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DECOMPOSITIONS OF SUBMEASURES

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In [**4**] we showed that one can tell whether a submeasure on a Boolean algebra has a control measure or is pathological by comparing the Fréchet-Nikodým topology it generates to the universal measure topology of Graves. We then wondered if a submeasure could be decomposed into a part with a control measure and a part which is pathological or zero. This led to the problem of finding a Lebesgue decomposition for a submeasure on an algebra of sets with respect to a Fréchet-Nikodým topology.

In [**6**] Drewnowski proved a Lebesgue decomposition theorem for exhaustive submeasures with respect to "additivities" and a similar theorem for exhaustive Fréchet-Nikodým topologies. He asked if an exhaustive Fréchet-Nikodým topology could be decomposed with respect to another Fréchet-Nikodým topology. In [**12**] Traynor showed that the answer is "yes".

Here we prove a Lebesgue decomposition theorem for exhaustive submeasures with respect to Fréchet-Nikodým topologies generalizing Drewnowski's results and deriving Traynor's theorem as a corollary. We then discuss the control measure problem of Maharam. A special case of our decomposition theorem, viewed in the light of [**4**], is a decomposition of an exhaustive submeasure into a part with a control measure and a part which is pathological or zero. We show that the part with a control measure has a control measure it dominates. Finally we give a counterexample to show that the hypothesis of exhaustivity is necessary.

1. Preliminaries. Let $\mathscr{A}$ be an algebra of subsets of a nonempty set X. We assume that $\mathscr{A}$ separates points. A *submeasure* on $\mathscr{A}$ is a map $\lambda:\mathscr{A}\to[0,\infty)$ such that

(1) $\lambda(\emptyset) = 0,$

(2) $\lambda(A) \leqq \lambda(B)$ whenever $A \subseteq B$ in $\mathscr{A}$,

(3) $\lambda(A \cup B) \leqq \lambda(A) + \lambda(B)$ for all A and B in $\mathscr{A}$.

Call λ *exhaustive* if $\lambda(A_n) \to 0$ whenever (A_n) is a disjoint sequence in $\mathscr{A}$.

A *Fréchet-Nikodým* (FN) *topology* on $\mathscr{A}$ is a topology making the map $(A, B) \to A \,\Delta\, B$ from $\mathscr{A} \times \mathscr{A}$ to $\mathscr{A}$ continuous and making the map

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$A \to A \cap B$ continuous at $\emptyset$ uniformly for B in $\mathscr{A}$. An FN topology on $\mathscr{A}$ makes $\mathscr{A}$ into a topological group in which intersection is uniformly continuous. FN topologies were introduced by Drewnowski [**5**].

For submeasures λ and μ, say $\lambda \leqq \mu$ if $\lambda(A) \leqq \mu(A)$ for all A in $\mathscr{A}$. Then the set of all submeasures on $\mathscr{A}$ is a Dedekind-complete lattice. Let E be a nonempty family of submeasures. Then

$$\left(\bigvee_{\lambda \in E} \lambda\right)(A) = \sup_{\lambda \in E} \lambda(A)$$

if E is bounded above, and

$$\left(\bigwedge_{\lambda \in E} \lambda\right)(A) = \inf \{\lambda_1(A_1) + \ldots + \lambda_n(A_n) \mid$$

$$(A_i)_{i=1}^{n} \text{ is a finite partition of } A \text{ and } \lambda_1, \ldots, \lambda_n \in E\}.$$

In particular,

$$(\lambda \vee \mu)(A) = \max \{\lambda(A), \mu(A)\}$$

and

$$(\lambda \wedge \mu)(A) = \inf_{B \subseteq A} \{\lambda(B) + \mu(A \backslash B)\}.$$

The set of all FN topologies on $\mathscr{A}$, ordered by inclusion, forms a complete lattice, whose greatest element is the discrete topology D and least element the indiscrete topology O. Let E be a nonempty family of FN topologies on $\mathscr{A}$. Then $\bigvee_{G \in E} G$ is the usual supremum topology [**13**] and

$$\bigwedge_{G \in E} G = \vee \{H \mid H \text{ is an FN topology and } H \subseteq G \text{ for all } G \in E\}.$$

If λ is a submeasure on $\mathscr{A}$, we may define a semimetric d_λ on $\mathscr{A}$ by

$$d_\lambda(A, B) = \lambda(A \,\Delta\, B).$$

Then the semimetric topology G_λ is an FN topology on $\mathscr{A}$.

1.1. PROPOSITION. *The map $\lambda \to G_\lambda$ is a lattice homomorphism.*

Proof. This is straightforward.

Drewnowski [**5**] has observed that every FN topology is generated by a family of submeasures. We give the details for the sake of completeness.

1.2. THEOREM. *Let G be an* FN *topology on $\mathscr{A}$. (1) For every G-neighborhood U of $\emptyset$ there exist a submeasure λ with $G_\lambda \subseteq G$ and $\delta > 0$ such that*

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$$\{A \in \mathscr{A} | \lambda(A) < \delta\} \subseteq U.$$

(2) $G = \vee\{G_\lambda | G_\lambda \subseteq G\}$.

