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TRANSVERSAL INFINITESIMAL AUTOMORPHISMS ON KÄHLER FOLIATIONS

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Abstract

Let $\mathcal{F}$ be a Kähler foliation on a compact Riemannian manifold M. If the transversal scalar curvature of $\mathcal{F}$ is nonzero constant, then any transversal conformal field is a transversal Killing field; and if the transversal Ricci curvature is nonnegative and positive at some point, then there are no transversally holomorphic fields.

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1. Introduction

Let $(M, \mathcal{F})$ be a Riemannian manifold with a Riemannian foliation $\mathcal{F}$ of codimension q. A transversal infinitesimal automorphism on M is an infinitesimal automorphism which preserves the leaves. A transversal infinitesimal automorphism is said to be a transversal Killing field, a transversal conformal field or a transversal projective field if it generates a one-parameter family of a transversal infinitesimal isometric, a transversal infinitesimal conformal or a transversal infinitesimal projective transformation, respectively. Such geometric objects give some important information about the leaf space $M/\mathcal{F}$. There are several results about infinitesimal automorphisms on Riemannian foliations [4–7, 9, 10]. Recently, Jung and Jung [4] studied properties of transversal infinitesimal automorphisms on a compact foliated Riemannian manifold $(M, \mathcal{F})$.

In this paper, we investigate properties of transversal infinitesimal automorphisms on Kähler foliations. The paper is organised as follows. In Section 2, we review basic results on Riemannian foliations. In Section 3, we review well-known results about infinitesimal automorphisms on Riemannian foliations. In Section 4, we prove that, on Kähler foliations, any transversal conformal (or projective) field is a transversal

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affine field. Note that on ordinary manifolds any affine field is a Killing field, but on Riemannian foliations a transversal affine field is not necessarily a transversal Killing field [4]. In fact, if any transversal conformal (or projective) field satisfies some mean curvature assumption, then it is a transversal Killing field. Moreover, we prove that on a Kähler foliation of nonzero constant transversal scalar curvature, any transversal conformal field is a transversal Killing field. In Section 5, we study transversally holomorphic fields, that is, infinitesimal holomorphic transformations which preserve the leaves. We give a vanishing theorem without making assumptions about the mean curvature vector. In [9], Nishikawa and Tondeur proved that there are no transversally holomorphic fields under the assumption that the foliation is minimal. In [2], Jung proved that there are no transversally holomorphic fields satisfying $\nabla_{\kappa^\sharp}\bar{Y} = 0$. Also, we prove that if the transversal Ricci curvature is nonnegative and positive at some point, then there are no transversally holomorphic fields.

2. Preliminaries

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$ [12]. Let ∇^M be the Levi-Civita connection with respect to g_M. Let TM be the tangent bundle of M, L its integrable subbundle given by $\mathcal{F}$, and $Q = TM/L$ the corresponding normal bundle. Then there exists an exact sequence of vector bundles

$$0 \longrightarrow L \longrightarrow TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \longrightarrow 0,$$

where $\sigma : Q \to L^\perp$ is a bundle map satisfying $\pi \circ \sigma = \mathrm{id}$. Let g_Q be the holonomy invariant metric on Q induced by $g_M = g_L + g_{L^\perp}$; that is,

$$g_Q(s, t) = g_M(\sigma(s), \sigma(t)) \quad \forall s, t \in \Gamma Q.$$

This means that $\theta(X) g_Q = 0$ for $X \in \Gamma L$, where $\theta(X)$ is the transverse Lie derivative. So we have an identification $L^\perp$ with Q via an isometric splitting $(Q, g_Q) \cong (L^\perp, g_{L^\perp})$. A transversal Levi-Civita connection ∇ in Q is defined [7, 12] by

$$\nabla_X s = \begin{cases} \pi([X, Y_s]) & \forall X \in \Gamma L \\ \pi(\nabla^M_X Y_s) & \forall X \in \Gamma L^\perp, \end{cases}$$

