Proceedings of the Edinburgh Mathematical Society (2010) **53**, 557–574 ©
DOI:10.1017/S0013091507000648 Printed in the United Kingdom

NORM FUNCTORS AND EFFECTIVE ZERO CYCLES

VLADIMIR BARANOVSKY

Department of Mathematics, University of California, Irvine, Irvine, CA 92697, USA (vbaranov@math.uci.edu)

(Received 3 November 2007)

Abstract We compare two known definitions for a relative family of effective zero cycles: one based on traces of functions and one based on norms of functions. In characteristic zero we show that both definitions agree. In the general setting we show that the norm map on functions can be expanded to a norm functor between certain categories of line bundles, thereby giving a third approach to families of zero cycles.

Keywords: zero cycles; determinant bundles; Chow varieties

2010 *Mathematics subject classification:* Primary 14C05
Secondary 14D23

1. Introduction

Let us start with the simple situation of a quasi-projective scheme X over a perfect field k. We want to understand the notion of a family of degree d effective zero cycles parametrized by a k-scheme T. When $T = \mathrm{Spec}(k)$ these are just finite formal linear combinations

$$\xi = \sum d_i x_i \tag{1.1}$$

of closed points $x_i \in X$ with $d_i \in \mathbb{Z}_{\geqslant 0}$. Such a cycle has degree $d = \sum_i d_i e_i$, where e_i is the degree of the field extension $k \subset k(x_i)$. We remark here that if $k \subset k(x_i)$ were not separable, one would have to work with rational d_i having powers of $p = \mathrm{char}\, k$ in denominators (see [**14**, § 8]).

To obtain a description that works over an arbitrary base T let f be a regular function defined on an open subset $U \subset X$ containing all x_i. Define the following elements in k:

$$\theta(f) = \sum_i d_i \theta_i(f(x_i)), \qquad n(f) = \prod_i n_i(f(x_i))^{d_i},$$

where θ_i and n_i stand for the trace and the norm of the field extension $k \subset k(x_i)$, respectively.

When ξ varies with $t \in T$, define $X_T = X \times_k T$ and let $\pi_T : X_T \to T$ be the canonical projection. The above construction gives *trace* and *norm* maps

$$\theta : (\pi_T)_* \mathcal{O}_{\hat{X}_T} \to \mathcal{O}_T, \qquad n : (\pi_T)_* \mathcal{O}_{\hat{X}_T} \to \mathcal{O}_T,$$

where $\hat{X}_T$ is the completion of X_T along the closed subset Z swept out by the points $x_i(t)$. Observe that θ is a morphism of $\mathcal{O}_T$-modules, while n is just a multiplicative map. The values on functions pulled back from T are given by $\theta(f) = d \cdot f$ and $n(f) = f^d$. Both the trace and the norm maps should be continuous, i.e. they should factor through $(\pi_T)_* \mathcal{O}_Y$ for some closed subscheme $Y \hookrightarrow X$ with support in Z. Both constructions should commute with base change $T' \to T$.

This suggests the idea that a family of zero cycles with base T can be defined by specifying an appropriate closed subset $Z \subset X_T$ and either a trace map θ or a norm map n, as above (the version with norm maps is apparently originally due to Grothendieck, who also applied it to non-effective cycles by restricting to multiplicative groups of $\mathcal{O}_{\hat{X}_T}$ and $\mathcal{O}_T$). Observe that for a base change $T' \to T$ the trace map automatically pulls back to T', being a morphism of $\mathcal{O}_T$-modules, while with the norm map n we have to *specify* the pullback of n. In other words, we should have a system of maps

$$n_{T'} : (\pi_{T'})_* \mathcal{O}_{\hat{X}_{T'}} \to \mathcal{O}_{T'}$$

for $T' \to T$ that agree with each other in a natural sense. This seems like an inconvenient detail. However, it turns out that one needs to impose further conditions on θ to get a trace map that comes from a geometric family of cycles (see Definition 3.2), while for n the existence of its extensions $n_{T'}$ plays the role of such a condition. In addition, the trace construction only works well when k has characteristic zero (or finite characteristic $p > d$).

The approach using traces was used in characteristic zero by Angeniol [**1**] and also by Buchstaber and Rees [**2**]. Angeniol extends his definition to cycles of higher dimensions, which leads to a construction of the Chow scheme of cycles. In the affine case the norm approach was used by Roby [**12**], but the global version of his construction was carried out only recently (more than 40 years later!) by Rydh [**13–15**]. Rydh deals with a general situation of a separated morphism of algebraic spaces $\pi : X \to S$. He considers higher-dimensional cycles as well, using an old idea of Barlet (also employed by Angeniol) that a family of n-dimensional cycles over an l-dimensional base can be represented locally as a family of zero cycles over an $(n+l)$-dimensional base.

In the above construction one should take into account that θ and n could factor through a completion along a smaller closed subset $Z' \subset Z$. In the additive case, Angeniol formulates a non-degeneracy condition ensuring that such Z' does not exist. In the multiplicative case, Rydh simply considers pairs (Y, n) consisting of a closed subscheme Y and a norm map $n : (\pi_T)_* \mathcal{O}_Y \to \mathcal{O}_T$, and then uses an equivalence relation that identifies (Y', n') and (Y'', n'') if n' and n'' factor through a norm map defined on a closed subscheme $Y \subset Y' \cap Y''$. In this paper we adopt the second approach, modifying and generalizing the trace definition.

In characteristic zero both functors of families of zero cycles are represented by the symmetric power $\mathrm{Sym}^d(X/S)$, i.e. the quotient of the d-fold Cartesian product of X over S by the action of the symmetric group Σ_d. In arbitrary characteristic the approach based on traces breaks down (for example, we are not able to distinguish $\xi = x$ from $\xi = (p+1)x$), while the norms approach leads to the *space of divided powers* $\Gamma^d(X/S)$. There

https://doi.org/10.1017/S0013091507000648 Published online by Cambridge University Press

is a natural morphism $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$, which is an isomorphism in characteristic zero, but in general it is only a universal homeomorphism [**13**]. Even for general schemes over a field k (not necessarily quasi-projective) both $\mathrm{Sym}^d(X/S)$ and $\Gamma^d(X/S)$ may not be schemes but rather algebraic spaces. Therefore, it is more natural to work in the category of algebraic spaces from the beginning.

The purpose of this paper is to explore a third approach to families of zero cycles: one that admits a reasonably straightforward generalization to higher-dimensional cycles. In the original set-up of a scheme X over k, choose a line bundle L defined on an open subset $U \subset X$ containing all x_i in (1.1) and assume for simplicity that all $k(x_i)$ are equal to k. Define a one-dimensional vector space

$$N(L) = \prod L_{x_i}^{\otimes d_i}$$

over k. When the zero cycle ξ varies over a base T, this gives a line bundle $N(L)$ on T. Obviously, an isomorphism of line bundles on X induces an isomorphism of bundles on T. However, non-negativity of the coefficients d_i is reflected in the fact that *any* morphism of invertible $\mathcal{O}_{\hat{X}_T}$-modules $\psi : L \to M$ gives a morphism of $\mathcal{O}_T$-modules $N(\psi) : N(L) \to N(M)$. We can further consider the line bundles defined only on a neighbourhood of Z. Thus, for a closed subscheme $Y \hookrightarrow X_T$ supported at Z we should have a *norm functor*

$$N : \mathrm{PIC}(Y) \to \mathrm{PIC}(T),$$

where 'PIC' is the category of line bundles and morphisms as $\mathcal{O}$-modules. Again, this functor should come with functorial pullbacks with respect to morphisms of schemes (or algebraic spaces) $T' \to T$. In practice, it suffices to restrict to those T' that are affine over T (or even to the full subcategory generated by $\mathbb{A}^n_T$, $n \geqslant 1$).

