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INTEGRAL POINTS ON SINGULAR DEL PEZZO SURFACES

ULRICH DERENTHAL[1] AND FLORIAN WILSCH[2]

[1] *Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany*
(derenthal@math.uni-hannover.de)

[2] *Institute of Science and Technology Austria, Am Campus 1, 3400 Klosterneuburg, Austria*
(florian.wilsch@ist.ac.at)

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Abstract In order to study integral points of bounded log-anticanonical height on weak del Pezzo surfaces, we classify weak del Pezzo pairs. As a representative example, we consider a quartic del Pezzo surface of singularity type $\mathbf{A}_1 + \mathbf{A}_3$ and prove an analogue of Manin's conjecture for integral points with respect to its singularities and its lines.

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1. Introduction

Del Pezzo surfaces over $\mathbb{Q}$ often contain infinitely many rational points. Over the past 20 years, Manin's conjecture [16, 22] for the asymptotic behavior of the number of rational points of bounded anticanonical height has been confirmed for some smooth and many singular del Pezzo surfaces (see [4, 5, 6] for some milestones and [1, § 6.4.1] for many further references), in most cases using universal torsors, often combined with advanced analytic techniques.

In recent years, a conjectural framework for the density of *integral* points has emerged in the work of Chambert-Loir and Tschinkel [7]. The purpose of this paper is to initiate a systematic investigation of integral points of bounded height on del Pezzo surfaces. Only a few of them are covered by general results for equivariant compactifications of vector groups [8] or the incomplete work on toric varieties [9] (see also [25]); most del Pezzo surfaces are out of reach of this *harmonic analysis approach* since they are not equivariant compactifications of algebraic groups [13, 14]. Del Pezzo surfaces are inaccessible to the *circle method*, which gives asymptotic formulas for integral points only on high-dimensional complete intersections [3], [7, § 5.4]. Therefore, we adapt the *universal torsor method* to integral points in order to confirm new cases of an integral analogue of Manin's conjecture. See also [24] for a three-dimensional example.

As rational and integral points coincide on a projective variety X, the study of the latter becomes interesting on its own on an integral model of the complement $X \setminus Z$ of an appropriate *boundary* Z. Our first result (Theorem 10 in Section 2) is a general treatment of possible boundaries on singular del Pezzo surfaces of low degree. For singular cubic surfaces, Z must be an $\mathbf{A}$-singularity; for singular quartic del Pezzo surfaces, Z must be an $\mathbf{A}$-singularity or a line passing only through $\mathbf{A}$-singularities. Furthermore, $\mathbf{A}_1$-singularities behave differently than other $\mathbf{A}$-singularities.

Therefore, a good starting point seems to be a quartic del Pezzo surface that contains an $\mathbf{A}_1$- and an $\mathbf{A}_3$-singularity and three lines, which is neither toric [12, Remark 6] nor a compactification of $\mathbb{G}_\mathrm{a}^2$ [13]. For each boundary Z admissible in the sense of Theorem 10, we get an associated counting problem and prove an asymptotic formula of the shape

$$cB(\log B)^{b-1}$$

(Theorem 1), encountering a range of different phenomena when dealing with the different types of boundary. These asymptotic formulas admit a geometric interpretation (Theorem 2). In particular, the leading constant c consists of Tamagawa numbers as defined in [7] and combinatorial constants (analogous to the constant α defined by Peyre for rational points) as defined in [9] for toric varieties and studied in greater generality in [25]; this is the first result applying this combinatorial construction in a nontoric setting.

1.1. The counting problem

Let $S \subset \mathbb{P}^4_\mathbb{Q}$ be the quartic del Pezzo surface defined by

$$x_0^2 + x_0x_3 + x_2x_4 = x_1x_3 - x_2^2 = 0 \tag{1}$$

over $\mathbb{Q}$, with an $\mathbf{A}_1$-singularity $Q_1 = (0:1:0:0:0)$ and an $\mathbf{A}_3$-singularity $Q_2 = (0:0:0:0:1)$. Let $\mathcal{S} \subset \mathbb{P}^4_\mathbb{Z}$ be its integral model defined by the same equations over $\mathbb{Z}$.