Proof. (1) Let U be a G-neighborhood of $\emptyset$. Since G is a commutative group topology there is a continuous invariant semimetric ρ on $\mathscr{A}$ with $\rho \leqq 1$ and $\delta > 0$ such that

$$\{A \in \mathscr{A} | \rho(A, \emptyset) < \delta\} \subseteq U.$$

For A in $\mathscr{A}$, put

$$\lambda(A) = \sup_{B \subseteq A} \rho(B, \emptyset).$$

Then it is easy to see that λ is a submeasure on $\mathscr{A}$, $G_\lambda \subseteq G$, and

$$\{A \in \mathscr{A} | \lambda(A) < \delta\} \subseteq U.$$

(2) follows immediately from (1).

Of course, $G_\lambda \subseteq G$ just means that λ is G-continuous on $\mathscr{A}$.

For submeasures λ and μ on $\mathscr{A}$, say that λ is *μ-continuous* if for every $\epsilon > 0$ there is $\delta > 0$ such that $\lambda(A) < \epsilon$ whenever $\mu(A) < \delta$. Call λ and μ *equivalent* and write $\lambda \sim \mu$ if λ is μ-continuous and μ is λ-continuous. Say that λ and μ are *singular* or *topologically orthogonal* and write $\lambda \perp \mu$ if for every $\epsilon > 0$ there is A in $\mathscr{A}$ such that $\lambda(A) < \epsilon$ and $\mu(X \backslash A) < \epsilon$. Then λ is μ-continuous if and only if $G_\lambda \subseteq G_\mu$, $\lambda \sim \mu$ if and only if $G_\lambda = G_\mu$, and $\lambda \perp \mu$ if and only if $\lambda \wedge \mu = 0$ if and only if $G_\lambda \wedge G_\mu = O$. Also note that $\lambda + \mu \sim \lambda \vee \mu$.

We end this section with two useful results.

1.3. PROPOSITION. *Let (λ_n) be a sequence of submeasures which is bounded above. Then*

$$\vee G_{\lambda_n} = G_\lambda, \text{ where } \lambda = \sum \frac{1}{2^n} \lambda_n.$$

Proof. This is straightforward.

1.4. THEOREM. *Let G be an* FN *topology on $\mathscr{A}$. Then the set I_G of all* FN *topologies H on $\mathscr{A}$ such that $G \wedge H = O$ is a complete ideal in the lattice of* FN *topologies.*

Proof. If $H_1 \subseteq H_2$ and H_2 is in I_G, then clearly H_1 is in I_G.

Next notice that if λ, λ_1 and λ_2 are submeasures and $\lambda \perp \lambda_1$ and $\lambda \perp \lambda_2$, then $\lambda \perp \lambda_1 \vee \lambda_2$.

Let E be a family in I_G and put

$$T = \bigvee_{H \in E} H.$$

Suppose $G_\lambda \subseteq G \wedge T$. Then $G_\lambda \subseteq G$, so $G_\lambda \wedge H = O$ for all H in E. Let $\epsilon > 0$. Since $G_\lambda \subseteq T$,

$$U = \{A \in \mathscr{A} | \lambda(A) < \epsilon\}$$

is a T-neighborhood of $\emptyset$. By definition of the supremum topology there are $H_1, \ldots, H_n$ in E and $U_1, \ldots, U_n$ such that, for $1 \leqq i \leqq n$, U_i is an H_i-neighborhood of $\emptyset$ and

$$U_I \cap \ldots \cap U_n \subseteq U.$$

By 1.2 (1) there exist submeasures $\lambda_1, \ldots, \lambda_n$ and positive numbers $\delta_1, \ldots, \delta_n$ such that $G_{\lambda_i} \subseteq H_i$ and

$$\{A \in \mathscr{A} | \lambda_i(A) < \delta_i\} \subseteq U_i \text{ for } 1 \leqq i \leqq n.$$

Put

$$\delta = \min \delta_1, \ldots, \delta_n \quad \text{and} \quad \mu = \lambda_1 \vee \ldots \vee \lambda_n.$$

If $\mu(A) < \delta$, then A is in $U_1 \cap \ldots \cap U_n$, so $\lambda(A) < \epsilon$. But for $1 \leqq i \leqq n$,

$$G_\lambda \wedge G_{\lambda_i} = O,$$

so $\lambda \perp \lambda_i$. Then $\lambda \perp \mu$. So there is C in $\mathscr{A}$ such that $\mu(C) < \delta$ and $\lambda(X \setminus C) < \epsilon$. Then $\lambda(C) < \epsilon$, so

$$\lambda(X) \leqq \lambda(C) + \lambda(X \setminus C) < 2\epsilon.$$

Then $\lambda = 0$. By 1.2 (2), $G \wedge T = 0$, so T is in I_G. Therefore, I_G is a complete ideal.

