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ON HOMOMORPHISMS OF AN ORTHOGONALLY DECOMPOSABLE HILBERT SPACE II

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Abstract

We present two more characterizations of maps which preserve orthogonal decompositions defined on Hilbert spaces ordered by natural cones.

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Let M be a von Neumann algebra on a Hilbert space H. We shall assume that there is a cyclic and separating vector $\xi_0 \in H$ for M. Then, by the Tomita-Takesaki theory, there are the conjugation operator J and the modular operator Δ associated with ξ_0 such that

$$H^+ = \overline{\{xj(x)\xi_0 : x \in M\}} = \overline{\{\Delta^{1/4}x\xi_0 : x \in M^+\}}$$

defines the "natural" positive cone of H, where $j(x) = JxJ$ and M^+ is the set of all positive elements of M. Then, every element ξ of H such that $\xi = J\xi$ admits a unique *orthogonal decomposition*: $\xi = \xi^+ - \xi^-$, $\xi^+ \in H^+$, $\xi^- \in H^+$ and $(\xi^+, \xi^-) = 0$. For the details of these facts, see [1] and [2]. A continuous linear operator ϕ: $H \to H$ is called an *o.d. homomorphism* if $\phi\xi = \phi\xi^+ - \phi\xi^-$ is also an orthogonal decomposition. This is equivalent to that $\phi(H^+) \subset H^+$ and $(\phi\xi, \phi\eta) = 0$ whenever $\xi \in H^+$, $\eta \in H^+$ and $(\xi, \eta) = 0$. The following fact has been proved in [3].

THEOREM 1. *Let ϕ: $H \to H$ be a continuous linear operator. Then, ϕ is an o.d. homomorphism if and only if $\phi(H^+) \subset H^+$ and $\phi^*\phi \in M \cap M'$ (the center of M).*

The aim of this note is to add two more characterizations of o.d. homomorphisms.

THEOREM 2. *Let ϕ: $H \to H$ be a continuous linear operator such that $\phi(H^+) \subset H^+$. The following conditions are equivalent.*

(1) *ϕ is an o.d. homomorphism.*

(2) *$\phi^* x \phi \in M \cap M'$ for every $x \in M \cap M'$.*

PROOF. (1) $\Rightarrow$ (2). Let $x \in (M \cap M')^+$. When $\xi \in H^+$, $\eta \in H^+$ and $(\xi, \eta) = 0$, it follows from the condition (1) that $\phi\xi \in H^+$, $\phi\eta \in H^+$ and $(\phi\xi, \phi\eta) = 0$. Furthermore,

$$\left(x^{1/2}\phi\xi, x^{1/2}\phi\eta\right) = \left(x^{1/2}p_{\phi\xi}\phi\xi, x^{1/2}p_{\phi\eta}\phi\eta\right) = \left(p_{\phi\xi}x^{1/2}\phi\xi, p_{\phi\eta}x^{1/2}\phi\eta\right) = 0,$$

where $p_{\phi\xi} = [M'\phi\xi]$ and $p_{\phi\eta} = [M'\phi\eta]$ are cyclic projections. Since $x^{1/2}(H^+) \subset H^+$, this implies that $x^{1/2}\phi$ is an o.d. homomorphism. Hence, by Theorem 1,

$$\phi^* x \phi = \left(\phi^* x^{1/2}\right)\left(x^{1/2}\phi\right) = \left(x^{1/2}\phi\right)^*\left(x^{1/2}\phi\right) \in M \cap M'.$$

(2) $\Rightarrow$ (1). For $x = 1$, the identity of M, we have $\phi^*\phi \in M \cap M'$. Hence, by Theorem 1, ϕ is an o.d. homomorphism.

COROLLARY 3. *Let ϕ: $H \to H$ be a continuous linear operator such that $\phi(H^+) \subset H^+$. Then, if $\phi^* x \phi \in M$ for every $x \in M$, ϕ is an o.d. homomorphism.*

PROOF. By the Tomita-Takesaki theory, we have $J^* = J$ and $M' = j(M)$. Since $\phi(H^+) \subset H^+$, we have $\phi J = J\phi$. Hence, for any $x' \in M'$, we can take $x \in M$ such that $x' = j(x)$ and

$$\phi^* x' \phi = \phi^* J x J \phi = J\left(\phi^* x \phi\right) J \in j(M) = M'.$$

It then follows from the assumption that $\phi^*(M \cap M')\phi \subset M \cap M'$. Hence, ϕ is an o.d. homomorphism by Theorem 2.

The second characterization has its origin in the following lemma which, when ϕ is a unitary operator, is due to [2].

LEMMA 4. *Let ϕ: $H \to H$ be a continuous linear bijection such that $\phi(H^+) = H^+$. Then, for any cyclic and separating vector $\xi \in H^+$ for M, there is a unital Jordan $*$-isomorphism $\alpha_{\phi,\xi}$ of M such that*

$$\Delta_{\phi\xi}^{1/4}\alpha_{\phi,\xi}(x)\phi\xi = \phi\left(\Delta_\xi^{1/4} x \xi\right) \quad \textit{for all } x \in M,$$

where Δ_ξ and $\Delta_{\phi\xi}$ are the modular operators associated with the cyclic and separating vectors ξ and $\phi\xi$ respectively.