The closure of every rational point $P \in S(\mathbb{Q})$ is an integral point $\overline{P} \in \mathcal{S}(\mathbb{Z})$; both are represented (uniquely up to sign) by coprime $(x_0,\dots,x_4) \in \mathbb{Z}^5 \setminus \{0\}$ satisfying the defining equations (1). Recall that studying integral points becomes interesting only when we choose a *boundary* $\mathcal{Z}$ to consider integral points on $\mathcal{S} \setminus \mathcal{Z}$, and that the types of boundaries

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in Theorem 10 for our case are the singularities and the lines; we start with the former. To do so, let $Z_1 = Q_1$, $Z_2 = Q_2$; in addition to these, we study the boundary $Z_3 = Q_1 \cup Q_2$, which goes beyond the setting of weak del Pezzo pairs described in the beginning of the following section. Let $\mathcal{Z}_i = \overline{Z_i}$ and $\mathcal{U}_i = \mathcal{S} \setminus \mathcal{Z}_i$. Hence, $\overline{P}$ lies in $\mathcal{U}_3(\mathbb{Z})$, say, if and only if it is does not reduce to one of the singularities modulo any prime p. In other words, a representative $(x_0,\ldots,x_4)$ of a point in $\mathcal{U}_i(\mathbb{Z})$ satisfies the *integrality condition*

$$\begin{aligned} &&\gcd(x_0,x_2,x_3,x_4)=1, &\qquad \text{if } i=1,\\ &&\gcd(x_0,x_1,x_2,x_3)=1, &\qquad \text{if } i=2, \text{ or}\\ \gcd(x_0,x_2,x_3,x_4)=1 &\quad \text{and} & \gcd(x_0,x_1,x_2,x_3)=1, &\qquad \text{if } i=3. \end{aligned} \tag{2}$$

Since the sets $\mathcal{U}_i(\mathbb{Z})$ of integral points are clearly infinite, we consider integral points of bounded height. We work with the *height functions*

$$\begin{aligned} H_1(\overline{P}) &= \max\{|x_0|,|x_2|,|x_3|,|x_4|\},\\ H_2(\overline{P}) &= \max\{|x_0|,|x_1|,|x_2|,|x_3|\}, \qquad \text{and}\\ H_3(\overline{P}) &= \max\{|x_0|,|x_2|,|x_3|,\min\{|x_1|,|x_4|\}\} \end{aligned} \tag{3}$$

because they can be interpreted as *log-anticanonical heights* on a minimal desingularization, as we shall see below (Lemma 14).

It turns out that the number of integral points of bounded height is dominated by the integral points on the three lines

$$L_1 = \{x_0 = x_2 = x_3 = 0\},\ L_2 = \{x_0 = x_1 = x_2 = 0\},\ L_3 = \{x_0 + x_3 = x_1 = x_2 = 0\}; \tag{4}$$

in fact, there are infinitely many integral points of height 1 on some of them. Therefore, we count integral points only in their complement $V = S \setminus \{x_2 = 0\}$. Hence, we are interested in the asymptotic behavior of

$$N_i(B) = \#\{\overline{P} \in \mathcal{U}_i(\mathbb{Z}) \cap V(\mathbb{Q}) \mid H_i(\overline{P}) \le B\}, \tag{5}$$

the number of integral points of bounded log-anticanonical height that are not contained in the lines. Explicitly, this is

$$N_i(B) = \#\{(x_0,\ldots,x_4) \in \mathbb{Z}^5 \setminus \{0\} \mid (1),\ (2),\ x_2 > 0,\ H_i(x_0 : \cdots : x_4) \le B\}. \tag{6}$$