2. A decomposition theorem. In this section we prove our main theorem, a Lebesgue decomposition theorem for exhaustive submeasures with respect to FN topologies.

A family E of submeasures on $\mathscr{A}$ is *uniformly exhaustive* if $\lambda(A_n) \to 0$ uniformly for λ in E whenever (A_n) is a disjoint sequence in $\mathscr{A}$. We begin with a key lemma, also proved by Drewnowski [Lemma 4.7. **7**].

2.1. LEMMA. *Let E be a nonempty uniformly exhaustive family of submeasures on $\mathscr{A}$. Then for every $\epsilon > 0$ there are a finite subset E' of E and $\delta > 0$ such that $\lambda(A) < \epsilon$ for all λ in E whenever $\lambda(A) < \delta$ for all λ in E'.*

Proof. Replace additive set functions by submeasures in the intricate and ingenious proof of Lemma 1.4 [**11**].

2.2. LEMMA. *Let E be a nonempty uniformly exhaustive family of submeasures which is bounded above. Then there is a sequence (λ_n) in E such that*

$$\bigvee_{\lambda \in E} \lambda \sim \sum \frac{1}{2^n} \lambda_n.$$

Proof. Set

$$\mu = \bigvee_{\lambda \in E} \lambda.$$

By 2.1 for every $k \geqq 1$ there are a finite subset E_k of E and $\delta_k > 0$ such that

$$\lambda(A_i) < \frac{1}{k} \text{ for all } \lambda \text{ in } E$$

whenever $\lambda(A) < \delta_k$ for all λ in E_k. Put $C = \cup E_k$. Write $C = (\lambda_n)$. Put

$$\nu = \sum \frac{1}{2^n} \lambda_n.$$

Then $\nu \leqq \mu$, so ν is μ-continuous.

Let $\epsilon > 0$. Find k such that $1/k < \epsilon$. Put

$$n_k = \max \{n | \lambda_n \in E_k\} \quad \text{and} \quad \delta = \frac{1}{2^{n_k}} \delta_k.$$

Suppose that $\nu(A) < \delta$. For each $n \leqq n_k$, we have

$$\frac{2^n}{2^{n_k}} \leqq 1,$$

so

$$\lambda_n(A) < \frac{2^n}{2^{n_k}} \delta_k \leqq \delta_k.$$

Then $\lambda_n(A) < \delta_k$ for all λ_n in E_k, so

$$\lambda(A) < \frac{1}{k} \text{ for all } \lambda \text{ in } E.$$

Then

$$\mu(A) \leqq \frac{1}{k} < \epsilon.$$

Thus μ is ν-continuous. Therefore $\mu \sim \nu$.

2.3. THEOREM. *Let E be a nonempty uniformly exhaustive family of submeasures which is bounded above. Then*

$$\bigvee_{\lambda \in E} G_\lambda = G \bigvee_{\lambda \in E} \lambda.$$

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Proof. Put

$$\mu = \bigvee_{\lambda \in E} \lambda.$$

Clearly

$$\bigvee_{\lambda \in E} G_\lambda \subseteq G_\mu.$$

By 2.2 there is a sequence (λ_n) in E such that

$$\mu \sim \sum \frac{1}{2^n} \lambda_n.$$

By 1.3,

$$G_\mu = \vee G_{\lambda_n} \subseteq \bigvee_{\lambda \in E} G_\lambda.$$

Therefore

$$G_\mu = \bigvee_{\lambda \in E} G_\lambda.$$

2.4. LEMMA. *Let λ be an exhaustive submeasure and G an* FN *topology on $\mathscr{A}$. Put*

$$\lambda_1 = \vee \{\alpha | \alpha \text{ is a submeasure, } \alpha \leqq \lambda \text{ and } G_\alpha \subseteq G\},$$

$$\lambda_2 = \vee \{\beta | \beta \text{ is a submeasure, } \beta \leqq \lambda \text{ and } G_\beta \wedge G = O\}.$$

Then $G_{\lambda_1} = G_\lambda \wedge G$ and $G_{\lambda_2} \wedge G = O$.

Proof. Since λ is exhaustive, $\{\alpha | \alpha \leqq \lambda \text{ and } G_\alpha \subseteq G\}$ and $\{\beta | \beta \leqq \lambda \text{ and } G_\beta \wedge G = O\}$ are nonempty, uniformly exhaustive, and bounded above.

By 1.2 (2),

$$G_\lambda \wedge G = \vee \{G_\mu | G_\mu \subseteq G_\lambda \text{ and } G_\mu \subseteq G\}.$$

Note that $G_\mu \subseteq G_\lambda$ if and only if $G_{\lambda \wedge \mu} = G_\mu$ if and only if $G_\mu = G_\alpha$ for some $\alpha \leqq \lambda$. Then

$$G_\lambda \wedge G = \vee \{G_\alpha | \alpha \leqq \lambda \text{ and } G_\alpha \subseteq G\}.$$

By 2.3, $G_\lambda \wedge G = G_{\lambda_1}$.