where $s \in \Gamma Q$ and $Y_s = \sigma(s) \in \Gamma L^\perp$ corresponding to s under the canonical isomorphism $Q \cong L^\perp$. The curvature R^∇ of ∇ is defined by $R^\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for $X, Y \in \Gamma TM$. Since $i(X) R^\nabla = 0$ for any $X \in \Gamma L$ [7, 12], we can define the transversal Ricci operator $\rho^\nabla : \Gamma Q \to \Gamma Q$ by

$$\rho^\nabla(s_x) = \sum_{a=1}^{q} R^\nabla(s, E_a) E_a,$$

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where $\{E_a\}_{a=1,\dots,q}$ is an orthonormal basic frame of Q. Then the transversal Ricci curvature Ric^∇ is given by $\mathrm{Ric}^\nabla(s_1, s_2) = g_Q(\rho^\nabla(s_1), s_2)$ for any $s_1, s_2 \in \Gamma Q$. The transversal scalar curvature σ^∇ is given by $\sigma^\nabla = \mathrm{Tr}\rho^\nabla$. The foliation $\mathcal{F}$ is said to be (transversally) *Einsteinian* if the model space is Einsteinian, that is,

$$\rho^\nabla = \frac{1}{q}\sigma^\nabla \cdot \mathrm{id}$$

with constant transversal scalar curvature σ^∇. The *mean curvature vector* $\kappa^\sharp$ of $\mathcal{F}$ is defined by

$$\kappa^\sharp = \pi\Big(\sum_{i=1}^{p} \nabla^M_{E_i} E_i\Big),$$

where $\{E_i\}$ is a local orthonormal basis of L. The foliation $\mathcal{F}$ is said to be *minimal* if $\kappa^\sharp = 0$. A differential form $\omega \in \Omega^r(M)$ is *basic* if $i(X)\omega = 0$ and $\theta(X)\omega = 0$ for all $X \in \Gamma L$. Let $\Omega^r_B(\mathcal{F})$ be the set of all basic r-forms on M. Then $\Omega^r(M) = \Omega^r_B(\mathcal{F}) \oplus \Omega^r_B(\mathcal{F})^\perp$ [1]. It is well known that the mean curvature form κ_B is closed, that is, $d\kappa_B = 0$, where κ_B is the basic part of κ. The *basic Laplacian* acting on $\Omega^*_B(\mathcal{F})$ is defined by

$$\Delta_B = d_B\delta_B + \delta_B d_B,$$

where δ_B is the formal adjoint of $d_B = d|_{\Omega^*_B(\mathcal{F})}$ [1, 3]. Let $\{E_a\}$ $(a = 1, \dots, q)$ be a local orthonormal basis of Q. We define $\nabla^*_{\mathrm{tr}}\nabla_{\mathrm{tr}} : \Omega^r_B(\mathcal{F}) \to \Omega^r_B(\mathcal{F})$ by

$$\nabla^*_{\mathrm{tr}}\nabla_{\mathrm{tr}} = -\sum_a \nabla^2_{E_a,E_a} + \nabla_{\kappa^\sharp_B},$$

where $\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla^M_X Y}$ for any $X, Y \in TM$. The operator $\nabla^*_{\mathrm{tr}}\nabla_{\mathrm{tr}}$ is positive definite and formally self-adjoint on the space of basic forms [3]. We define the bundle map $A_Y : \Lambda^r Q^* \to \Lambda^r Q^*$ for any $Y \in V(\mathcal{F})$ [8] by

$$A_Y\phi = \theta(Y)\phi - \nabla_Y\phi,$$

where $\theta(Y)$ is the transverse Lie derivative. It has been proved [8] that, for any vector field $Y \in V(\mathcal{F})$,

$$A_Y s = -\nabla_{Y_s}\bar{Y}, \tag{2.1}$$

where $Y_s = \sigma(s) \in \Gamma TM$. So A_Y depends only on $\bar{Y} = \pi(Y)$ and is a linear operator. Since $\theta(X)\phi = \nabla_X\phi$ for any $X \in \Gamma L$, A_Y preserves the basic forms and depends only on $\bar{Y}$. Then we have the generalised Weitzenböck formula as given in the following theorem.