However, the correspondence $\psi \mapsto N(\psi)$ on morphisms is no longer $\mathcal{O}_T$-linear, but rather satisfies $N(\psi)(\pi_T^*(f)s) = f^d N(\psi)(s)$, where s is a local section of $N(L)$ and f is a local section of $\mathcal{O}_T$. Morphisms of modules with this property were also considered by Roby in [**12**], where they are called *homogeneous polynomial laws of degree d*. As with norm maps, we should also specify a functorial extension of $\psi \mapsto N(\psi)$ with respect to base changes $T' \to T$. The fact that N is a functor means that $\psi \mapsto N(\psi)$ is multiplicative, since compositions should go to compositions. In addition, N should agree with tensor products of line bundles and, similarly to identities,

$$\theta(\pi_T^*(f)) = df, \qquad n(\pi_T^*(f)) = f^d,$$

we should have an isomorphism of functors

$$\eta : N \circ \pi_T^* \simeq \{L \mapsto L^{\otimes d}\}$$

agreeing with base change. This rigidification also ensures that N does not have any non-trivial functor automorphisms.

Besides the generalization to higher-dimensional cycles based on the work of Ducrot [**4**], Elkik [**5**] and Muñoz-Garcia [**11**], this approach to zero cycles can also be used to define

https://doi.org/10.1017/S0013091507000648 Published online by Cambridge University Press

the Uhlenbeck compactification of the moduli stack of vector bundles on a surface. The standard constructions like Hilbert–Chow morphisms, sums of cycles and Chow forms are also rather simple in the language of norm functors.

Norms of line bundles were earlier considered in [**7**] and [**3**]. More general norms of quasi-coherent sheaves were studied in [**6**] and [**14**].

This work is organized as follows. In § 2 we recall the basic results on polynomial laws, divided powers and norms for finite flat morphisms. In § 3 we define the functor of families in terms of norms and traces and prove that the two definitions are equivalent when $d!$ is invertible. The norm definition is essentially the one given in [**13**]—in particular, the corresponding functor is represented by the space of divided powers—while the trace definition is a generalization of the one given in [**1**]. We also obtain a formula for the tangent space to a point in the symmetric power, which appears to be new. In § 4 we prove that divided powers of a line bundle give a line bundle, define norm functors and use them to formulate a third definition for families of zero cycles. We prove that it is equivalent to the definition via norm maps. Finally, in § 5 we interpret in terms of norm functors such standard constructions as Hilbert–Chow morphisms, sums and direct images of cycles and Chow forms. Quite naturally, our descriptions are closely related to those of [**14**].

2. Preliminaries

2.1. Polynomial laws and divided powers

We recall some definitions from [**12**] (see also [**10**]). In this subsection all rings and algebras will be assumed to be commutative and with identity, although the theory can be developed in greater generality (see [**10**]). Let M, N be two modules over a ring A. Denote by $\mathcal{F}_M$ the functor

$$\mathcal{F}_M : A\text{-alg} \to \text{Sets}, \qquad A' \to A' \otimes_A M,$$

where A-alg is the category of (commutative) A-algebras.

Definition 2.1. A *polynomial law* from M to N is a natural transformation $F : \mathcal{F}_M \to \mathcal{F}_N$, i.e. for every A-algebra A' it defines a map $F_{A'} : A' \otimes_A M \to A' \otimes_A N$ and for any morphism $A' \to A''$ of A-algebras the natural agreement condition is satisfied. The polynomial law F is *homogeneous of degree* d if $F_{A'}(ax) = a^d F_{A'}(x)$ for any $a \in A'$ and $x \in A' \otimes_A M$. If B and C are A-algebras, then $F : \mathcal{F}_B \to \mathcal{F}_C$ is *multiplicative* if $F_{A'}(1) = 1$ and $F_{A'}(xy) = F_{A'}(x)F_{A'}(y)$ for $x, y \in B \otimes_A A'$.

Denote by $\mathrm{Pol}^d(M, N)$ the set of homogeneous polynomial laws of degree d from M to N. By Theorem IV.1 in [**12**, p. 266], the functor $\mathrm{Pol}^d(M, \cdot)$ is representable: there exists an A-module $\Gamma^d_A(M)$, called a *module of degree d divided powers*, and an isomorphism of functors in N:

$$\mathrm{Pol}^d(M, N) \simeq \mathrm{Hom}_A(\Gamma^d_A(M), N). \tag{2.1}$$

Moreover, if B, C are A-algebras, then each $\Gamma^d_A(B)$ is also an A-algebra and multiplicative laws in $\mathrm{Pol}^d(B, C)$ correspond precisely to A-algebra morphisms $\Gamma^d_A(B) \to C$ (see Theorem 7.11 in [**10**] or Proposition 2.5.1 in [**6**]).

Explicitly, the direct sum $\Gamma_A(M) = \bigoplus_{d\geqslant 0} \Gamma_A^d(M)$ may be defined as a unital graded commutative A-algebra with product $\times$, degree-d generators $\gamma^d(x)$, $x \in M$, $d \geqslant 0$, and relations

$$\gamma^0(x) = 1, \qquad \gamma^d(xa) = \gamma^d(x)a^d, \qquad \gamma^d(x) \times \gamma^e(x) = \binom{d+e}{e}\gamma^{d+e}(x),$$
$$\gamma^d(x+y) = \sum_{d_1+d_2=d} \gamma^{d_1}(x) \times \gamma^{d_2}(y).$$

In particular, $\Gamma_A^0(M) \simeq A$ and $\Gamma_A^1(M) \simeq M$ with $\gamma^1(x)$ given by x. We briefly summarize the properties of this construction as follows.

(1) $\Gamma_A(\cdot)$ is a covariant functor from the category of A-modules to the category of graded A-algebras that commutes with base change $A \to A'$.

(2) If B is an A-algebra, then the A-algebra $\Gamma_A^d(B)$ satisfies $\gamma^d(xy) = \gamma^d(x)\gamma^d(y)$ for any $x, y \in B$. Below we will also use a formula for arbitrary products which can be found in [**6**, Formula (2.4.2)].

(3) The map $\gamma^d : M \to \Gamma_A^d(M)$ is a homogeneous polynomial law of degree d. The isomorphism of (2.1) is obtained by composing an A-module homomorphism $\Psi_n : \Gamma_A^d(M) \to N$ with γ^d to obtain a polynomial law $n : M \to N$:

$$n = \Psi_n \circ \gamma^d.$$

(4) When M is flat over A or $d!$ is invertible in A, $\Gamma_A^d(M)$ is isomorphic to the module of symmetric tensors $TS_A^d(M)$, i.e. the submodule of invariants $[T_A^d(M)]^{\Sigma_d}$ in the tensor power. If this isomorphism holds for a triple of values $d = d_1$, d_2 and d_1+d_2, then it takes the $\times$ product to the shuffle product on the symmetric tensors. When $d!$ is invertible we can further identify both modules with the symmetric power $S_A^d(M)$ (i.e. the *quotient* of the tensor power $T_A^d(M)$ by the obvious relations).

(5) If $F \in \mathrm{Pol}^d(M,N)$ and we evaluate F at the A-algebra $A' = A[t_1,\dots,t_k]$, then $F(t_1m_1+\cdots+t_km_d) \in A'\otimes_A N$ is a sum of degree-d monomials in $t_1,\dots,t_k$ and the coefficient of $t_1^{\alpha_1}\cdots t_k^{\alpha_k}$ is the value of the corresponding A-module homomorphism $\Psi_F : \Gamma_A^d(M) \to N$ at $\gamma^{\alpha_1}(x_1)\times\cdots\times\gamma^{\alpha_k}(x_k)$. This explains the term 'degree d homogeneous polynomial law'.

2.2. Norms and traces for finite flat morphisms

Let $\pi : Y \to S$ be a finite flat morphism of algebraic spaces and assume that $\pi_*\mathcal{O}_Y$ is locally free of constant rank d. We have a natural morphism of $\mathcal{O}_S$-modules

$$\pi_*\mathcal{O}_Y \to \mathrm{End}_{\mathcal{O}_S}(\pi_*\mathcal{O}_Y)$$

and taking the composition with trace and determinant we obtain two maps:

$$\theta : \pi_*\mathcal{O}_Y \to \mathcal{O}_S, \qquad n : \pi_*\mathcal{O}_Y \to \mathcal{O}_S.$$

https://doi.org/10.1017/S0013091507000648 Published online by Cambridge University Press

It is easy to see that θ is a morphism of $\mathcal{O}_S$-modules and that n extends to a homogeneous polynomial law of degree d and therefore defines a section $\sigma : S \to \mathrm{Spec}(\Gamma^d_{\mathcal{O}_S}(\pi_*\mathcal{O}_Y)) =: \Gamma^d(Y/S)$. For any line bundle L on Y we also define its *norm* as an invertible sheaf

$$N(L) = \mathrm{Hom}_{\mathcal{O}_S}(\Lambda^d(\pi_*\mathcal{O}_Y), \Lambda^d(\pi_* L)).$$

On the other hand, if $S = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ are affine and L is given by an invertible B-module M, then $\Gamma^d_A(M)$ is naturally a $\Gamma^d_A(B)$-module [**10**, **12**]. Gluing the modules $\Gamma^d_A(M)$ on the elements of an affine cover of a general (i.e. non-affine) S, we obtain a quasi-coherent sheaf on $\Gamma^d(Y/S)$, which we denote by $\Gamma^d(L)$. One can show that $\Gamma^d(L)$ is an invertible $\mathcal{O}_{\Gamma^d(Y/S)}$-module whenever L is (see § 4.1). We have the following important result.