By 2.3 and 1.4, $G_{\lambda_2} \wedge G = O$.

We shall need an ordinary Lebesgue decomposition for exhaustive submeasures, due to Drewnowski.

2.5. THEOREM. *Let* λ *be an exhaustive submeasure and* η *a submeasure on* $\mathscr{A}$. *Then there are submeasures* λ_1 *and* λ_2 *on* $\mathscr{A}$ *such that* $\lambda_1 \leqq \lambda$, $\lambda_2 \leqq \lambda$, λ_1 *is* η*-continuous,* $\lambda_2 \perp \eta$, *and* $\lambda \sim \lambda_1 + \lambda_2 \sim \lambda_1 \vee \lambda_2$. *The submeasures* λ_1 *and* λ_2 *are unique up to equivalence.*

Proof. See 4.3 (2), [**6**].

2.6. THEOREM. *Let* λ *be an exhaustive submeasure and* G *an* FN *topology on* $\mathscr{A}$. *Then there are submeasures* λ_1 *and* λ_2 *on* $\mathscr{A}$ *such that* $\lambda_1 \leqq \lambda$, $\lambda_2 \leqq \lambda$, $G_{\lambda_1} \subseteq G$, $G_{\lambda_2} \wedge G = O$, *and* $\lambda \sim \lambda_1 \vee \lambda_2$. *The submeasures* λ_1 *and* λ_2 *are unique up to equivalence.*

Proof. Define λ_1 and λ_2 as in 2.4. Then

$$\lambda_1 \leqq \lambda,\ \lambda_2 \leqq \lambda,\ G_{\lambda_1} \subseteq G, \quad \text{and} \quad G_{\lambda_2} \wedge G = O.$$

Since $\lambda_1 \vee \lambda_2 \leqq \lambda$,

$$G_{\lambda_1 \vee \lambda_2} \subseteq G_\lambda.$$

To show the reverse, we decompose λ with respect to $\lambda_1 \vee \lambda_2$. By 2.5 there are submeasures α_1 and α_2 such that $\alpha_1 \leqq \lambda$, $\alpha_2 \leqq \lambda$, α_1 is $\lambda_1 \vee \lambda_2$-continuous, $\alpha_2 \perp \lambda_1 \vee \lambda_2$, and $\lambda \sim \alpha_1 \vee \alpha_2$. Then $\alpha_2 \perp \lambda_1$ and $\alpha_2 \perp \lambda_2$. By 2.4,

$$G_{\alpha_2} \wedge G = (G_{\alpha_2} \wedge G_\lambda) \wedge G = G_{\alpha_2} \wedge (G_\lambda \wedge G)$$
$$= G_{\alpha_2} \wedge G_{\lambda_1} = O.$$

So $\alpha_2 \leqq \lambda_2$. Then $\alpha_2 = 0$. So $\lambda \sim \alpha_1$. Then

$$G_\lambda = G_{\alpha_1} \subseteq G_{\lambda_1 \vee \lambda_2}.$$

Therefore $\lambda \sim \lambda_1 \vee \lambda_2$.

Suppose that β_1 and β_2 are submeasures such that $G_{\beta_1} \subseteq G$, $G_{\beta_2} \wedge G = O$, and $\lambda \sim \beta_1 \vee \beta_2$. Then

$$G_{\beta_1} \subseteq G_\lambda \wedge G = G_{\lambda_1}.$$

By 2.5 there are submeasures γ_1 and γ_2 such that γ_1 is β_1-continuous, $\gamma_2 \perp \beta_1$, and $\lambda_1 \sim \gamma_1 \vee \gamma_2$. Then $G_{\gamma_2} \subseteq G_{\lambda_1} \subseteq G$, so

$$G_{\gamma_2} \wedge G_{\beta_2} = O.$$

Then

$$G_{\gamma_2} = G_{\gamma_2} \wedge G_\lambda = G_{\gamma_2} \wedge (G_{\beta_1} \vee G_{\beta_2}) = O$$

by 1.4. So $\gamma_2 = 0$. Then $\lambda_1 \sim \gamma_1$, so

$$G_{\lambda_1} = G_{\gamma_1} \subseteq G_{\beta_1}.$$

Thus $\lambda_1 \sim \beta_1$.

Again by 2.5 there are submeasures δ_1 and δ_2 such that δ_1 is

λ_2-continuous, $\delta_2 \perp \lambda_2$, and $\beta_2 \sim \delta_1 \vee \delta_2$. Since $G_{\beta_1} \wedge G_{\beta_2} = O$,

$$G_{\lambda_1} \wedge G_{\delta_2} = G_{\beta_1} \wedge G_{\delta_2} = 0.$$

Then

$$G_{\delta_2} = G_{\delta_2} \wedge G_\lambda = G_{\delta_2} \wedge (G_{\lambda_1} \vee G_{\lambda_2}) = 0$$

by 1.4. So $\delta_2 = 0$. Then $\beta_2 \sim \delta_1$, so

$$G_{\beta_2} = G_{\delta_1} \subseteq G_{\lambda_2}.$$

Similarly

$$G_{\lambda_2} \subseteq G_{\beta_2}.$$

Thus $\lambda_2 \sim \beta_2$.