THEOREM 2.1 [3]. *On a Riemannian foliation $\mathcal{F}$,*

$$\Delta_B\phi = \nabla^*_{\mathrm{tr}}\nabla_{\mathrm{tr}}\phi + F(\phi) + A_{\kappa^\sharp_B}\phi, \quad \phi \in \Omega^r_B(\mathcal{F}),$$

where $F(\phi) = \sum_{a,b} \theta^a \wedge i(E_b)R^\nabla(E_b, E_a)\phi$. If ϕ is a basic 1-form, then $F(\phi)^\sharp = \rho^\nabla(\phi^\sharp)$.

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From Theorem 2.1, for any $\phi \in \Omega_B^r(\mathcal{F})$,

$$\tfrac{1}{2}\Delta_B|\phi|^2 = \langle \Delta_B\phi, \phi\rangle - |\nabla_{\rm tr}\phi|^2 - \langle F(\phi), \phi\rangle - \langle A_{\kappa_B^\sharp}\phi, \phi\rangle. \tag{2.2}$$

We now recall the following generalised maximum principle.

LEMMA 2.2 [5]. *Let $\mathcal{F}$ be a Riemannian foliation on a compact Riemannian manifold (M, g_M). If $(\Delta_B - \kappa_B^\sharp)f \geq 0$ (or ≤ 0) for any basic function f, then f is constant.*

Let $V(\mathcal{F})$ be the space of all vector fields Y on M satisfying $[Y, Z] \in \Gamma L$ for all $Z \in \Gamma L$. An element of $V(\mathcal{F})$ is called an *infinitesimal automorphism* of $\mathcal{F}$ [8]. Let

$$\bar{V}(\mathcal{F}) = \{\bar{Y} = \pi(Y) \mid Y \in V(\mathcal{F})\}.$$

It is trivial that an element s of $\bar{V}(\mathcal{F})$ satisfies $\nabla_X s = 0$ for all $X \in \Gamma L$ [8]. Hence $\bar{V}(\mathcal{F}) \cong \Omega_B^1(\mathcal{F})$.

3. Transversal infinitesimal automorphisms

If $Y \in V(\mathcal{F})$ satisfies $\theta(Y)g_Q = 0$, then $\bar{Y}$ is called a *transversal Killing field* of $\mathcal{F}$. If $Y \in V(\mathcal{F})$ satisfies $\theta(Y)g_Q = 2f_Y g_Q$ for a basic function f_Y depending on Y, then $\bar{Y}$ is called a *transversal conformal field* of $\mathcal{F}$. Equivalently, for any $X, Z \in V(\mathcal{F})$,

$$g_Q(\nabla_X\bar{Y}, Z) + g_Q(X, \nabla_Z\bar{Y}) = 2f_Y g_Q(\bar{X}, \bar{Z}). \tag{3.1}$$

In this case,

$$f_Y = \frac{1}{q}\,\mathrm{div}_\nabla\,\bar{Y},$$

where $\mathrm{div}_\nabla\bar{Y}$ is the transversal divergence of $\bar{Y}$. A transversal conformal field $\bar{Y}$ is *homothetic* if f_Y is constant. For any vector fields $Y, Z \in V(\mathcal{F})$ and $X \in \Gamma Q$, we have [4]

$$(\theta(Y)\nabla)(Z, X) = R^\nabla(\bar{Y}, \bar{Z})X + \nabla_{\bar{Z}}\nabla_X\bar{Y} - \nabla_{\nabla_Z X}\bar{Y}. \tag{3.2}$$