Lemma 2.2. *In the notation introduced above, $N(L) \simeq \sigma^*\Gamma^d(L)$. We also have canonical isomorphisms*

$$N(L \otimes_{\mathcal{O}_Y} F) \simeq N(L) \otimes_{\mathcal{O}_S} N(F), \qquad N(\pi^*(H)) \simeq H^{\otimes d},$$

where F is an invertible $\mathcal{O}_Y$-module and H is an invertible $\mathcal{O}_S$-module.

Proof. The first two assertions are proved in Proposition 3.3 in [**6**] (in fact, F can be a coherent sheaf on Y if the norm is understood as in [**6**]) and the third follows from the above definition of $N(L)$ and the projection formula. □

3. Families of zero cycles via norm and trace maps

3.1. The relationship between traces and norms

First assume that $S = \mathrm{Spec}(A)$, that $X = \mathrm{Spec}(B)$ and that $d!$ is invertible in A. Then $\Gamma^d_A(B)$ is isomorphic to the algebra of symmetric tensors $TS^d_A(B)$ (see [**10**, Theorem 4.6]). On the one hand, A-algebra homomorphisms $\Gamma^d_A(B) \to A$ correspond to homogeneous multiplicative polynomial laws $B \to A$ of degree d. On the other hand, homomorphisms $TS^d_A(B) \to A$ are described by certain 'trace morphisms' $\theta : B \to A$ of A-modules (see [**1**] and [**2**]; in the latter they are called Frobenius d-homomorphisms). We briefly outline the relationship between polynomial laws and trace morphisms. It does not seem to appear in the literature as explicitly as we state it here, although many ingredients can be found in [**1**], in [**2**] and in Iversen's formalism of linear determinants [**8**]. Two more results are relevant here: the description of a general polynomial law via determinants, which is given in [**17**] (in fact, our discussion below generalizes the well-known relationship between the trace and the determinant of a matrix), and the classical result of Weyl that the divided power algebras for invertible $d!$ are generated by elements of the form $\gamma^{d-1}(1) \times \gamma^1(b)$ (which in the setting described below means that any trace defines *at most one* norm).

The polynomial law n gives, in particular, a map $n_{A[t]} : B[t] \to A[t]$. Imitating the relationship between the determinant and the trace of a linear operator (see also § 2.2), we define $\theta : B \to A$ as the map that sends b to the coefficient of t in $n_{A[t]}(1 + bt)$. More generally, for $k \geqslant 1$ define a map

$$\Theta_k : B^{\times k} \to A$$

https://doi.org/10.1017/S0013091507000648 Published online by Cambridge University Press

by sending $(b_1,\dots,b_k)$ to the coefficient of $t_1\cdots t_k$ in $n_{A[t_1,\dots,t_k]}(1+t_1b_1+\cdots+t_kb_k)$. In terms of the morphism of A-algebras $\Psi_n : \Gamma^d_A(B)\to A$ corresponding to n, by property (5) in § 2.1 we have

$$\Theta_k(b_1,\dots,b_k)=\Psi_n(\gamma^{d-k}(1)\times\gamma^1(b_1)\times\cdots\times\gamma^1(b_k)).$$

Lemma 3.1. *For any degree-d polynomial law n, the maps Θ_k, $k\geqslant 1$, satisfy the following properties.*

(1) *$\Theta_k=0$ for $k>d$.*

(2) *$\Theta_1=\theta$ and $\Theta_d(x,\dots,x)=d!n(x)$.*

(3) *Θ_k is symmetric in its arguments and A-linear in each of them, i.e. it descends to an A-module morphism from the kth symmetric power $S^k_A(B)\to A$.*

(4) *If, in addition, n is multiplicative, then the following formula holds for all $k\geqslant 1$:*

$$\begin{aligned}\Theta_{k+1}(b_1,\dots,b_{k+1}) &:= \theta(b_1)\Theta_k(b_2,\dots,b_{k+1})-\Theta_k(b_1b_2,b_3,\dots,b_{k+1})\\ &\quad -\Theta_k(b_2,b_1b_3,\dots,b_{k+1})-\cdots-\Theta_k(b_2,b_3,\dots,b_1b_{k+1}).\end{aligned}$$

Proof. Properties (1) and (2) immediately follow from the definitions and the identity $\gamma^1(x)^{\times d}=d!\gamma^d(x)$. In (3), symmetry also follows immediately from the definitions. Part (4) follows from multiplicativity of Ψ_n and the product formula (see [**6**, Formula (2.4.2)]):

$$\begin{aligned}&[\gamma^{d-1}(1)\times\gamma^1(b_1)][\gamma^{d-k}(1)\times\gamma^1(b_2)\times\cdots\times\gamma^1(b_{k+1})]\\ &\qquad =\gamma^{d-k-1}(1)\times\gamma^1(b_1)\times\cdots\times\gamma^1(b_{k+1})\\ &\qquad\quad +\sum_{i=2}^{k+1}\gamma^{d-k}(1)\times\gamma^1(b_2)\times\cdots\times\gamma^1(b_1b_i)\times\cdots\times\gamma^1(b_{k+1}).\end{aligned}$$

Finally, A-multilinearity in (3) follows from the linearity of γ^1 and the linearity of Ψ_n. □

Definition 3.2. Let B be an A-algebra. A morphism of A-modules $\theta : B\to A$ is a *degree-d trace* if

$$\theta(1)=d,\qquad \Theta_{d+1}\equiv 0,$$

where the Θ_k are constructed from $\theta =: \Theta_1$ using formula (4) in Lemma 3.1.

Lemma 3.3. *The operations*

$$\theta(b)\mapsto n(b)=\frac{1}{d!}\Theta_d(b,\dots,b)$$

and

$$n(b)\mapsto\theta(b)=\Psi_n(\gamma^{d-1}(1)\times\gamma^1(b))=\frac{1}{\varepsilon}(n_{A[\varepsilon]/\varepsilon^2}(1+\varepsilon b)-1)$$

define mutually inverse bijections between the set of degree-d traces and the set of degree-d norm maps.

https://doi.org/10.1017/S0013091507000648 Published online by Cambridge University Press

Proof. The formula $n(x) = (1/d!)\Theta_d(x, \dots, x)$ defines a polynomial law n: since θ has canonical pullbacks $\theta \otimes 1 : B \otimes A' \to A'$ for all A-algebras A', so does n. By part (3) of Lemma 3.1, $n(x)$ is homogeneous of degree d and it is multiplicative by [**1**, Theorem 1.5.3] or [**2**, Theorem 2.8]. Let us show that the two constructions are mutually inverse to each other.