As a corollary we obtain a theorem of Traynor [Theorem 4.2, **12**]. Say that an FN topology G is *exhaustive* if $A_n \to \emptyset$ with respect to G whenever (A_n) is a disjoint sequence in $\mathscr{A}$.

2.7. COROLLARY. *Let G be an exhaustive* FN *topology and H an* FN *topology on $\mathscr{A}$. Then there are unique* FN *topologies G_1 and G_2 on $\mathscr{A}$ such that $G_1 \subseteq H$, $G_2 \wedge H = O$, and $G = G_1 \vee G_2$. Moreover, $G_1 = G \wedge H$.*

Proof. Put

$$G_1 = \vee \{G_\lambda | G_\lambda \subseteq G \text{ and } G_\lambda \subseteq H\},$$

$$G_2 = \vee \{G_\lambda | G_\lambda \subseteq G \text{ and } G_\lambda \wedge H = O\}.$$

Then

$$G_1 = \vee \{G_\lambda | G_\lambda \subseteq G \wedge H\} = G \wedge H$$

by 1.2 (2). Clearly $G_1 \subseteq H$. By 1.4, $G_2 \wedge H = O$. Clearly $G_1 \vee G_2 \subseteq G$. Suppose $G_\lambda \subseteq G$. By 2.6 there are submeasures λ_1 and λ_2 such that

$$G_{\lambda_1} \subseteq H,\ G_{\lambda_2} \wedge H = O, \text{ and } \lambda \sim \lambda_1 \vee \lambda_2.$$

Then

$$G_\lambda = G_{\lambda_1} \vee G_{\lambda_2} \subseteq G_1 \vee G_2.$$

By 1.2 (2), $G \subseteq G_1 \vee G_2$. Therefore $G = G_1 \vee G_2$.

Suppose that T_1 and T_2 are FN topologies such that $T_1 \subseteq H$, $T_2 \wedge H = O$, and $G = T_1 \vee T_2$. Then

$$T_1 \subseteq G \wedge H = G_1.$$

If $G_\lambda \subseteq T_2$, then $G_\lambda \subseteq G$ and $G_\lambda \wedge H = O$, so $G_\lambda \subseteq G_2$. Then $T_2 \subseteq G_2$ by 1.2 (2).

To show the reverse inclusions is a little harder. Suppose $G_\lambda \subseteq G$. By 2.6 there are submeasures α and β such that

$$G_\alpha \subseteq T_1,\ G_\beta \wedge T_1 = O, \text{ and } \lambda \sim \alpha \vee \beta.$$

Again by 2.6 there are submeasures γ and δ such that

$$G_\gamma \subseteq T_2,\ G_\delta \wedge T_2 = O, \text{ and } \beta \sim \gamma \vee \delta.$$

Since $G_\delta \subseteq G_\beta$, $G_\delta \wedge T_1 = O$. Then

$$G_\delta = G_\delta \wedge G = G_\delta \wedge (T_1 \vee T_2) = O$$

by 1.4. So $G_\beta = G_\gamma \subseteq T_2$. Now if also $G_\lambda \subseteq H$, then since $G_\beta \subseteq T_2$ and $T_2 \wedge H = O$, $G_\beta = O$. Then $G_\lambda = G_\alpha \subseteq T_1$. On the other hand, if $G_\lambda \wedge H = O$, then since $G_\alpha \subseteq T_1 \subseteq H$, $G_\alpha = O$. Then $G_\lambda = G_\beta \subseteq T_2$. Thus $G_1 \subseteq T_1$ and $G_2 \subseteq T_2$. Therefore $G_1 = T_1$ and $G_2 = T_2$.

3. Applications to the control measure problem. In this section we apply 2.6 to split an exhaustive submeasure on a Boolean algebra into a part which has a control measure and a part which is pathological or zero. To do this, we use the universal measure topology of Graves [**9**].

Let $\mathscr{A}$ be an algebra of sets and $S(\mathscr{A})$ the vector space of all complex-valued $\mathscr{A}$-simple functions. A finitely additive map ϕ from $\mathscr{A}$ to a locally convex space W is *strongly bounded* if $\phi(A_n) \to 0$ whenever (A_n) is a disjoint sequence in $\mathscr{A}$, and *strongly countably additive* if it is strongly bounded and countably additive. Each finitely additive ϕ from $\mathscr{A}$ to W induces a linear map $\tilde{\phi}$ from $S(\mathscr{A})$ to W defined by

$$\tilde{\phi}(f) = \int f d\phi.$$

Let τ be the weakest topology on $S(\mathscr{A})$ making $\tilde{\phi}$ continuous for every strongly countably additive ϕ from $\mathscr{A}$ to W for every locally convex W. Then τ is a locally convex topology, called the *universal measure topology*. The restriction of τ to (the image of) $\mathscr{A}$ is an exhaustive FN topology. The *universal measure space* $\mathscr{L}(\mathscr{A})$ is the τ-completion of $S(\mathscr{A})$. For information about $\mathscr{L}(\mathscr{A})$, see [**2**], [**3**], and [**9**].