If $Y \in V(\mathcal{F})$ satisfies $\theta(Y)\nabla = 0$, then $\bar{Y}$ is called a *transversal affine field* of $\mathcal{F}$. If $Y \in V(\mathcal{F})$ satisfies

$$(\theta(Y)\nabla)(X, Z) = \alpha_Y(X)Z + \alpha_Y(Z)X \tag{3.3}$$

for any $X, Z \in \Gamma Q$, where α_Y is a basic 1-form on M, then $\bar{Y}$ is called a *transversal projective field* of $\mathcal{F}$; in this case, it is trivial that

$$\alpha_Y = \frac{1}{q+1} d_B\,\mathrm{div}_\nabla\,\bar{Y}. \tag{3.4}$$

Let $\{E_a\}_{a=1,\ldots,q}$ be a local orthonormal basic frame in Q such that $(\nabla E_a)_x$ for $x \in M$. From now on, all the computations in this paper will be made in such charts. For any $Y \in V(\mathcal{F})$, from (3.2),

$$(\theta(Y)R^\nabla)(E_a, E_b)E_c = (\nabla_a\theta(Y)\nabla)(E_b, E_c) - (\nabla_b\theta(Y)\nabla)(E_a, E_c),$$

where $\nabla_a = \nabla_{E_a}$. Then we have the following lemma.

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Lemma 3.1 [4]. *Let $\mathcal{F}$ be a Riemannian foliation of codimension q on a Riemannian manifold (M, g_M). If $\bar{Y} \in \bar{V}(\mathcal{F})$ is a transversal conformal field, that is, $\theta(Y)g_Q = 2f_Y g_Q$, then*

$$g_Q((\theta(Y)\nabla)(E_a, E_b), E_c) = \delta_b^c f_a + \delta_a^c f_b - \delta_a^b f_c, \tag{3.5}$$

$$g_Q((\theta(Y)R^\nabla)(E_a, E_b)E_c, E_d) = \delta_b^d \nabla_a f_c - \delta_b^c \nabla_a f_d - \delta_a^d \nabla_b f_c + \delta_a^c \nabla_b f_d, \tag{3.6}$$

$$\theta(Y)\sigma^\nabla = 2(q-1)(\Delta_B f_Y - \kappa_B^\sharp(f_Y)) - 2f_Y \sigma^\nabla, \tag{3.7}$$

where $f_a = \nabla_a f_Y$.

From (3.5), it is trivial that any transversal homothetic field is a transversal affine field. On the other hand, from (3.3) and (3.5), we have the following lemma.

Lemma 3.2. *Let $\mathcal{F}$ be as in Lemma 3.1. If $\bar{Y} \in \bar{V}(\mathcal{F})$ is a transversal projective field, then*

$$\begin{aligned}(\theta(Y)R^\nabla)(E_a, E_b)E_c = {} & (\nabla_a \alpha_Y)(E_b)E_c + (\nabla_a \alpha_Y)(E_c)E_b \\ & -(\nabla_b \alpha_Y)(E_a)E_c - (\nabla_b \alpha_Y)(E_c)E_a.\end{aligned}$$

We now define the operator $B_Y^\mu : \Gamma Q \to \Gamma Q (\mu \in \mathbb{R})$ for any $Y \in V(\mathcal{F})$ by

$$B_Y = A_Y + A_Y^t + \mu \cdot \mathrm{div}_\nabla \bar{Y}\, \mathrm{id}. \tag{3.8}$$

It is well known [8] that $\bar{Y}$ is a transversal conformal (respectively, transversal Killing) field if and only if $B_Y^{2/q} = 0$ (respectively, $B_Y^0 = 0$). Then from (3.1) and the transversal divergence theorem [14], we have the following proposition.

Proposition 3.3. *Let $\mathcal{F}$ be a Riemannian foliation on a closed orientable Riemannian manifold (M, g_M). If $\bar{Y}$ is transversally homothetic, that is, $\mathrm{div}_\nabla \bar{Y}$ is constant, then*

$$\int_M g_Q(B_Y^\mu \bar{Y}, \kappa_B^\sharp) = \Big(\mu - \frac{2}{q}\Big)(\mathrm{div}_\nabla \bar{Y})^2 \mathrm{vol}(M). \tag{3.9}$$

We now recall the following relationships among infinitesimal automorphisms on a Riemannian foliation.