First assume that $n(x) = (1/d!)\Theta_d(x, \dots, x)$ and let us show that the trace constructed from n coincides with the original $\theta = \Theta_1$. If the Θ_k are defined from θ using formula (4) in Lemma 3.1, one can show that the Θ_k are symmetric and multilinear (see, for example, Definition 1.3.1 in [**1**], where the Θ_k are denoted by P^k_θ). The polynomial law $n(x)$ gives an A-algebra homomorphism $\Psi_n : \Gamma^d_A(B) \to A$ and

$$\Theta_d(b, \dots, b) = \Psi_n(d!\gamma^d(b)) = \Psi_n(\gamma^1(b) \times \cdots \times \gamma^1(b)).$$

Since $\gamma^1(b)$ is A-linear in b and $d!$ is invertible in A, by an easy polarization argument we conclude that $\Theta_d(b_1, \dots, b_d) = \Psi_n(\gamma^1(b_1) \times \cdots \times \gamma^1(b_d))$. Using the recursive definition of Θ_k we obtain

$$\Theta_{k+1}(1, b_2, \dots, b_{k+1}) = (d-k)\Theta_k(b_2, \dots, b_{k+1})$$

and by descending induction on k we conclude that $\Theta_k(b_1, \dots, b_k) = \Psi_n(\gamma^{d-1}(1) \times \gamma^1(b_1) \times \cdots \times \gamma^1(b_k))$. In particular,

$$\theta(b) = \Psi_{(1/d!)\Theta_d(b,\dots,b)}(\gamma^{d-1}(1) \times \gamma^1(b)).$$

On the other hand, if we start with a polynomial law $n(x)$ and set $\theta(b) = \Psi_n(\gamma^{d-1}(1) \times \gamma^1(b))$, then Lemma 3.1 tells us that $n(b) = (1/d!)\Theta_d(b, \dots, b)$, as required. □

Example 3.4. Let $f_i : B \to A$ be A-algebra homomorphisms for $i = 1, \dots, d$. Then the product $n = f_1 \cdots f_d$ is a degree-d homogeneous polynomial law and $\theta = f_1 + \cdots + f_d$ is the degree-d trace corresponding to it, while Θ_k is given by the kth elementary symmetric function in the f_i (up to a scalar).

Example 3.5. Let $A = k$ be a field of characteristic p with $p = 0$ or $p > d$ and Q a k-point of $\operatorname{Spec}(B)$ with $k(Q) = k$, corresponding to the evaluation homomorphism $B \to k$, $b \mapsto b(Q)$. Consider the polynomial law $b \mapsto b(Q)^d$ corresponding to the effective cycle $[dQ] \in \operatorname{Spec}(\Gamma^d_k(B)) \simeq \operatorname{Sym}^d(\operatorname{Spec}(B))$. If $\mathfrak{m}$ is the maximal ideal of Q in B, we have the following formula for the dual of the tangent space at $[dQ]$:

$$T^\vee_{[dQ]} \simeq \mathfrak{m}/\mathfrak{m}^{d+1}.$$

In fact, by assumption on k, an element of $T_{[dQ]}$ corresponds to a degree-d trace

$$\theta = \theta' + \varepsilon\theta'' : B \to k[\varepsilon]/\varepsilon^2 = k \oplus \varepsilon k$$

with $\theta'(f) = d \cdot f(Q)$. Since $\theta(1) = d$, θ'' vanishes on the subspace of constants $k \subset B$ and we can therefore identify it with a linear function of $\mathfrak{m}$. Let us show that the condition $\Theta_{d+1}(b_1, \dots, b_{d+1}) = 0$ for all $b_i \in B$ is equivalent to $\theta''(\mathfrak{m}^{d+1}) = 0$. In fact, since

Θ_{d+1} is multilinear, we can assume that each b_i is either 1 or in $\mathfrak{m}$. If at least one of the b_i is 1, by symmetry we can assume that $b_1 = 1$ and then $\theta(1) = d$ together with formula (4) in Lemma 3.1 immediately give the vanishing of Θ_{d+1}. If all the arguments $b_1, \dots, b_{d+1}$ are in $\mathfrak{m}$, then it is easy to show by induction using the same formula that $\Theta_{l+1}(b_1, \dots, b_{l+1}) = (-1)^l l! \varepsilon\theta''(b_1 \cdots b_{l+1})$, with $l \geqslant 0$ and the usual convention $0! = 1$. In particular, $\Theta_{d+1}(b_1, \dots, b_{d+1}) = (-1)^d d! \varepsilon\theta''(b_1 \cdots b_{d+1})$ and, using our assumption on k again, we see that $\Theta_{d+1} \equiv 0$ if and only if θ'' vanishes on $(d+1)$-fold products of elements in $\mathfrak{m}$, i.e. it descends to a linear function on $\mathfrak{m}/\mathfrak{m}^{d+1}$, which proves the assertion.

The isomorphism above also has a direct (but less conceptual) definition. Let $\mathfrak{M}$ be the maximal ideal of the point $[dQ] \in \mathrm{Sym}^d(\mathrm{Spec}(B))$. For any $b(x) \in \mathfrak{m}$, the expression $b(x_1) + \cdots + b(x_d)$ can be viewed as a symmetric element in the d-fold tensor power or in B. This defines a k-linear map

$$\mathfrak{m} \to \mathfrak{M}$$

and the above argument with polynomial laws essentially shows that the composition $\mathfrak{m} \to \mathfrak{M}/\mathfrak{M}^2$ is surjective with kernel $\mathfrak{m}^{d+1}$ (it is also possible, although a little harder, to check this directly).

3.2. Functors of zero cycles

Definition 3.6. Let $\pi : X \to S$ be a separated morphism of algebraic spaces [**9**]. Let $\mathrm{Chow}^n_{\pi,d}$ be the functor on the category of algebraic spaces over S, sending $T \to S$ to a set $\mathrm{Chow}^n_{\pi,d}(T)$ of equivalence classes of pairs (Y, n), where $Y \hookrightarrow X_T$ is a closed algebraic subspace that is integral over T, and $n : (\pi_T)_*\mathcal{O}_Y \to \mathcal{O}_T$ is a multiplicative polynomial law of degree d. Recall that Y is integral over T if it is affine and locally over T every regular function on Y satisfies a monic polynomial with coefficients in $\mathcal{O}_T$. Two pairs (Y_1, n_1) and (Y_2, n_2) are called equivalent if there is a third pair (Y, n) such that Y is an algebraic subspace in $Y_1 \cap Y_2$ that is integral over T and n_i is equal to the composition of the natural morphism $(\pi_T)_*\mathcal{O}_{Y_i} \to (\pi_T)_*\mathcal{O}_Y$ with n, for $i = 1, 2$. The inverse image of (Y, n) with respect to an S-morphism $\phi : T' \to T$ is given by (Y', n'), where $Y' = Y \times_T T'$ and n' is described on elements of an étale covering $T' = \bigcup \mathrm{Spec}(A_i)$ by restricting n to those $\mathcal{O}_T$-algebras that factor through A_i.

Definition 3.7. Let $\pi : X \to S$ be as before and assume that $d!$ defines an invertible regular function on S. Let $\mathrm{Chow}^\theta_{\pi,d}$ be the functor on the category of algebraic spaces over S, which is given by equivalence classes of pairs (Y, θ), where $Y \hookrightarrow X_T$ is a closed algebraic subspace of X_T that is integral over T and $\theta : (\pi_T)_*\mathcal{O}_Y \to \mathcal{O}_T$ is a degree-d trace. Equivalence of such pairs is defined in a similar way. Note that for a pullback (Y', θ') with respect to $\phi : T' \to T$, we can define θ' simply as $\theta \otimes_{\mathcal{O}_T} \mathcal{O}_{T'}$.

Proposition 3.8. *Assume that $d!$ is an invertible regular function on S. There exists an isomorphism of functors* $\mathrm{Chow}^n_{\pi,d} \simeq \mathrm{Chow}^\theta_{\pi,d}$.

Proof. The proof follows immediately from Lemma 3.3. □

https://doi.org/10.1017/S0013091507000648 Published online by Cambridge University Press

Remark 3.9. The definition of $\mathrm{Chow}^n_{\pi,d}$ is a restatement of Definition 3.1.1 in [**13**]. Therefore, this functor is represented by the space of divided powers $\Gamma^d(X/S)$ and the above proposition can be easily derived from the results of [**13**]. The definition of $\mathrm{Chow}^\theta_{\pi,d}$ is a version of the definition that appears on p. 7 of [**1**] applied to zero cycles, but it is stated in greater generality here.

Remark 3.10. Let $n : B \to A$ be a degree-d norm map. Then, following [**13**, Definition 2.1.5] we define the characteristic polynomial of $b \in B$ by the formula

$$\chi_{n,b}(t) := n_{A[t]}(b - t) = \sum_{k=0}^{d} (-1)^k \Psi_n(\gamma^k(1) \times \gamma^{d-k}(b)) t^k \in A[t].$$

In the notation of Lemma 3.1 we have

$$\chi_{n,b}(t) = \sum_{k=0}^{d} (-1)^k \frac{\Theta_{d-k}(b, \dots, b)}{(d-k)!} t^k.$$

Now let $J_n \subset B$ be the ideal generated by $\chi_{n,b}(b)$ for all $b \in B$, which is called the *Cayley–Hamilton ideal* of n. Then, by Proposition 2.1.6 in [**13**], the norm map n factors through the quotient B/J_n. Similarly, for a degree-d trace $\theta : B \to A$, [**1**, § 1.6] defines an ideal $J_\theta \subset B$ such that θ factors through B/J_θ.