Recall that the second type of boundary is a line, resulting in $Z_4 = L_1, Z_5 = L_2, Z_6 = L_3$ with the notation in equation (4). Let $\mathcal{Z}_i = \overline{Z_i}$ in $\mathcal{S}$ and $\mathcal{U}_i = \mathcal{S} \setminus \mathcal{Z}_i$ for $i = 4,5,6$. Analogously to the first three cases, a point $(x_0 : \cdots : x_4) \in S$ with coprime $x_0,\ldots,x_4 \in \mathbb{Z}$ lies in $\mathcal{U}_i(\mathbb{Z})$ if and only if

$$\begin{aligned} \gcd(x_0,x_2,x_3) &= 1, \qquad \text{if } i=4,\\ \gcd(x_0,x_1,x_2) &= 1, \qquad \text{if } i=5, \text{ or}\\ \gcd(x_0+x_3,x_1,x_2) &= 1, \qquad \text{if } i=6. \end{aligned} \tag{7}$$

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We work with the heights

$$H_4(\overline{P}) = \max\{|x_0|,|x_2|,|x_3|\},$$
$$H_5(\overline{P}) = \max\{|x_0|,|x_1|,|x_2|\}, \qquad \text{and}$$
$$H_6(\overline{P}) = \max\{|x_0+x_3|,|x_1|,|x_2|\},$$

which will again turn out to be log-anticanonical on a minimal desingularization. Let $N_i(B)$ for $i=4,5,6$ be defined as in equation (5). They satisfy descriptions as in equation (6), with the integrality condition (2) replaced by condition (7).

Our second result consists of asymptotic formulas for these counting problems.

Theorem 1. *As $B \to \infty$, we have*

$$N_1(B) = \frac{13}{4320}\left(\prod_p \left(1-\frac{1}{p}\right)^5\left(1+\frac{5}{p}\right)\right) B(\log B)^5 + O(B(\log B)^4 \log\log B),$$

$$N_2(B) = \frac{1}{32}\left(\prod_p \left(1-\frac{1}{p}\right)^3\left(1+\frac{3}{p}\right)\right) B(\log B)^4 + O(B(\log B)^3 \log\log B),$$

$$N_3(B) = \frac{1}{8}\left(\prod_p \left(1-\frac{1}{p}\right)^2\left(1+\frac{2}{p}-\frac{1}{p^2}\right)\right) B(\log B)^3 + O(B(\log B)^2 \log\log B),$$

$$N_4(B) = 2\left(\prod_p \left(1-\frac{1}{p}\right)\left(1+\frac{1}{p}\right)\right) B(\log B)^2 + O(B\log B \log\log B), \quad \textit{and}$$

$$N_5(B) = N_6(B) = \frac{7}{24}\left(\prod_p \left(1-\frac{1}{p}\right)^2\left(1+\frac{2}{p}\right)\right) B(\log B)^3 + O(B(\log B)^2 \log\log B).$$

Cases 5 and 6 are symmetric: the involutive automorphism

$$(x_0,x_1,x_2,x_3,x_4) \mapsto (x_0+x_3,-x_1,x_2,-x_3,x_4) \tag{8}$$

of S exchanges the lines L_2 and L_3 and the height functions H_5 and H_6, while leaving $V = S \setminus \{x_2 = 0\}$ invariant, whence $N_5(B) = N_6(B)$.

1.2. The expected asymptotic formula

Similarly to the case of rational points [2, 22], our asymptotic formulas for the number of integral points of bounded height should be interpreted on a desingularization $\rho\colon \widetilde{S} \to S$. Here, $\widetilde{S}$ is a *weak del Pezzo surface*, that is, a smooth projective surface whose anticanonical bundle $\omega_{\widetilde{S}}^\vee$ is big and nef (but not ample in our case).