Theorem 3.4 [4]. *Let $\mathcal{F}$ be a Riemannian foliation on a closed orientable Riemannian manifold (M, g_M). Then:*

(1) *any transversal Killing field is a transversal affine field;*
(2) *any transversal affine field with $\int_M g_Q(B_Y^0 \bar{Y}, \kappa^\sharp) = 0$ is a transversal Killing field;*
(3) *any transversal conformal field (or projective field) $\bar{Y}$ with the properties:*
    (i) $\int_M g_Q(B_Y^0 \bar{Y}, \kappa^\sharp) \geq 0$;
    (ii) $d_B \mathrm{div}_\nabla \bar{Y} = 0$
    is a transversal Killing field.

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Note that on $\mathcal{F}$ with a constant transversal scalar curvature σ^∇, if $\mathcal{F}$ admits a transversal conformal field $\bar{Y}$ with $f_Y \neq 0$, then σ^∇ is nonnegative [4, Corollary 5.6]. Equivalently, on $\mathcal{F}$ with a negative constant σ^∇, there is no nonisometric transversal conformal field. Hence we have the following proposition.

Proposition 3.5. *Let $\mathcal{F}$ be a Riemannian foliation of codimension q on a closed, connected orientable Riemannian manifold (M, g_M). Assume that the transversal scalar curvature σ^∇ is negative constant. Then any transversal conformal field is a transversal Killing field.*

Theorem 3.6 [6]. *Let $(M, g_M, \mathcal{F})$ be a compact orientable Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. Assume that the transversal Ricci curvature ρ^∇ is nonpositive and negative at some point. Then:*

(1) *there are no transversal Killing fields on M;*
(2) *if $\delta_B\kappa_B = 0$, then there are no transversal conformal fields.*

Remark. From Proposition 3.5 and Theorem 3.6, it is well known that on a transversally Einstein foliation with negative scalar curvature, there are no transversal conformal fields without the condition $\delta_B\kappa_B = 0$. For more relations among infinitesimal automorphisms on a Riemannian foliation, see [4, 10].

4. Transversal conformal and projective field on Kähler foliations

We now study the infinitesimal automorphisms on Kähler foliations. Let $\mathcal{F}$ be a Kähler foliation of codimension $q = 2m$ on a Riemannian manifold (M, g_M) [9]. Note that, for any $X, Y \in \Gamma Q$,

$$\Omega(X, Y) = g_Q(X, JY)$$

defines a basic 2-form Ω, which is closed, where $J : Q \to Q$ is an almost complex structure on Q. Then

$$\Omega = -\frac{1}{2}\sum_{a=1}^{2m} \theta^a \wedge J\theta^a,$$

where θ^a is a dual form of E_a. Moreover, we have the following identities:

$$R^\nabla(X, Y)J = JR^\nabla(X, Y), \quad R^\nabla(JX, JY) = R^\nabla(X, Y) \tag{4.1}$$

for any $X, Y \in \Gamma Q$. Then we have the following proposition.

Proposition 4.1. *Let $\mathcal{F}$ be a Kähler foliation of codimension $q = 2m$ on a Riemannian manifold (M, g_M) and let $\bar{Y}$ be a transversal conformal field, that is, $\theta(Y)g_Q = 2f_Y g_Q$. Then*

$$\Delta_B f_Y - \kappa_B^\sharp(f_Y) = 0. \tag{4.2}$$

Moreover, if M is compact, then f_Y is constant, that is, $\bar{Y}$ is transversally homothetic.

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PROOF. Let f_Y be a basic function with $\theta(Y)g_Q = f_Y g_Q$. Fix $x \in M$ and let $\{E_a\}$ be a local orthonormal basic frame such that $(\nabla E_a)_x = 0$. Then, at x, from (3.6),

$$\sum_{a,b=1}^{2m} g_Q((\theta(Y)R^\nabla)(E_a, E_b)E_a, E_b) = 2q \sum_{a=1}^{2m} E_a E_a(f_Y),$$
$$\sum_{a,b=1}^{2m} g_Q((\theta(Y)R^\nabla)(JE_a, JE_b)E_a, E_b) = 2 \sum_{a=1}^{2m} E_a E_a(f_Y).$$