We observe here that $J_n = J_\theta$ if n and θ are related by the bijection of Lemma 3.1. In fact, by Definition 1.6.2.2 of [**1**], J_θ is generated by values of the polarized version of $\chi_{n,b}$ involving the quantities $\Theta_{d-k}(b_1, \dots, b_{d-k})$, which are defined recursively using the formula from part (4) of Lemma 3.1. In the case of J_n, the values $\Theta_{d-k}(b, \dots, b)$ are given by $\Psi_n(d!\gamma^d(b))$. Since $d!$ is assumed to be invertible in A, the values of the polarized version generate the same ideal as values of $\chi_{n,b}$ itself, and the assertion $J_\theta = J_n$ reduces to the equivalence of the two formulae for Θ_{d-k}.

Remark 3.11. In [**1**] traces were defined on the completion $\hat{X}_T$ at a closed subset $Z \subset X_T$ that is proper and of pure relative dimension zero over T. But by Corollary 1.6.3 in [**1**] a degree-d trace $\theta : (\pi_T)_* \mathcal{O}_{\hat{X}_T} \to \mathcal{O}_T$ descends to the subscheme Y in X_T defined by the ideal sheaf $\mathcal{J}_\theta$. Since such a Y is integral over T, we can restrict to integral subschemes in the definition.

4. Families of zero cycles via norm functors

4.1. Divided powers of line bundles

Let $\pi : X \to S$ be an affine morphism of algebraic spaces. For any quasi-coherent sheaf L on X, the sheaf $\Gamma^d_{\mathcal{O}_S}(\pi_* L)$ is a module over the $\mathcal{O}_S$-algebra $\Gamma^d_{\mathcal{O}_S}(\pi_* \mathcal{O}_X)$ [**10**]. This gives a quasi-coherent sheaf $\Gamma^d(L)$ on $\Gamma^d(X/S)$.

We now recall a construction from [**6**, § 3], presenting it here in a sheafified version. Let L' and L'' be two quasi-coherent sheaves on X. There is a unique functorial morphism of $\mathcal{O}_S$-modules

$$\Gamma^d_{\mathcal{O}_S}(\pi_* L') \otimes_{\mathcal{O}_S} \Gamma^d_{\mathcal{O}_S}(\pi_* L'') \to \Gamma^d_{\mathcal{O}_S}(\pi_* L' \otimes_{\mathcal{O}_S} \pi_* L'')$$

that sends $\gamma^d(x) \otimes \gamma^d(y)$ to $\gamma^d(x \otimes y)$ (see [**12**], [**6**] and [**10**, (7.1.7)] in the affine case). The image on a general element is given by Formula (2.4.2) in [**6**]. The composition of this map with $\Gamma^d_{\mathcal{O}_S}(\pi_* L' \otimes_{\mathcal{O}_S} \pi_* L'') \to \Gamma^d_{\mathcal{O}_S}(\pi_*(L' \otimes_{\mathcal{O}_X} L''))$ descends to a morphism of $\Gamma^d_{\mathcal{O}_S}(\pi_* \mathcal{O}_X)$-modules

$$\Gamma^d_{\mathcal{O}_S}(\pi_* L') \otimes_{\Gamma^d_{\mathcal{O}_S}(\pi_* \mathcal{O}_X)} \Gamma^d_{\mathcal{O}_S}(\pi_* L'') \to \Gamma^d_{\mathcal{O}_S}(\pi_*(L' \otimes_{\mathcal{O}_X} L'')). \tag{4.1}$$

The following result does not seem to appear in the literature.

Lemma 4.1. *The morphism (4.1) is an isomorphism if at least one of the sheaves L', L'' is an invertible $\mathcal{O}_X$-module. In particular, $\Gamma^d(L)$ is an invertible module on $\Gamma^d(X/S)$ if L is an invertible module on X. The map $L \mapsto \Gamma^d(L)$ extends to a functor $N_\Gamma : \mathrm{PIC}(X) \to \mathrm{PIC}(\Gamma^d(X/S))$ between the categories PIC of invertible modules (and morphisms as $\mathcal{O}$-modules). The functor N_Γ is equipped with isomorphisms*

$$\Gamma^d(L') \otimes_{\mathcal{O}_{\Gamma^d(X/S)}} \Gamma^d(L'') \simeq \Gamma^d(L' \otimes_{\mathcal{O}_X} L'')$$

that agree with commutativity and associativity isomorphisms for tensor product of line bundles. If $\pi^d : \Gamma^d(X/S) \to S$ is the canonical morphism, then the induced map

$$\pi_* \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(L', L'') \to \pi^d_* \mathcal{H}\mathrm{om}_{\mathcal{O}_{\Gamma^d(X/S)}}(\Gamma^d(L'), \Gamma^d(L''))$$

extends canonically to a homogeneous polynomial law of degree d. When π is finite and flat of rank d and L is a line bundle on X, one has a canonical isomorphism

$$N(L) \simeq \sigma^* N_\Gamma(L),$$

where $N(L)$ is the norm defined in § 2.2 and $\sigma : S \to \Gamma^d(X/S)$ is the canonical section.

Proof. Here we prove the first two assertions, since the statements about the PIC functor follow from a routine check based on the definitions involved and the isomorphism of two norms is proved in Lemma 2.2.

To prove that (4.1) is an isomorphism, we can assume that $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are affine and that L' is given by a projective B-module P of rank 1. Observe that for any finite subset of closed points $x_1, \dots, x_l$ there is an affine open subset $U \subset X$ containing these points, such that $L|_U$ is trivial. In fact, we can assume that no x_i is in the closure of another x_j (otherwise we can erase x_j from the list, since trivialization of L in a neighbourhood of x_i will also give a trivialization for x_j). Since X is affine, we can choose sections $l_1, \dots, l_d$ in $H^0(X, L)$, generating the stalks $L_{x_1}, \dots, L_{x_d}$, respectively. Also, by the Chinese Remainder Theorem we can choose functions $f_1, \dots, f_d$ in $H^0(X, \mathcal{O}_X)$ such that each (image of) f_i generates the stalk $\mathcal{O}_{x_i}$ and vanishes at x_j for $i \neq j$. The section $l = l_1 f_1 + \dots + l_d f_d$ then generates the stalks of L at $x_1, \dots, x_d$ and hence defines a trivialization of L in an affine neighbourhood U of $x_1, \dots, x_d$.

Choose and fix a closed point $\beta \in \Gamma^d(X/S)$. It suffices to prove that (4.1) is an isomorphism in a neighbourhood of β. If $x_1, \dots, x_l$, $l \leqslant d$, is the support of β in X (see, for example, [**13**, Theorem 2.4.6]), then choosing an open affine neighbourhood U

of $x_1, \dots, x_l$ and a trivialization $L'|_U$ as above, we obtain an open affine neighbourhood $\Gamma^d(U/S) \subset \Gamma^d(X/S)$ of β an isomorphism of $\Gamma^d(L')|_{\Gamma^d(U/S)}$ with the structure sheaf. Thus, on $\Gamma^d(U/S)$ the map (4.1) is an isomorphism and $\Gamma^d(L')$ is locally free of rank 1. □

Remark 4.2. Note that, once we know the isomorphism (4.1), the fact that $\Gamma^d(L')$ is invertible may also be proved by choosing L'' to be the dual of L'.

Remark 4.3. By Equation (2.4.3.1) of [**6**], for any invertible $\mathcal{O}_S$-module P and any $\mathcal{O}_X$-module F one has

$$\Gamma^d(\pi^* P \otimes_{\mathcal{O}_X} F) \simeq (\pi^d)^*(P^{\otimes d}) \otimes_{\mathcal{O}_{\Gamma^d(X/S)}} \Gamma^d(F)$$

as $\Gamma^d(X/S)$-modules.