Let $\text{ba}(\mathscr{A})$ denote the Banach space of complex-valued bounded additive maps on $\mathscr{A}$ and $\text{sca}(\mathscr{A})$ the closed subspace of complex-valued strongly countably additive maps on $\mathscr{A}$. Let $\text{ba}(\mathscr{A})^+$ and $\text{sca}(\mathscr{A})^+$ denote the sets of nonnegative elements in $\text{ba}(\mathscr{A})$ and $\text{sca}(\mathscr{A})$ respectively.

In [**4**] we considered submeasures λ for which $G_\lambda \subseteq \tau$. All results in section 2 of [**4**] remain true if $\mathscr{C}$ (the algebra of clopen subsets of a compact T_2 totally disconnected space) is replaced by an algebra of sets $\mathscr{A}$. We record one theorem here.

3.1. THEOREM. *Let λ be a submeasure on $\mathscr{A}$. Then $G_\lambda \subseteq \tau$ on $\mathscr{A}$ if and only if there is μ in* sca $(\mathscr{A})^+$ *such that $\lambda \sim \mu$.*

A simpler description of the topology τ on $\mathscr{A}$ follows.

3.2. Corollary. *On* $\mathscr{A}$, $\tau = \vee \{G_\mu | \mu \in \text{sca}(\mathscr{A})^+\}$.

Proof. For each μ in $\text{sca}(\mathscr{A})^+$, $G_\mu \subseteq \tau$. Now use 1.2 (2) and 3.1.

Next we describe the control measure problem, or Maharam submeasure problem, raised by Maharam in [**10**]. Let λ be a submeasure on a Boolean algebra $\mathscr{B}$. Say λ has a *control measure* if there is a nonnegative bounded additive μ on $\mathscr{B}$ such that $\lambda \sim \mu$. Say λ is *pathological* if $\lambda \neq 0$ but λ dominates no nonzero nonnegative bounded additive μ on $\mathscr{B}$. These statements are equivalent:

(1) If λ is an exhaustive submeasure on a Boolean algebra $\mathscr{B}$ then λ has a control measure.

(2) If λ is an exhaustive submeasure on a Boolean algebra $\mathscr{B}$, then λ is not pathological.

See [**8**] for a long list of equivalent statements, including (1) and (2). Whether these statements are true is still an open question.

Let $\mathscr{B}$ be a Boolean algebra and $\mathscr{C}$ the algebra of clopen subsets of its Stone space. Then submeasures on $\mathscr{B}$ are in one to one correspondence with submeasures on $\mathscr{C}$.

3.3. Theorem. *Let* λ *be a nonzero submeasure on* $\mathscr{B}$.

(1) λ *has a control measure if and only if* $G_\lambda \subseteq \tau$ *on* $\mathscr{C}$.

(2) λ *is pathological if and only if* $G_\lambda \wedge \tau = O$ *on* $\mathscr{C}$.

Proof. See 2.8 and 3.1 of [**4**].

Now 3.3 gives meaning to a special case of 2.6.

3.4. Theorem. *Let* λ *be an exhaustive submeasure on* $\mathscr{B}$. *Then there are submeasures* λ_1 *and* λ_2 *on* $\mathscr{B}$ *such that* $\lambda_1 \leqq \lambda$, $\lambda_2 \leqq \lambda$, λ_1 *has a control measure,* λ_2 *is pathological or zero, and* $\lambda \sim \lambda_1 \vee \lambda_2$. *The submeasures* λ_1 *and* λ_2 *are unique up to equivalence.*

Proof. Use 2.6 with $G = \tau$ on $\mathscr{C}$ and 3.3.

Of course, if every exhaustive submeasure has a control measure, 3.4 says nothing. But we can say more about the submeasure λ_1 in 3.4. To do this, we need a lemma of Aleksyuk [Lemma 1.2, **1**], for which we give a short proof.

3.5. Lemma. *Let* λ *be an exhaustive submeasure on* $\mathscr{C}$. *If*

$$\lambda \sim \vee \{\mu \in \text{ba}(\mathscr{C})^+ | \mu \leqq \lambda\},$$

then there is ν *in* $\text{ba}(\mathscr{C})^+$ *such that* $\nu \leqq \lambda$ *and* $\lambda \sim \nu$.