From (4.1),

$$2(q-1)\sum_{a=1}^{2m} E_a E_a(f_Y) = 0.$$

Since $q > 1$, $\sum_{a=1}^{2m} E_a E_a(f_Y) = 0$. Hence $\Delta_B f_Y = \kappa_B^\sharp(f_Y)$, which proves (4.2). Moreover, if M is compact, by Lemma 2.2, f_Y is constant. □

COROLLARY 4.2. *Let $\mathcal{F}$ be a Kähler foliation on a compact Riemannian manifold (M, g_M). Then any transversal conformal field is a transversal affine field.*

PROOF. Let $\bar{Y}$ be a transversal conformal field such that $\theta(Y)g_Q = f_Y g_Q$. By Proposition 4.1, f_Y is constant. Therefore, from (3.5) in Lemma 3.1, $\theta(Y)\nabla = 0$. So $\bar{Y}$ is the transversal affine field. □

REMARK. On a compact Kähler manifold, any conformal field is always a Killing field, because any affine field is a Killing field [11]. For the foliated manifold, this does not hold because of Theorem 3.4(2).

COROLLARY 4.3. *Let $\mathcal{F}$ be a Kähler foliation on a closed orientable Riemannian manifold (M, g_M). Then any transversal conformal field $\bar{Y}$ with the property*

$$\Big(\mu - \frac{2}{q}\Big)\int_M g_Q(B_Y^\mu \bar{Y}, \kappa_B^\sharp) \le 0 \quad \Big(\mu \ne \frac{2}{q}\Big)$$

is a transversal Killing field.

PROOF. From Proposition 4.1, $\bar{Y}$ is transversally homothetic. Therefore, Proposition 3.3 implies that $\operatorname{div}_\nabla \bar{Y} = 0$. So $\bar{Y}$ is transversal Killing. □

THEOREM 4.4. *Let $\mathcal{F}$ be a Kähler foliation on a compact Riemannian manifold (M, g_M). Assume that the transversal scalar curvature σ^∇ is nonzero constant. Then any transversal conformal field is a transversal Killing field.*

PROOF. Let $\bar{Y}$ be a transversal conformal field such that $\theta(Y)g_Q = f_Y g_Q$. Since $\sigma^\nabla \ne 0$ is constant, from (3.7) in Lemma 3.1 and Proposition 4.1, $f_Y = 0$. Therefore, $\bar{Y}$ is a transversal Killing field. □

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We now study the transversal projective field on a Kähler foliation. From Lemma 3.2, we have the following proposition.

Proposition 4.5. *Let $\mathcal{F}$ be a Kähler foliation of codimension $q = 2m$ $(m \geq 2)$ on a Riemannian manifold (M, g_M) and let $\bar{Y}$ be a transversal projective field. Then*

$$\Delta_B g_Y - \kappa_B^\sharp(g_Y) = 0,$$

where $g_Y = \mathrm{div}_\nabla(\bar{Y})$. If M is compact, then g_Y is constant.

Proof. Let $\bar{Y}$ be a transversal projective field. Let $\{E_a\}$ be a local orthonormal basic frame such that $(\nabla E_a)_x = 0$ at $x \in M$. Then, from Lemma 3.2,

$$\begin{aligned}(q+1)(\theta(Y)R^\nabla)(E_a, E_b)E_c &= E_aE_b(\mathrm{div}_\nabla \bar{Y})E_c + E_aE_c(\mathrm{div}_\nabla \bar{Y})E_b \\ &\quad - E_bE_a(\mathrm{div}_\nabla \bar{Y})E_c - E_bE_c(\mathrm{div}_\nabla \bar{Y})E_a.\end{aligned}$$

Hence

$$(q+1)\sum_{a,b=1}^{2m} g_Q((\theta(Y)R^\nabla)(E_a, E_b)E_a, E_b) = (q-1)\sum_{a=1}^{2m} E_aE_a(f_Y),$$

$$(q+1)\sum_{a,b=1}^{2m} g_Q((\theta(Y)R^\nabla)(JE_a, JE_b)E_a, E_b) = \sum_{a=1}^{2m} E_aE_a(f_Y).$$

From (4.1),

$$(q-2)\sum_{a=1}^{2m} E_aE_a(\mathrm{div}_\nabla \bar{Y}) = 0.$$

Since $q > 2$, we have $(\Delta_B - \kappa_B^\sharp)g_Y = \sum_{a=1}^{2m} E_aE_a(g_Y) = 0$. Moreover, if M is compact, by Lemma 2.2, g_Y is constant. □

From Proposition 4.5, we have the following corollary.