Remark 4.4. Suppose that S can be covered by affine open subsets V_i such that L is trivial on $\pi^{-1}(V_i)$ (for instance, that we are in the situation of § 2.2). Then $\Gamma^d(L)$ is trivial on the open subset $(\pi^d)^{-1}(V_i)$. If the ϕ_{ij} are the transition functions for L, then their norms $\gamma^d(\phi_{ij})$ are the transition functions of $\Gamma^d(L)$. This construction was originally given for the setting of § 2 by Grothendieck in [**7**, 6.5].

Remark 4.5. The construction of $\Gamma^d(L)$ is globalized in § 10 of [**14**] so that it applies to any separated morphism $\pi : X \to S$ of algebraic spaces. The multiplicativity of N_Γ remains valid in this setting, as does the isomorphism $\Gamma^d(\pi^* P) \simeq (\pi^d)^* P{\otimes}d$, which follows from Remark 4.3.

4.2. Norm functors

Definition 4.6. Let $\pi : Y \to S$ be a morphism of algebraic spaces. Denote by $\mathrm{PIC}(Y/S)$ the category with objects given by (S', L), where $S' \to S$ is an algebraic space over S and L is a line bundle on $Y_{S'} = Y \times_S S'$. A morphism $(\xi, \rho) : (S_1, L_1) \to (S_2, L_2)$ in the category $\mathrm{PIC}(Y/S)$ is given by a morphism $\xi : S_1 \to S_2$ of algebraic spaces over S, plus a morphism $\rho : L_1 \to \xi^*(L_2)$ of coherent sheaves on Y_{S_1}. There is an obvious forgetful functor $p_Y : \mathrm{PIC}(Y/S) \to \mathrm{Sp}/S$ to the category of algebraic spaces over S, given by $(T, L) \mapsto T$ and $(\xi, \rho) \mapsto \xi$. When $Y = S$ and π is the identity morphism, we write $\mathrm{PIC}(S)$ instead of $\mathrm{PIC}(S/S)$. There is a natural functor $\mathrm{PIC}(S) \to \mathrm{PIC}(Y/S)$ given by the pullback of L from S' to $Y_{S'}$. We denote this functor simply by π^*.

Definition 4.7. Let $\pi : Y \to S$ be as above. A *norm functor of degree d over π* is a triple $\mathcal{N} = (N, \mu, \epsilon)$, where N is a functor $\mathrm{PIC}(Y/S) \to \mathrm{PIC}(S)$ such that $p_S \circ N = p_Y$. In other words, a pair $(S', L) \in \mathrm{Ob}(\mathrm{PIC}(Y/S))$ is sent to a pair $(S', M) \in \mathrm{Ob}(\mathrm{PIC}(S))$ and sometimes we will abuse notation by dropping S' and writing $M = N(L)$. Furthermore, for any pair (S', L_1), (S', L_2) of objects in $\mathrm{PIC}(Y/S)$ with the same S', we require an isomorphism

$$\mu_{S', L_1, L_2} : N(L_1) \otimes_{\mathcal{O}_{S'}} N(L_2) \simeq N(L_1 \otimes_{\mathcal{O}_{Y_{S'}}} L_2)$$

such that the system of isomorphisms $\mu = \mu_{\{S', \cdot, \cdot\}}$ agrees with the base change and the standard symmetry and associativity isomorphisms for line bundles on $Y_{S'}$ and S', respectively (see, for example, the last two diagrams on p. 36 of [**4**]). Finally, ϵ is an isomorphism of functors $\mathrm{PIC}(S) \to \mathrm{PIC}(S)$,

$$\epsilon : N \circ \pi^* \simeq (\cdot)^{\otimes d},$$

such that $\mu_{\{S',\cdot,\cdot\}} \circ (N\pi^* \otimes N\pi^*)$ is given by the canonical isomorphism

$$L_1^{\otimes d} \otimes_{\mathcal{O}_{S'}} L_2^{\otimes d} \simeq (L_1 \otimes_{\mathcal{O}_{S'}} L_2)^{\otimes d}.$$

Definition 4.8. Let $\pi : X \to S$ be a morphism of algebraic spaces. Let $\mathrm{Chow}^N_{\pi,d}$ be the functor from the category Sp/S of algebraic spaces over S to sets, sending $T \to S$ to equivalence classes of data $(Y, \mathcal{N})$, where $Y \hookrightarrow X_T$ is a closed algebraic subspace, integral over T and quasi-finite over it (but not necessarily locally of finite type), and $\mathcal{N}$ is a degree-d norm functor over $(\pi_T)|_Y$. Two pairs $(Y_1, \mathcal{N}_1)$ and $(Y_2, \mathcal{N}_2)$ are called equivalent if there is a third subspace $Y \subset Y_1 \cap Y_2$ and a degree-d norm functor $\mathcal{N}$ over $(\pi_T)|_Y$ together with isomorphisms between $N_i : \mathrm{PIC}(Y_i/S) \to \mathrm{PIC}(S)$ and the composition of $N : \mathrm{PIC}(Y/S) \to \mathrm{PIC}(S)$ with restriction from Y_i to Y, which are also required to agree with ϵ_i and μ_i in the obvious sense.

Remark 4.9. Since by definition a norm functor is local on S, we obtain a map

$$(\pi_{S'})_* \mathcal{H}\mathrm{om}_{\mathcal{O}_{Y_{S'}}}(L_1, L_2) \to \mathcal{H}\mathrm{om}_{\mathcal{O}_{S'}}(N(L_1), N(L_2)). \tag{4.2}$$

This map is not $\mathcal{O}_{S'}$-linear; it is rather a polynomial law of degree d. To show this, it suffices to assume that $L_1 = \mathcal{O}_{Y_{S'}}$. In fact, by definition N preserves tensor products and using $N(\mathcal{O}_{Y_{S'}}) \simeq N(\pi^*_{S'}\mathcal{O}_{S'}) \simeq \mathcal{O}_{S'}^{\otimes d} \simeq \mathcal{O}_{S'}$ we have $N(L^\vee) = N(L)^\vee$. The left-hand side can be rewritten as $(\pi_{S'})_* \mathcal{H}\mathrm{om}_{\mathcal{O}_{Y_{S'}}}(\mathcal{O}_{Y_{S'}}, L_1^\vee \otimes L_2)$, while the right-hand side becomes

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{S'}}(\mathcal{O}_{S'}, N(L_1)^\vee \otimes N(L_2)) \simeq \mathcal{H}\mathrm{om}_{\mathcal{O}_{S'}}(\mathcal{O}_{S'}, N(L_1^\vee \otimes L_2)).$$

A local section f of $\mathcal{O}_{S'}$ acts on $(\pi_{S'})_* \mathcal{H}\mathrm{om}_{\mathcal{O}_{Y_{S'}}}(\mathcal{O}_{Y_{S'}}, L_1^\vee \otimes L_2)$ by composition with the 'multiplication by f' endomorphism of $\mathcal{O}_{Y_{S'}} \simeq \pi^*_{S'}\mathcal{O}_{S'}$. By definition of ϵ, the norm functor sends it to multiplication by f^d.

Lemma 4.10. *Let $\pi : Y \to S$ be an integral morphism of algebraic spaces that is quasi-finite (but not necessarily locally of finite type) and let L be a line bundle on Y. Any point $s \in S$ has an étale neighbourhood $U \subset S$ such that the restriction of L on $\pi^{-1}(U)$ is trivial.*

Proof. It suffices to assume that S, and hence also Y, are affine. Since the fibre $\pi^{-1}(s)$ is finite, by repeating the argument in Lemma 4.1 we can find a section l of L on Y that generates the stalks of L at each of the points in $\pi^{-1}(s)$. The subset $W \subset Y$ of points where l *fails* to generate the stalk of L is closed in Y and is disjoint from the fibre $\pi^{-1}(s)$. Its image $\pi(W)$ is closed in Y, since π is integral, and does not contain s. Hence s admits an affine Zariski neighbourhood $U \subset (Y \setminus \pi(W))$ such that on $\pi^{-1}(U)$ the line bundle L is trivialized by the section l. $\square$

Lemma 4.11. *If $\pi : Y \to S$ is as in the previous lemma, then no norm functor has any non-trivial automorphisms.*

Proof. A functor automorphism is given by a family of isomorphisms $\phi_{(T,L)} : N(L) \to N(L)$ for all objects (T, L) of $\mathrm{PIC}(Y/S)$. If L is pulled back from T, this automorphism has to be the identity since it has to respect ϵ. By the previous lemma, we can find an étale open cover $\{U_i\}$ of T such that L is trivial on the preimage of each U_i in Y_T. The restriction of $\phi_{(T,L)}$ to each U_i is then the identity due to the agreement with ϵ, and hence $\phi_{(T,L)}$ is itself the identity. □

Remark 4.12. For a general π the previous result fails. One possible example is the situation when Y_0 and S are over a field k, $Y = Y_0 \times_{\mathrm{Spec}(k)} S$ and there exists a non-trivial group homomorphism $\mathrm{Pic}(Y_0) \to \mathcal{O}_S^*$ (where 'Pic' is the group of isomorphism classes of line bundles on Y_0). For instance, if Y_0 is the complement to a smooth degree-n hypersurface in $\mathbb{P}^n$, then it is well known that $\mathrm{Pic}(Y_0) \simeq \mathbb{Z}_n$ and such a homomorphism indeed exists.