Proof. Set

$$E = \{\mu \in \text{ba}(\mathscr{C})^+ | \mu \leqq \lambda\} \text{ and } \alpha = \bigvee_{\mu \in E} \mu.$$

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Then $\alpha \sim \lambda$ and E is nonempty, uniformly exhaustive, and bounded above. By 2.2 there is a sequence (μ_n) in E such that

$$\alpha \sim \sum \frac{1}{2^n} \mu_n.$$

Put

$$\nu = \sum \frac{1}{2^n} \mu_n.$$

Then ν is in E and $\lambda \sim \nu$.

3.6. THEOREM. *Let* λ *be a submeasure on* $\mathscr{C}$. *The following are equivalent*:

(1) $G_\lambda \subseteq \tau$ *on* $\mathscr{C}$.

(2) λ *is exhaustive and* $\lambda \sim \vee \{\mu \in \text{ba}(\mathscr{C})^+ | \mu \leqq \lambda\}$.

(3) *There is* ν *in* $\text{ba}(\mathscr{C})^+$ *such that* $\nu \leqq \lambda$ *and* $\lambda \sim \nu$.

Proof. (1) implies (2): Suppose $G_\lambda \subseteq \tau$. Then λ is exhaustive. Put

$$\alpha = \vee \{\mu \in \text{ba}(\mathscr{C})^+ | \mu \leqq \lambda\}.$$

Then $\alpha \leqq \lambda$. By 2.5 there are submeasures β_1 and β_2 such that $\beta_1 \leqq \lambda$, $\beta_2 \leqq \lambda$, β_1 is α-continuous, $\beta_2 \perp \alpha$, and $\lambda \sim \beta_1 \vee \beta_2$. Suppose that μ is in $\text{ba}(\mathscr{C})^+$ and $\mu \leqq \beta_2$. Then $\mu \leqq \lambda$ so $\mu \leqq \alpha$. Then $\beta_2 \perp \mu$, so $\mu = 0$. Then β_2 is pathological or zero. By 3.3 (2),

$$G_{\beta_2} \wedge \tau = O.$$

But

$$G_{\beta_2} \subseteq G_\lambda \subseteq \tau.$$

Then $\beta_2 = 0$. It follows that $\lambda \sim \alpha$.

By 3.5, (2) implies (3), and (3) implies (1) is clear.

3.7. COROLLARY. *Let* λ, λ_1 *and* λ_2 *be as in* 3.4. *Then* λ_1 *has a control measure* ν *such that* $\nu \leqq \lambda$.

Proof. By construction $G_{\lambda_1} \subseteq \tau$ on $\mathscr{C}$. Now use 3.6.

4. A counterexample. In this section we show that if $\mathscr{A}$ is an infinite algebra, then the universal measure topology τ on $\mathscr{A}$ has no complement in the lattice of FN topologies. Thus 2.6 and 2.7 are false without the hypothesis of exhaustivity.

Let $\mathscr{A}$ be an algebra and G and G' FN topologies on $\mathscr{A}$. Say that G' is a *complement* for G if $G \wedge G' = O$ and $G \vee G' = D$, where D is the discrete topology.

4.1. PROPOSITION. *Let G and H be* FN *topologies on $\mathscr{A}$ such that $G \subseteq H$ and H is exhaustive. If H has a complement, then so does G.*

Proof. Suppose that H' is a complement for H. By 2.7 there are unique FN topologies H_1 and H_2 such that

$$H_1 \subseteq G,\ H_2 \wedge G = O, \text{ and } H = H_1 \vee H_2.$$

In fact $H_1 = G \wedge H = G$. Put $G' = H_2 \vee H'$. Since $G \subseteq H$, $G \wedge H' = O$. By 1.4, $G \wedge G' = O$. Also

$$G \vee G' = G \vee (H_2 \vee H') = (H_1 \vee H_2) \vee H' = H \vee H' = D.$$

Therefore G' is a complement for G.

4.2. LEMMA. *Let μ be in* $\text{sca}(\mathscr{A})^+$. *If G_μ has a complement, then it has a complement of the form G_λ, where λ is a submeasure on $\mathscr{A}$.*

Proof. Let G' be a complement for G_μ. Then $G_\mu \vee G' = D$. Since $\{\emptyset\}$ is a D-neighborhood of $\emptyset$ there exist a G_μ-neighborhood U of $\emptyset$ and a G'-neighborhood V of $\emptyset$ such that $U \cap V = \{\emptyset\}$. By 1.2 (1) there are a submeasure λ such that $G_\lambda \subseteq G'$, $\delta > 0$ and $\epsilon > 0$ such that

$$\{A \in \mathscr{A} | \mu(A) < \epsilon\} \subseteq U \text{ and } \{A \in \mathscr{A} | \lambda(A) < \delta\} \subseteq V.$$

Put $r = \min \{\delta, \epsilon\}$. Then

$$\{A \in \mathscr{A} | (\lambda \vee \mu)(A) < r\} = \{\emptyset\}.$$

So

$$G_\lambda \vee G_\mu = G_{\lambda \vee \mu} = D.$$

Since $G_\lambda \subseteq G'$ and $G_\mu \wedge G' = O$,

$$G_\lambda \wedge G_\mu = O.$$

Thus G_λ is a complement for G_μ.