To interpret the number of points on $\mathcal{U}_i = \mathcal{S} \setminus \mathcal{Z}_i$, we study a desingularization $\widetilde{U}_i = \widetilde{S} \setminus D_i$ of U_i, where $D_i = \rho^{-1}(Z_i)$ is a reduced effective divisor with strict normal crossings. In the context of integral points, the log-anticanonical bundle $\omega_{\widetilde{S}}(D_i)^\vee$ assumes the role of the anticanonical bundle. From this point of view, Theorem 1 can be interpreted in the framework described in [7].

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The minimal desingularization $\rho : \widetilde{S} \to S$ is an iterated blowup of $\mathbb{P}^2_{\mathbb{Q}}$ in five points. The analogous blowup of $\mathbb{P}^2_{\mathbb{Z}}$ results in an integral model $\rho\colon \widetilde{\mathcal{S}} \to \mathcal{S}$ (see section Section 3 for more details). Then D_1, D_2 are the divisors above Q_1,Q_2, respectively, and $D_3 = D_1 + D_2$ is the one over both; see Figure 1 for their dual graph (Dynkin diagram). Our discussion is simplified by the fact that the pairs $(\widetilde{S}, D_i)$ are split, in the sense that $\operatorname{Pic}\widetilde{S} \to \operatorname{Pic}\widetilde{S}_{\overline{\mathbb{Q}}}$ is an isomorphism and [18, Definition 1.6] holds, and by the fact that we are working over $\mathbb{Q}$. Let $\widetilde{U}_i, \widetilde{\mathcal{U}}_i$ be the complement of $D_i, \overline{D_i}$ in $\widetilde{S}, \widetilde{\mathcal{S}}$, respectively, where $\overline{D_i}$ is the Zariski closure of D_i in $\widetilde{\mathcal{S}}$. The preimage of the complement V of the lines on S is the complemenent $\widetilde{V}$ of all negative curves on $\widetilde{S}$.

This leads to the reinterpretation of our counting problem as

$$N_i(B) = \#\{\overline{P} \in \widetilde{\mathcal{U}}_i(\mathbb{Z}) \cap \widetilde{V}(\mathbb{Q}) \mid H_i(\rho(\overline{P})) \le B\}$$

on the minimal desingularization, and we prove in Lemma 14 that $H_i \circ \rho$ is a log-anticanonical height function on $\widetilde{\mathcal{U}}_i(\mathbb{Z}) \cap \widetilde{V}(\mathbb{Q})$. Note that the log-anticanonical bundle $\omega_{\widetilde{S}}(D_i)^\vee$ is big and nef for $i = 1,2,4,5,6$ but big and *not* nef for $i = 3$ (Lemma 12); the unusual shape of H_3 is clearly related to this.

From the shape of asymptotic formulas in previous results [9, 8, 23, 24] and the study of volume asymptotics in [7], we expect that

$$N_i(B) \sim c_{i,\mathrm{fin}} c_{i,\infty} B (\log B)^{b_i - 1},$$

where the leading constant can be decomposed into a finite part $c_{i,\mathrm{fin}}$ and an Archimedean part $c_{i,\infty}$ that we shall describe and determine in Section 6 precisely.

The finite part

$$c_{i,\mathrm{fin}} = \prod_p \left(1 - \frac{1}{p}\right)^{\operatorname{rk}\operatorname{Pic}\widetilde{U}_i} \tau_{(\widetilde{S},D_i),p}(\widetilde{\mathcal{U}}_i(\mathbb{Z}_p)), \tag{9}$$

which behaves similarly as in the case of rational points, is defined as an Euler product of convergence factors and p-adic Tamagawa numbers. We compute the latter as p-adic integrals over $\widetilde{\mathcal{U}}_i(\mathbb{Z}_p)$ (Lemma 24); they turn out to be simply $\#\widetilde{\mathcal{U}}_i(\mathbb{F}_p)/p^{\dim S}$. This reflects the fact that integral points should be distributed evenly in the set $\widetilde{\mathcal{U}}_i(\mathbb{Z}_p)$, which has positive and finite volume with respect to the modified Tamagawa measure $\tau_{(\widetilde{S},D_i),p}$ defined in [7]. (However, we do not prove such an equidistribution result here.)