Proposition 4.13. *If $\pi : X \to S$ is a separated morphism of algebraic spaces, the functor $\mathrm{Chow}^N_{\pi,d}$ is isomorphic to $\mathrm{Chow}^n_{\pi,d}$ and is therefore represented by the space of divided powers $\Gamma^d(X/S)$.*

Proof. An S-morphism $T \to \Gamma^d(X/S)$ induces a norm functor by Lemma 4.1 and Remark 4.5, since one can take the pullback of line bundles $\Gamma^d(L)$ to T. Conversely, taking $L_1 = L_2 = \mathcal{O}_Y$ in (4.2) we obtain a norm map $(\pi_T)_*\mathcal{O}_{Y_T} \to \mathcal{O}_T$.

It is obvious that $\mathrm{Chow}^n_{\pi,d} \to \mathrm{Chow}^N_{\pi,d} \to \mathrm{Chow}^n_{\pi,d}$ is the identity since we are essentially expanding the data involved in the definition of $\mathrm{Chow}^n_{\pi,d}$ and then forgetting the extra data constructed.

In the opposite direction, suppose we have a closed subspace $Y \subset X_T$, integral and quasi-finite over T, and a norm functor $\mathcal{N} = (N, \mu, \epsilon)$ over $(\pi_T)|_Y$ that we use to extract the polynomial law $(\pi_T)_*\mathcal{O}_Y \to \mathcal{O}_T$ and thus obtain an S-morphism $\sigma : T \to \Gamma^d(Y/S)$. We need to construct isomorphisms $N(L) \simeq \sigma^*(\Gamma^d(L))$ for all line bundles L, which commute with pullbacks, agree with multiplicativity isomorphisms and give identity on $L^{\otimes d}$ for those L which are pulled back from T to Y. In other words, we need to prove an isomorphism

$$\mathcal{O}_T \otimes_{\Gamma^d_{\mathcal{O}_T}((\pi_T)_*\mathcal{O}_T)} \Gamma^d_{\mathcal{O}_T}((\pi_T)_*L) \simeq N(L).$$

It suffices to construct a morphism from the left-hand side to the right-hand side and then apply Lemma 4.1 to find a Zariski open covering $\{U_i\}$ of T such that L is trivial on the preimage of U_i, in which case the isomorphism becomes a tautology. To that end, observe that (4.2) gives a polynomial law $(\pi_T)_*L \to N(L)$ by choosing L_1 to be the trivial bundle and $L_2 = L$ and using that the norm of a trivial bundle is trivial (see the lines after (4.2)) and that Y is integral (and hence affine) over T. Thus we get a morphism of $\mathcal{O}_T$-modules

$$\mu_L : \Gamma^d_{\mathcal{O}_T}((\pi_T)_*L) \to N(L).$$

https://doi.org/10.1017/S0013091507000648 Published online by Cambridge University Press

The fact that μ_L descends to the above tensor product is equivalent to the formula

$$\mu_L(f \cdot s) = \mu_{\mathcal{O}_T}(f) \cdot \mu_L(s), \tag{4.3}$$

where f (respectively s) is a local section of $\Gamma^d_{\mathcal{O}_T}((\pi_T)_*\mathcal{O}_Y)$ (respectively $\Gamma^d_{\mathcal{O}_T}((\pi_T)_*L)$), and the module structure of the left-hand side is given, for example, by [**10**, Formula (7.6.1)]. But (4.3) follows from the fact that N is a functor, i.e. it maps compositions of morphisms to compositions of morphisms, and the fact that after a faithfully flat base change the $\mathcal{O}_T$-module $\Gamma^d_{\mathcal{O}_T}((\pi_T)_*L)$ is generated by $\gamma^d((\pi_T)_*L)$ (see Lemma 2.3.1 in [**6**]).

Agreement of $\sigma^*(\Gamma^d(L)) \simeq N(L)$ with multiplicativity isomorphisms with ϵ also follows from the functor property of N. □

4.3. Non-homogeneous norm functors

One can also give a definition of a non-homogeneous norm functor as a triple (N, μ, ϵ), where N and μ are as before and ϵ is an isomorphism

$$\epsilon : N(\mathcal{O}_Y) \simeq \mathcal{O}_S$$

that sends the identity endomorphism of $\mathcal{O}_Y$ to the identity endomorphism of $\mathcal{O}_S$.

Observe that the proof of Lemma 4.10 still works in this case, and hence non-homogeneous norm functors form a set. As in the homogeneous case, any such functor gives a multiplicative polynomial law

$$\pi_*\mathcal{O}_Y \to \mathcal{O}_S,$$

and hence by [**14**, § 2] it defines a section

$$S \to \Gamma^\star(Y/S) = \coprod_{d\geqslant 0} \Gamma^d(Y/S).$$

Repeating the argument of the previous subsection one shows that the functor of zero cycles $\mathrm{Chow}^N_{\pi,\star}$ defined via non-homogeneous norm functors is isomorphic to the functor of zero cycles defined via non-homogeneous norm maps $\mathrm{Chow}^n_{\pi,\star}$. Therefore, $\mathrm{Chow}^N_{\pi,\star}$ is represented by the space of effective zero cycles $\Gamma^\star(X/S)$. Details are left to the motivated reader.

5. Standard constructions

5.1. Hilbert–Chow morphisms

If $\pi : Y \to S$ is finite and flat and $\pi_*\mathcal{O}_Y$ is locally free of constant rank d on S, the construction in § 2.2 gives the norm of a line bundle L on Y. Lemma 2.2 shows that the norm of line bundles defines a norm functor $\mathrm{PIC}(Y/S) \to \mathrm{PIC}(S)$ inducing a morphism of representing spaces:

$$\mathrm{Hilb}^d(X/S) \to \mathrm{Chow}^N_{\pi,d}(X/S).$$

5.2. Sums of cycles

If $(Y_1, \mathcal{N}_1)$ and $(Y_2, \mathcal{N}_2)$ are two families of zero cycles of degrees d_1 and d_2, respectively, then we define their sum as follows. Let $Y_1 \cup Y_2$ be the subspace of X corresponding to the sheaf of ideals $\mathcal{J}_1 \cap \mathcal{J}_2$, where $\mathcal{J}_i$ is the ideal sheaf of Y_i, $i = 1, 2$. There is then a norm functor of degree $d_1 + d_2$ given by $(Y_1 \cup Y_2, (\mathcal{N}_1 \circ i_1^*) \otimes (\mathcal{N}_2 \circ i_2^*))$, where $(i_l)_*$ is the functor defined by restriction of line bundles from $Y = Y_1 \cup Y_2$ to Y_l for $l = 1, 2$. Observe that $Y_1 \cup Y_2$ may not be integral over S but, after taking the quotient by the Cayley–Hamilton ideal discussed in Remark 3.11, we will obtain a closed subspace $(Y_1 \cup Y_2)'$ that is integral over S. This induces the sum morphism

$$\pi_{d_1,d_2} : \Gamma^{d_1}(X/S) \times_S \Gamma^{d_2}(X/S) \to \Gamma^{d_1+d_2}(X/S).$$

5.3. Universal families

For $T = \Gamma^d(X/S)$ consider the base change morphism $\pi_T : X_T \to T$. Set $Y = \Gamma^{d-1}(X/S) \times_S X$, which maps to T via the addition morphism $\pi_{d-1,1}$. By [**14**], Y is integral over T and can be identified with a closed subspace of $\Gamma^d(X/S) \times_S X$ via the morphism $(\xi, x) \mapsto (\xi + x, x)$. There does not seem to be a easy definition of the corresponding universal norm functor $N : \mathrm{PIC}(Y/T) \to \mathrm{PIC}(T)$, as is also the case with the universal norm map $(\pi_{d-1,1})_* \mathcal{O}_Y \to \mathcal{O}_T$. However, if $\eta : \Gamma^{d-1}(X/S) \times_S X \to X$ is the canonical projection, it follows easily that the composition

$$\mathrm{PIC}(X) \xrightarrow{\eta^*} \mathrm{PIC}(Y/T) \xrightarrow{N} \mathrm{PIC}(T) = \mathrm{PIC}(\Gamma^d(X/S))$$

is given simply by the functor $L \mapsto \Gamma^d(L)$.

