap \pi_2(p)$ is a half-line (see Figure 3). Then $C_p \times T_1^c(p)$ $(= C_p \times T_2^c(p))$ separates U into two cylinders, a big one (with aperture 2π at p) and a small one, in the neighbourhood of which each ball is contained. To fix the ideas, suppose that B_1 is the 'big ball' in U, and consider symplectic coordinates in U such that $p = 0$, $\pi_1(p) = \{z_2 = x_2 + iy_2 = 0\}$ and $\pi_1(p) \cap \pi_2(p) = \mathrm{Span}_\mathbb{R}(\partial/\partial y_1)$. Note that $\pi_2(p)$ is transverse to $\{z_1 = 0\}$ because it is symplectic and contains $\partial/\partial y_2$. So the projection of S on $\pi_1(p)$ along $T_1^c(p)$ inside a small bidisc Q_ε is a covering map, injective by Lemma 5.4. The intersection $\partial Q_\varepsilon \cap S$ is henceforth the graph of a piecewise $\mathcal{C}^1$-smooth complex-valued function over $\partial \mathbb{D}_\varepsilon$. It is easy to see that this map has bounded derivatives and small $\mathcal{C}^0$-norm. The same procedure as in case (a) above produces the smoothening.

Assume now that p lies in the boundary of three balls B_1, B_2, B_3. In a sufficiently small neighbourhood U of p, each intersection $B_i \cap B_j \cap U$ is a smooth open arc ℓ_{ij} ending at p. Reasoning as in cases (a) and (b), it is easy to smoothen S at p when $\pi_1(p) = \pi_2(p) = \pi_3(p)$ or $\pi_1(p) = \pi_2(p) \neq \pi_3(p)$. Thus it only remains to investigate the situation of three different tangent planes for $D_i(p)$ at p (see Figure 4). The curves ℓ_{ij} then have well-defined tangencies at p:

$$T_p \ell_{ij} = \pi_i(p) \cap \pi_j(p).$$

Consider a parameterization of the ℓ_{ij} by smooth maps $\gamma_{ij} : [0, \varepsilon[\longrightarrow U \subset \mathbb{C}^2$ with $\gamma_{ij}(0) = p$. Notice that $\omega(\dot{\gamma}_{ij}(0), \dot{\gamma}_{jk}(0)) > 0$ because $\pi_j(p) = \mathrm{Span}_\mathbb{R}(\dot{\gamma}_{ij}(0), \dot{\gamma}_{jk}(0))$ is a symplectic plane. Applying a
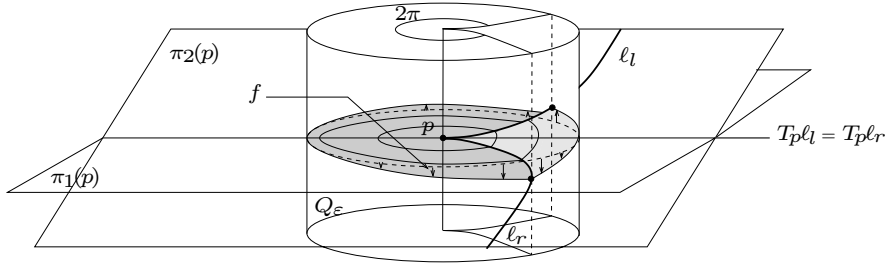
1573

FIGURE 3. Singularity of type $\pi_1(p) \neq \pi_2(p)$ and C_p is not smooth.

symplectic linear change of coordinates and a rescaling of the parameterizations, we can suppose that

$$\dot\gamma_{12}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \dot\gamma_{23}(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \dot\gamma_{31}(0) = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \subset \mathbb{C}^2 \approx \mathbb{R}^4.$$

Looking sufficiently close to p, there is a diffeomorphism Φ that is $\mathcal{C}^1$-close to the identity taking S to the cone over the triangle:

$$\Sigma := \left\{ \begin{pmatrix} z\varphi_1(\theta) \\ z \\ 0 \end{pmatrix}, 0 < h < 1 \right\},$$

where $\varphi_1(\theta)$ is the counterclockwise parameterization of the triangle T_1 spanned by the points $(1, 0)$, $(0, 1)$, $(-1, -1)$ of $\mathbb{R}^2$. Consider now the polar parameterizations φ_ρ of a family of convex curves T_ρ with $T_\rho = \{x^2 + y^2 = \rho^4\}$ for $\rho \ll 1$, $T_\rho = \rho \cdot T_1$ for $\rho \simeq 1$ and the angles of the singularities of T_ρ vary smoothly with ρ (see Figure 4). Then the surface

$$\widetilde{\sigma} := \{(\varphi_h(\theta), h, 0)\}$$

is smooth near the origin, has tangent planes far from being Lagrangian and coincides with Σ in a neighbourhood of $h = \{y_2 = 1\}$. Cutting Σ from S and replacing it by $\Phi^{-1}(\widetilde{\Sigma})$, we get a symplectic smoothening of S at p. We were also sufficiently cautious in the extrapolation from T_1 to T_ρ, $\rho \simeq 0$, to ensure that the remaining singularities have the form (5).

After a small perturbation of S, we thus obtain a symplectic surface with singularities along circles or segments, of the type required to apply Lemma 5.2. The surface S can therefore be perturbed to a smooth symplectic surface, and applying Lemma 5.1 proves Theorem 4. □
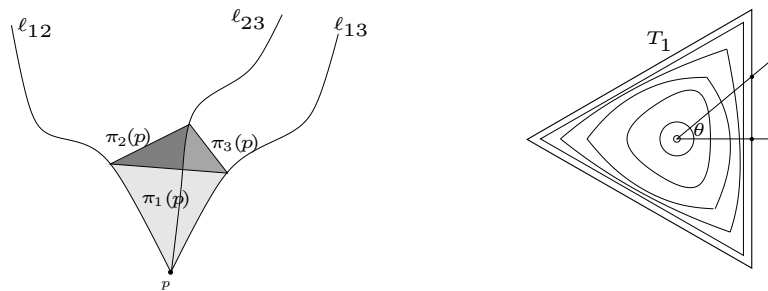
Figure 4. Singularity of S and extrapolation between T_1 and 0.

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Emmanuel Opshtein   opshtein@math.u-strasbg.fr

IRMA, 7 rue René Descartes, 67084 Strasbourg cedex, France