On the other hand, 100% of the integral points are arbitrarily close to the boundary with respect to the real-analytic topology, ordered by height. This makes the analysis of $c_{i,\infty}$ much more delicate than for rational points. More precisely, the points close to the minimal strata of the boundary—that is, the intersection of a maximal set of intersecting components of D_i—should dominate the counting function. These strata are encoded in the *(analytic) Clemens complex* $\mathcal{C}^{\mathrm{an}}_{\mathbb{R}}(D_i)$. For a split surface, the vertices of this Clemens complex correspond to the irreducible components of the boundary divisor D_i, and there is an edge for each intersection point of two divisors. The Archimedean constant

$$c_{i,\infty} = \sum_A \alpha_{i,A} \tau_{i,D_A,\infty}(D_A(\mathbb{R})) \tag{10}$$

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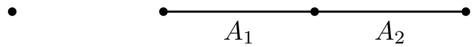

Figure 1. The Clemens complex of D_3 is the disjoint union of those of D_1 (left) and D_2 (right). It is the Dynkin diagram of the $\mathbf{A}_1$- and $\mathbf{A}_3$-singularities Q_1, Q_2.

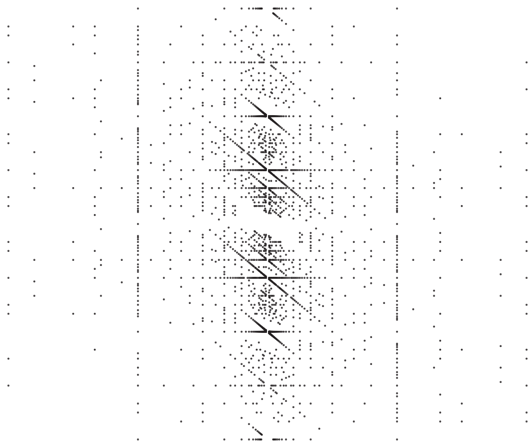

Figure 2. Integral points on $\widetilde{\mathcal{U}}_1$ of height ≤ 90. The boundary divisor is the central vertical line. Some horizontal and diagonal lines look accumulating, but in fact are not: They contain $\sim c'B$ points, which is less than the $cB(\log B)^5$ points on U; the constants c' can however be up to 2, while the constant c in our main theorem is numerically ≈ 0.0003.

is a sum over the faces A of maximal dimension of the Clemens complex, which correspond to the minimal strata D_A of D_i. For each maximal-dimensional face A, we have a product of a rational factor $\alpha_{i,A}$ and an Archimedean Tamagawa number $\tau_{i,D_A,\infty}(D_A(\mathbb{R}))$ coming from a *residue measure* as defined in [7]. This measure can be interpreted as a real density, which is supported on $D_A(\mathbb{R})$ and should measure the distribution of points in neighborhoods of open subsets of $D_A(\mathbb{R})$. From another point of view, the set $\widetilde{S}(\mathbb{R})$ has infinite volume with respect to a modified measure $\tau_{(\widetilde{S},D_i),\infty}$ as above, and $\tau_{i,D_A,\infty}(D_A(\mathbb{R}))$ appears in the leading constant of the asymptotic volume of *height balls* with respect to said measure (cf. [7, Propositions 2.5.1, 4.2.4]).

In the first case, the Clemens complex consists of only one vertex corresponding to the boundary divisor above the $\mathbf{A}_1$-singularity Q_1, and integral points accumulate near it (Figure 2). In the second and third case, the maximal-dimensional faces A_1, A_2 of the Clemens complex correspond to the two intersection points D_{A_1}, D_{A_2} of the divisors above the $\mathbf{A}_3$-singularity, and 'most' integral points are very close to these two intersection points (Figure 3). Correspondingly, the Archimedean Tamagawa number is the volume of the boundary divisor in the first case, and it is the volume of the two intersection points in the second and third cases. In the remaining cases, it similarly is a volume of intersection points (Lemma 25).