Now we consider two cases. Since $\mathscr{A}$ separates points, an atom in $\mathscr{A}$ is just a singleton.

4.3. LEMMA. *If $\mathscr{A}$ contains a sequence of distinct atoms, then τ has no complement.*

Proof. Let $(\{x_n\})$ be a sequence of distinct atoms. Set

$$\mu = \sum \frac{1}{2^n} \delta_{x_n},$$

where δ_{x_n} is the unit mass at x_n. Then μ is in $\text{sca}(\mathscr{A})^+$, so $G_\mu \subseteq \tau$.

Suppose that G_μ has a complement of the form G_λ. Since

$$G_{\lambda \vee \mu} = G_\lambda \vee G_\mu = D,$$

there is $\epsilon > 0$ such that

$$\{A \in \mathscr{A} \mid (\lambda \vee \mu)(A) < \epsilon\} = \{\emptyset\}.$$

Put $r = \min \{\epsilon, \frac{1}{2}\}$. If A is nonempty, then $\lambda(A) \geqq r$ or $\mu(A) \geqq r$. Since

$$G_{\lambda \wedge \mu} = G_\lambda \wedge G_\mu = O,$$

$\lambda \perp \mu$. Then there is C in $\mathscr{A}$ such that $\lambda(C) < r$ and $\mu(X \setminus C) < r$.

Let k be the smallest positive integer such that $1/2^k < r$. Since $r \leqq \frac{1}{2}$, $k \geqq 2$. If $n \geqq k$, then

$$\mu(\{x_n\}) = \frac{1}{2^n} \leqq \frac{1}{2^k} < r,$$

so $\lambda(\{x_n\}) \geqq r$. Then x_n is in $X \setminus C$. For each $N \geqq k$,

$$\mu(X \setminus C) \geqq \mu(\{x_k, \ldots, x_N\}) = \sum_{n=k}^{N} \frac{1}{2^n}.$$

Then

$$\mu(X \setminus C) \geqq \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k-1}} \geqq r.$$

By 4.2, this contradiction shows that G_μ has no complement.

By 4.1, τ has no complement.

4.4. LEMMA. *If $\mathscr{A}$ is infinite but contains at most finitely many atoms, then τ has no complement.*

Proof. If $\mathscr{A}$ has no atoms, set $B = X$. If $\mathscr{A}$ has atoms $\{x_1\}, \ldots, \{x_n\}$, set

$$B = X \setminus \{x_1, \ldots, x_n\}.$$

Then B is infinite. Find μ in $\text{sca}(\mathscr{A})^+$ such that $\mu(B) = \mu(X) = 1$. Then $G_\mu \subseteq \tau$.

Suppose that G_μ has a complement of the form G_λ. As in the proof of 4.3 there is $\epsilon > 0$ such that

$$\{A \in \mathscr{A} \mid (\lambda \vee \mu)(A) < \epsilon\} = \{\emptyset\}.$$

Put $r = \min \{\epsilon, 1\}$. If A is nonempty, then $\lambda(A) \geqq r$ or $\mu(A) \geqq r$. Again as in the proof of 4.3 there is C in $\mathscr{A}$ such that $\lambda(C) < r$ and $\mu(X \setminus C) < r$.

Since $\mu(X \setminus B) = 0$,

$$\mu(X\backslash(B \cap C)) \leqq \mu(X\backslash B) + \mu(X\backslash C) < r,$$

while

$$\lambda(B \cap C) \leqq \lambda(C) < r.$$

Since $r \leqq \mu(X)$, $B \cap C$ is nonempty. Since $B \cap C$ contains no atom, there is a strictly decreasing sequence (C_n) in $\mathscr{A}$ such that $C_1 = B \cap C$. For $n \geqq 1$, put $A_n = C_n\backslash C_{n+1}$. Then (A_n) is a disjoint sequence of nonempty subsets of $B \cap C$. Since μ is strongly bounded, $\mu(A_n) \rightarrow 0$. Then $\mu(A_N) < r$ for some N. Since A_N is nonempty, $\lambda(A_N) \geqq r$. But then $\lambda(B \cap C) \geqq r$. By 4.2, this contradiction shows that G_μ has no complement.

By 4.1, τ has no complement.

4.5. THEOREM. *If A is an infinite algebra, then τ has no complement.*

If $\mathscr{A}$ is an infinite algebra, then $\mathscr{A}$ must contain a disjoint sequence of nonempty sets. Define the *discrete submeasure* λ_d on $\mathscr{A}$ by $\lambda_d(\emptyset) = 0$ and $\lambda_d(A) = 1$ if A is nonempty. Then $G_{\lambda_d} = D$ and neither λ_d nor D is exhaustive. Thus 4.5 shows that exhaustivity is necessary in 2.6 and 2.7.

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