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(Since ϕ is bijective and $\phi(H^+) = H^+$, $\phi\xi$ is also a cyclic and separating vector for M by [2], Lemma 4.3.)

PROOF. Let $H_\xi = \{\eta \in H\colon -\lambda\xi \leqslant \eta \leqslant \lambda\xi$ for some $\lambda > 0\}$, $H_{\phi\xi} = \{\eta \in H\colon -\phi\xi \leqslant \eta \leqslant \phi\xi$ for some $\lambda > 0\}$ and M^h be the set of all self-adjoint elements of M. Since ϕ is bijective and $\phi(H^+) = H^+$, ϕ maps H_ξ onto $H_{\phi\xi}$ bijectively. On the other hand, by [1], Lemma 2.5.40, and [2], Proposition 1.2, there are bijective order isomorphisms

$$b_\xi\colon M^h \to H_\xi \quad \text{and} \quad b_{\phi\xi}\colon M^h \to H_{\phi\xi}$$

defined by

$$b_\xi(x) = \Delta_\xi^{1/4} x\xi \quad \text{and} \quad b_{\phi\xi}(x) = \Delta_{\phi\xi}^{1/4} x\phi\xi$$

for all $x \in M^h$. Hence, we can define a bijection $\alpha_{\phi,\xi}\colon M^h \to M^h$ by

$$\Delta_{\phi\xi}^{1/4}\alpha_{\phi,\xi}(x)\phi\xi = \phi\left(\Delta_\xi^{1/4}x\xi\right) \quad \text{for all } x \in M^h.$$

By the linearity, $\alpha_{\phi,\xi}(x)$ is defined for all $x \in M$. It satisfies $\alpha_{\phi,\xi}(1) = 1$ and $\alpha_{\phi,\xi}(M^+) = M^+$. Therefore, by a theorem of Kadison [4] (see also [1], Theorem 3.2.3), $\alpha_{\phi,\xi}$ is a Jordan *-isomorphism.

We shall prove that a continuous linear operator $\phi\colon H \to H$ such that $\phi(H^+) = H^+$ is an o.d. homomorphism if and only if $\alpha_\phi = \alpha_{\phi,\xi}$ for every cyclic and separating vector $\xi \in H^+$ for M, where $\alpha_\phi = \alpha_{\phi,\xi_0}$. It is known that the equality $\alpha_\phi = \alpha_{\phi,\xi}$ holds for a special class of o.d. homomorphisms. For example, it has been shown in [2], Theorem 3.2 (see also [1], Theorem 3.2.15) that, when u is a unitary operator such that $u(H^+) = H^+$, we have the equality $\alpha_u = \alpha_{u,\xi}$ for every cyclic and separating vector $\xi \in H^+$ for M, and, conversely, for any unital Jordan *-isomorphism $\alpha_\phi\colon M \to M$, there is a unique unitary operator u_α such that $u_\alpha(H^+) = H^+$ and

$$u_\alpha\left(\Delta_\xi^{1/4}x\xi\right) = \Delta_{u_\alpha\xi}^{1/4}\alpha_\phi(x)u_\alpha\xi \quad \text{for all } x \in M$$

for all cyclic and separating vector $\xi \in H^+$ for M. These facts and the symbol u_α will be used in the following discussion.

By definition, an o.d. isomorphism is a continuous linear bijection $\phi\colon H \to H$ such that ϕ and ϕ^{-1} are both o.d. homomorphisms. It has been proved in [3], (3.1), that bijective o.d. homomorphisms are o.d. isomorphisms. Obviously, a unitary operator u is an o.d. isomorphism if and only if $u(H^+) = H^+$.

THEOREM 5. *Let $\phi\colon H \to H$ be a continuous linear bijection such that $\phi(H^+) = H^+$. The following conditions are equivalent.*

(1) *ϕ is an o.d. isomorphism.*

(2) *For the polar decomposition $\phi = u|\phi|$,*

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(i) $|\phi|$ *is an o.d. isomorphism and* $\alpha_{|\phi|,\xi} = 1$ *for all cyclic and separating vector* $\xi \in H^+$ *for* M.

(ii) u *is an o.d. isomorphism,* $\alpha_\phi = a_u$ *and* $u = u_\alpha$.

(3) $\alpha_\phi = \alpha_{\phi,\xi}$ *for every cyclic and separating vector* $\xi \in H^+$.

(4) $\|\phi^{-1}\|^{-1} u_\alpha \xi \leqslant \phi\xi \leqslant \|\phi\| u_\alpha \xi$ *for all* $\xi \in H^+$.