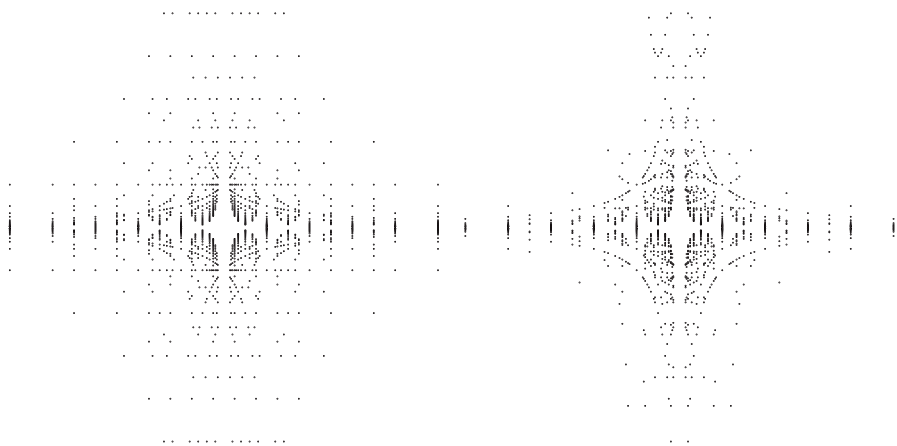

Figure 3. Integral points on $\widetilde{\mathcal{U}}_2$ of height ≤ 60 in neighborhoods of D_{A_1} (left) and D_{A_2} (right). Most points are close to the three boundary divisors, which are the central horizontal line and two vertical lines here.

The rational factor $\alpha_{i,A}$ is particularly interesting in our examples. It is introduced in [9] for toric varieties and generalized in [25] to be

$$\alpha_{i,A} = \operatorname{vol}\{x \in (\operatorname{Eff} \widetilde{U}_{i,A})^\vee \mid \langle x, \omega_{\widetilde{S}}(D_i)^\vee|_{\widetilde{U}_{i,A}}\rangle = 1\}, \tag{11}$$

where $\widetilde{U}_{i,A}$ is the subvariety consisting of $\widetilde{U}_i$ and the divisors corresponding to A. For vector groups [8] and wonderful compactifications [23], the effective cone is generated by the boundary divisors and simplicial, which makes the treatment of this factor easy. In [24], it behaves similarly as Peyre's α for projective varieties since the boundary has just one component; it is also much simpler since the Picard number is 2. Our second and following cases behave in a different way since the Clemens complex is not a simplex, providing the first nontrivial treatment of this factor for a nontoric variety. Here, it turns out that the resulting polytopes for the different maximal faces fit together to one polytope whose volume appears in the leading constant of the counting problem (Lemma 28). In case 4, one of the polytopes has volume 0, making this an example for the obstruction [25, Theorem 2.4.1 (i)] to the existence of integral points near the corresponding minimal stratum of the boundary (Remark 27).

The exponent of $\log B$ is expected to be $b_i - 1$, where

$$b_i = \operatorname{rk} \operatorname{Pic} \widetilde{U}_i - \operatorname{rk} \mathbb{Q}[U_i]/\mathbb{Q}^\times + \dim \mathcal{C}_{\mathbb{R}}^{\mathrm{an}}(D_i) + 1. \tag{12}$$

Here, $\dim \mathcal{C}_{\mathbb{R}}^{\mathrm{an}}(D_i) + 1$ is the maximal number of components of the boundary divisor D_i that meet in the same point, and $\mathbb{Q}[U_i]^\times = \mathbb{Q}^\times$ in each case. While the obstruction described in [25] can lead to this number being smaller than expected if it affects all maximal-dimensional faces of the Clemens complex, this does not happen in our fourth case as there are three unobstructed faces remaining.