Corollary 4.6. *Let $\mathcal{F}$ be a Kähler foliation of codimension $q = 2m$ $(m \geq 2)$ on a compact Riemannian manifold (M, g_M). Then any transversal projective field is a transversal affine field.*

Proof. Let $\bar{Y}$ be a transversal projective field. From Proposition 4.5, $g_Y = \mathrm{div}_\nabla \bar{Y}$ is constant. Hence, from (3.4), $\alpha_Y = 0$. From (3.3), $\bar{Y}$ is transversal affine. □

From Theorem 3.4 (3), we have the following corollary.

Corollary 4.7. *Let $\mathcal{F}$ be a Kähler foliation of codimension $q = 2m$ $(m \geq 2)$ on a closed orientable Riemannian manifold (M, g_M). Then any transversal projective field $\bar{Y}$ with the property*

$$\int_M g_Q(B_Y^0 \bar{Y}, \kappa_B^\sharp) \geq 0$$

is a transversal Killing field.

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Remark. (1) For the point foliation, since any transversal affine field is a transversal Killing field [13], Corollaries 4.2 and 4.6 yield [11, Theorems 1 and 4], respectively, in an ordinary manifold.

(2) From Corollaries 4.3 and 4.7, it is trivial that on Kähler foliations, Theorem 3.4(3) holds without condition (ii).

5. Transversally holomorphic fields

Let $\mathcal{F}$ be a Kähler foliation of codimension $q = 2m$ on a Riemannian manifold (M, g_M). Let Y be an infinitesimal automorphism of $\mathcal{F}$. Then a vector field $\bar{Y}$ is said to be a *transversally holomorphic field* [9] if

$$\theta(Y)J = 0,$$

or equivalently, if for all $Z \in \Gamma L^{\perp}$,

$$\nabla_{JZ}\bar{Y} = J\nabla_Z\bar{Y}.$$

Let $\{E_\alpha, JE_\alpha\}$ $(\alpha = 1, \dots, m)$ be a local orthonormal basis of $\Gamma L^{\perp}$. Then we recall the following well-known result.

Lemma 5.1 [9]. *On a Kähler foliation of codimension $q = 2m$,*

$$\rho^{\nabla}(X) = \sum_{\alpha=1}^{m} JR^{\nabla}(E_\alpha, JE_\alpha)X. \tag{5.1}$$

Then we have the following theorem.

Theorem 5.2. *Let $\mathcal{F}$ be a Kähler foliation $\mathcal{F}$ of codimension $q = 2m$ on a closed orientable Riemannian manifold M. Then $\bar{Y}$ is transversally holomorphic, that is, $\theta(Y)J = 0$ if and only if:*

(i) $\nabla^*_{\mathrm{tr}}\nabla_{\mathrm{tr}}\bar{Y} - \rho^{\nabla}(\bar{Y}) + A_Y\kappa_B^{\sharp} = 0$;

(ii) $\int_M g_Q((\theta(Y)J)\kappa_B^{\sharp}, J\bar{Y}) = 0$.