5.4. Direct images of cycles

Let $\pi' : Z \to S$ be another separated morphism of algebraic spaces and let $f : X \to Z$ be a morphism over S. Take a family of zero cycles on X represented by a pair $(Y, \mathcal{N})$. By [**13**, Theorem 2.4.6] we can assume that Y has universally topologically finite fibres over S and hence, by the appendix to [**13**], $f(Y)$ is a well-defined algebraic subspace of Z that is integral over S. One can also give a more direct proof of this result using the approximation results in [**16**, Theorem D]. The direct image cycle is defined by $(f(Y), \mathcal{N} \circ f^*)$, which induces a morphism

$$\mathrm{Chow}^N_{\pi,d}(X/S) \to \mathrm{Chow}^N_{\pi',d}(Z/S).$$

5.5. Chow forms

Assume that $X = \mathrm{Proj}(\mathcal{A})$, where $\mathcal{A} = \bigoplus_{l \geqslant 0} \mathcal{A}_l$ is a graded quasi-coherent $\mathcal{O}_S$-algebra generated over $\mathcal{A}_0 = \mathcal{O}_S$ by its first component $\mathcal{A}_1$. The natural sheaf $\mathcal{O}(1)$ on X is then invertible.

Let $(Y, \mathcal{N})$ be a pair representing an element in $\mathrm{Chow}^N_{\pi,d}(T)$ with $\xi : T \to S$ and denote the inverse image of $\mathcal{O}(1)$ on Y by L. By assumption, a local section of $\xi^* \mathcal{A}_l$ on $U \subset T$

https://doi.org/10.1017/S0013091507000648 Published online by Cambridge University Press

gives a section of $L^{\otimes l}$ on $\pi_T^{-1}(U)$ and hence, by (4.2), a section of $N(L^{\otimes l}) \simeq N(L)^{\otimes l}$ on U itself. Therefore, we obtain a degree-d homogenous polynomial law

$$\xi^* \mathcal{A}_l \to N(L)^{\otimes l},$$

and therefore a morphism of $\mathcal{O}_S$-modules

$$\Omega_l : \Gamma^d_{\mathcal{O}_T}(\xi^* \mathcal{A}_l) \to N(L)^{\otimes l},$$

which we call the lth *Chow form* of $(Y, \mathcal{N})$. It is easy to see that for any point $t \in T$ a local section ϕ of $\mathcal{A}_l$ that does not vanish at t gives a local section of $N(L)^{\otimes l}$ that does not vanish at t. Therefore, Ω_l is a surjective morphism of sheaves. Moreover, by multiplicativity of N for a section ϕ_l of $\mathcal{A}_k$ and a section ϕ_m of $\mathcal{A}_m$, we have equality

$$\Omega_{m+l}(\phi_l \phi_m) = \Omega_l(\phi_l) \otimes \Omega_m(\phi_m)$$

of local sections of $N(L)^{\otimes (m+l)}$. Therefore, we obtain a surjective morphism of $\mathcal{O}_S$-algebras

$$\xi^* \biggl(\bigoplus_{l \geqslant 0} \Gamma^d_{\mathcal{O}_S}(\mathcal{A}_l) \biggr) \simeq \bigoplus_{l \geqslant 0} \Gamma^d_{\mathcal{O}_T}(\xi^* \mathcal{A}_l) \to \bigoplus_{l \geqslant 0} N(L)^{\otimes l},$$

and hence an S-morphism

$$T \to \operatorname{Proj} \biggl(\bigoplus_{l \geqslant 0} \Gamma^d_{\mathcal{O}_S}(\mathcal{A}_l) \biggr).$$

Lemma 5.1. *In the situation described above,*

$$\operatorname{Proj} \biggl(\bigoplus_{l \geqslant 0} \Gamma^d_{\mathcal{O}_S}(\mathcal{A}_l) \biggr) \simeq \Gamma^d(X/S).$$

Moreover, if $\mathcal{A}$ is locally generated by at most $r+1$ elements and $l \geqslant r(d-1)$, then $\Gamma^d(X/S)$ is isomorphic to a closed subscheme of $\mathbb{P}(\Gamma^d(\mathcal{A}_l))$. When S is a scheme over $\mathbb{Q}$, the assertion holds for any $l \geqslant 1$.

Proof. See Corollary 3.2.8 and Proposition 3.2.9 in [**15**]. □

Corollary 5.2. *A family of cycles $(Y, \mathcal{N})$ is uniquely determined by its lth Chow form $\Omega_l : \Gamma^d_{\mathcal{O}_T}(\xi^* A_l) \to N(L)^{\otimes l}$, where l is given by the previous lemma.*

Acknowledgements. This work was supported by the Sloan Research Fellowship. The author is grateful to V. Drinfeld, from whom he learned Grothendieck's approach to zero cycles, and to D. Rydh for his useful remarks.

References

1. B. Angéniol, *Familles de cycle algébriques—schéma de Chow*, Lecture Notes in Mathematics, Volume 896 (Springer, 1981).
2. V. M. Buchstaber and E. G. Rees, The Gelfand map and symmetric products, *Selecta Math.* **8**(4) (2002), 523–535.
3. P. Deligne, *Cohomologie a support propre, SGA 4*, Exposé XVII, Lecture Notes in Mathematics, Volume 305 (Springer, 1972).
4. F. Ducrot, Cube structures and intersection bundles, *J. Pure Appl. Alg.* **195** (2005), 33–73.
5. R. Elkik, Fibrés d'intersections et intégrales de classes the Chern, *Annales Scient. Éc. Norm. Sup.* **22** (1989), 195–226.
6. D. Ferrand, Un foncteur norme, *Bull. Soc. Math. France* **126** (1998), 1–49.
7. A. Grothendieck, *Élements de géométrie algébrique*, Publications Mathématiques de l'IHÉS, No. 8 (IHÉS, Bures-sur-Yvette, 1961).
8. B. Iversen, *Linear determinants with applications to the Picard scheme of a family of algebraic curves*, Lecture Notes in Mathematics, Volume 174 (Springer, 1970).
9. D. Knutson, *Algebraic spaces*, Lecture Notes in Mathematics, Volume 203 (Springer, 1971).
10. D. Laksov, Notes on divided powers, preprint (available at www.math.kth.se/~laksov/notes/divpotensall.pdf).
11. E. Muñoz-Garcia, Fibrés d'intersections, *Compositio Math.* **124** (2000), 219–252.
12. N. Roby, Lois polynomes et lois formelles en théorie des modules, *Annales Scient. Éc. Norm. Sup.* **80** (1963), 213–348.
13. D. Rydh, Families of zero cycles and divided powers, I, Representability, preprint (arXiv:0803.0618).
14. D. Rydh, Families of zero cycles and divided powers, II, The universal family, preprint (available at www.math.kth.se/~dary).
15. D. Rydh, Hilbert and Chow schemes of points, symmetric products and divided powers, preprint (available at www.math.kth.se/~dary).
16. D. Rydh, Noetherian approximation of algebraic spaces and stacks, preprint (arXiv:math/0904.0227).
17. F. Vaccarino, Homogeneous multiplicative polynomial laws are determinants, *J. Pure Appl. Alg.* **213**(7) (2009), 1283–1289.