PROOF. (1) $\Rightarrow$ (2). (i). Since $|\phi| \in (M \cap M')^+$ by Theorem 1, we have $|\phi|(H^+) = |\phi|^{1/2} j(|\phi|^{1/2})(H^+) \subset H^+$. Hence, it follows from Theorem 1 that $|\phi|$ is an o.d. homomorphism. Since it is bijective, it is, in fact, an o.d. isomorphism. Now, let $\xi \in H^+$ be a cyclic and separating vector for M. Then, $|\phi|\xi$ is also a cyclic and separating vector in H^+ and, since $|\phi|$ is an invertible element of $(M \cap M')^+$, we have $\Delta^{1/4}_{|\phi|\xi} = \Delta^{1/4}_\xi$. Hence,

$$(\#) \qquad |\phi|\left(\Delta_\xi^{1/4} x\xi\right) = \Delta^{1/4}_{|\phi|\xi} x|\phi|\xi \quad \text{for all } x \in M.$$

This is equivalent to $\alpha_{|\phi|,\xi}(x) = x$ for all $x \in M$. To prove (ii), we first note that $u = \phi|\phi|^{-1}$ is an o.d. isomorphism because ϕ and $|\phi|^{-1}$ are. Then, by $(\#)$,

$$\begin{aligned}\Delta^{1/4}_{\phi\xi_0}\alpha_\phi(x)\phi\xi_0 &= \phi\left(\Delta^{1/4}_{\xi_0} x\xi_0\right) = u|\phi|\left(\Delta^{1/4}_{\xi_0} x\xi_0\right) \\ &= u\left(\Delta^{1/4}_{|\phi|\xi_0} x|\phi|\xi_0\right) = \Delta^{1/4}_{\phi\xi_0}\alpha_u(x)\phi\xi_0\end{aligned}$$

for all $x \in M$. Therefore, $\alpha_\phi = \alpha_u$. Furthermore, for every $x \in M$,

$$\begin{aligned}u_\alpha\left(\Delta^{1/4}_{\xi_0} x\xi_0\right) &= \Delta^{1/4}_{u_\alpha\xi_0}\alpha_\phi(x)u_\alpha\xi_0 = \Delta^{1/4}_{u_\alpha\xi_0}\alpha_u(x)u_\alpha\xi_0 \\ &= u\left(\Delta^{1/4}_{u^*u_\alpha\xi_0} xu^*u_\alpha\xi_0\right),\end{aligned}$$

that is,

$$u^*u_\alpha\left(\Delta^{1/4}_{\xi_0} x\xi_0\right) = \Delta^{1/4}_{u^*u_\alpha\xi_0} xu^*u_\alpha\xi_0,$$

where u^*u_α is a unitary operator such that $u^*u_\alpha(H^+) = H^+$. This equation shows that the unital Jordan *-isomorphism determined by u^*u_α is the identity map. Hence, $u^*u_\alpha = 1$, or, $u = u_\alpha$.

(2) $\Rightarrow$ (3). Let $\xi \in H^+$ be a cyclic and separating vector for M. Then, since $\alpha_{|\phi|,\xi} = 1$,

$$\begin{aligned}\Delta^{1/4}_{\phi\xi}\alpha_{\phi,\xi}(x)\phi\xi &= \phi\left(\Delta^{1/4}_\xi x\xi\right) = u|\phi|\left(\Delta^{1/4}_\xi x\xi\right) \\ &= u\left(\Delta^{1/4}_{|\phi|\xi} x|\phi|\xi\right) = \Delta^{1/4}_{\phi\xi}\alpha_u(x)\phi\xi\end{aligned}$$

for all $x \in M$. This implies $\alpha_{\phi,\xi} = \alpha_u = \alpha_\phi$.

(3) $\Rightarrow$ (4). For any cyclic and separating vector $\xi \in H^+$,

$$\phi\left(\Delta^{1/4}_\xi x\xi\right) = \Delta^{1/4}_{\phi\xi}\alpha_\phi(x)\phi\xi = u_\alpha\left(\Delta^{1/4}_{u_\alpha^*\phi\xi} xu_\alpha^*\phi\xi\right)$$

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for every $x \in M$. Therefore,

$$\left\| \Delta^{1/4}_{u_\alpha^* \phi \xi} x u_\alpha^* \phi \xi \right\| \leqslant \|\phi\| \left\| \Delta_\xi^{1/4} x \xi \right\| \quad \text{for every } x \in M.$$

By [2], Lemma 3.13, this inequality is equivalent to

$$u_\alpha^* \phi \xi \leqslant \|\phi\| \xi.$$

Since this inequality holds for every cyclic and separating vector $\xi \in H^+$ and such vectors are dense in H^+, we have

$$u_\alpha^* \phi \xi \leqslant \|\phi\| \xi \quad \text{for every } \xi \in H^+.$$

Since $u_\alpha(H^+) = H^+$, this is equivalent to

$$\phi \xi \leqslant \|\phi\| u_\alpha \xi \quad \text{for every } \xi \in H^+.$$

Starting with ϕ^{-1} instead of ϕ, we arrive at

$$\phi^{-1} \xi \leqslant \|\phi^{-1}\| u_\alpha^* \xi \quad \text{for every } \xi \in H^+.$$

(4) $\Rightarrow$ (1). We only need to show that $(\phi\xi, \phi\eta) = 0$ whenever $\xi \in H^+$, $\eta \in H^+$ and $(\xi, \eta) = 0$. However, this is obvious because u_α satisfies this condition.

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