Proof. Let $\bar{Y}$ be transversally holomorphic, that is, $\nabla_{JZ}\bar{Y} = J\nabla_Z\bar{Y}$ for any $Z \in \Gamma Q$. Then a lengthy calculation leads to

$$\nabla^*_{\mathrm{tr}}\nabla_{\mathrm{tr}}\bar{Y} = \sum_{\alpha=1}^{m} JR^{\nabla}(E_\alpha, JE_\alpha)\bar{Y} + \nabla_{\kappa_B^{\sharp}}\bar{Y}.$$

From (2.1) and (5.1),

$$\nabla^*_{\mathrm{tr}}\nabla_{\mathrm{tr}}\bar{Y} - \rho^{\nabla}(\bar{Y}) + A_Y\kappa_B^{\sharp} = 0.$$

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Hence (i) and (ii) are proved. Conversely, by direct calculation,

$$\int_M |\theta(Y)J|^2 = 2\int_M g_Q(\nabla_{\rm tr}^*\nabla_{\rm tr}\bar Y - \rho^\nabla(\bar Y) + A_Y\kappa_B^\sharp, \bar Y) \\ + 2\int_M \sum_{a=1}^{2m} E_a g_Q(\nabla_{E_a}\bar Y + J\nabla_{JE_a}\bar Y, \bar Y).$$

Now we choose $X \in \Gamma Q$ by $g_Q(X, Z) = g_Q(\nabla_Z\bar Y + J\nabla_{JZ}\bar Y, \bar Y)$ for any $Z \in \Gamma Q$. Then, by the transversal divergence theorem [14],

$$\int_M \sum_{a=1}^{2m} E_a g_Q(\nabla_{E_a}\bar Y + J\nabla_{JE_a}\bar Y, \bar Y) = \int_M \mathrm{div}_\nabla(X) \\ = \int_M g_Q(\nabla_{\kappa_B^\sharp}\bar Y + J\nabla_{J\kappa_B^\sharp}\bar Y, \bar Y).$$

Hence

$$\frac{1}{2}\int_M |\theta(Y)J|^2 = \int_M g_Q(\nabla_{\rm tr}^*\nabla_{\rm tr}\bar Y - \rho^\nabla(\bar Y) + A_Y\kappa_B^\sharp, \bar Y) + \int_M g_Q((\theta(Y)J)\kappa_B^\sharp, J\bar Y).$$

Thus the converse is proved. □

Corollary 5.3 [9]. *On a harmonic Kähler foliation $\mathcal{F}$ on a compact manifold (M, g_M), the following statements are equivalent.*

(1) *$\bar Y$ is transversally holomorphic, $\theta(Y)J = 0$.*
(2) *$\bar Y$ is a transversal Jacobi field of $\mathcal{F}$, that is, $\nabla_{\rm tr}^*\nabla_{\rm tr}\bar Y - \rho^\nabla(\bar Y) = 0$.*

Moreover, we have the following vanishing theorem.

Theorem 5.4. *Let $\mathcal{F}$ be a Kähler foliation on a compact Riemannian manifold (M, g_M). Assume that the transversal Ricci operator is nonpositive and negative at some point. Then any infinitesimal automorphism Y with a transversally holomorphic field $\bar Y$ satisfies $Y \in \Gamma L$, that is, $\bar Y = 0$.*

Proof. Let $\bar Y$ be a transversally holomorphic field. Then, by Theorem 5.2(i),

$$\Delta_B|\bar Y|^2 = 2g_Q(\nabla_{\rm tr}^*\nabla_{\rm tr}\bar Y, \bar Y) - 2|\nabla_{\rm tr}\bar Y|^2 \\ = 2g_Q(\rho^\nabla(\bar Y), \bar Y) - 2|\nabla_{\rm tr}\bar Y|^2 + \kappa_B^\sharp|\bar Y|^2.$$

Since the transversal Ricci curvature ρ^∇ is nonpositive, $(\Delta_B - \kappa_B^\sharp)|\bar Y|^2 \le 0$. Hence, by Lemma 2.2, $|\bar Y|$ is constant. Moreover, since ρ^∇ is negative at some point, $\bar Y$ is zero, that is, Y is tangential to $\mathcal{F}$. □

Remark. In [9], Nishikawa and Tondeur proved Theorem 5.4 when the foliation is minimal. In [2], Jung proved Theorem 5.4 when the transversally holomorphic field $\bar Y$ is parallel along the mean curvature vector, that is, $\nabla_{\kappa^\sharp}\bar Y = 0$.

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