We can reformulate Theorem 1 as follows.

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Theorem 2. *For $i \in \{1,\dots,6\}$, we have*

$$N_i(B) = c_{i,\infty} c_{i,\mathrm{fin}} B(\log B)^{b_i - 1}(1+o(1)) \tag{13}$$

as $B \to \infty$, where the constants $c_{i,\infty}$, $c_{i,\mathrm{fin}}$ and b_i are as in equations (9), (10) and (12), respectively.

This confirms the expectations extracted from [7, 9, 25].

1.3. Strategy of the proof

In Section 2, we define and classify *weak del Pezzo pairs* $(\widetilde{S},D)$, which have big and nef log-anticanonical bundle $\omega_{\widetilde{S}}(D)^\vee$ (Theorem 10).

In Section 3, we describe a universal torsor on the minimal desingularization of S, we show that our height functions are log-anticanonical, and we describe them in terms of Cox coordinates. This leads to a completely explicit counting problem on the universal torsor (Lemma 15), with a $(2^{\operatorname{rk}\operatorname{Pic}\widetilde{U}_i} : 1)$-map to our set of integral points of bounded height: roughly, the torsor variables corresponding to the boundary divisors must be ± 1, and in the case of a big and base point free (whence nef) log-anticanonical class, the height function H_i is given by monomials in the Cox ring of log-anticanonical degree. The third case seems to be one of the first examples of the universal torsor method with respect to a height for a divisor class that is big and *not* nef.

In Section 4, we estimate the number of points in our counting problem on the universal torsor using analytic techniques. Here, we approximate summations over the torsor variables by real integrals $V_{i,0}(B)$; the coprimality conditions lead to an Euler product that agrees with $c_{i,\mathrm{fin}}$ (Lemmas 16 and 17). This step is similar to the case of rational points treated in [11]; hence, we shall be very brief.

In Section 5, to complete the proof of Theorem 1, our goal is to transform $V_{i,0}(B)$ into $2^{\operatorname{rk}\operatorname{Pic}\widetilde{U}_i} C_i B(\log B)^{b_i-1}$, where C_i is the product of the volume of a polytope (which turns out to be $\sum \alpha_{i,A}$) and a real density (which agrees with the Archimedean Tamagawa numbers $\tau_{i,D_A,\infty}(D_A(\mathbb{R}))$), up to a negligible error term. In the first case, there is a complication due to an inhomogeneous expression (with respect to the grading by the Picard group) in the domain of $V_{1,0}$ (Lemma 18 and more importantly Lemma 19); here, a subtle estimation is necessary. In the third case, we modify the height function H_3 to H_3' (which coincides essentially with H_2) as in Lemma 20. These extra complications have never appeared in the universal torsor method for rational points; we believe that they are typical for integral points and nonnef heights.

In Section 6, we prove Theorem 2 by explicitly computing the expected constants discussed in Section 1.2.

2. Classification of weak del Pezzo pairs

For us, a *weak del Pezzo pair* $(\widetilde{S},D)$ consists of a smooth projective surface $\widetilde{S}$ with a reduced effective divisor D with strict normal crossings such that the log-anticanonical bundle $\omega_{\widetilde{S}}(D)^\vee$ is big and nef. The aim of this section is to study the possible choices of divisors D on a weak del Pezzo surface $\widetilde{S}$ that render the pair $(\widetilde{S},D)$ weak del Pezzo in this sense.

Remark 3. Considering *pairs* (X,D) is standard when studying integral points: While rational and integral points coincide on complete varieties as a consequence of the valuative criterion for properness, the study of integral points becomes a distinct problem on an integral model $\mathcal{U}$ of a noncomplete variety U. Then one passes to a compactification, more precisely, a smooth projective variety X containing U such that the boundary $D = X \setminus U$ is a reduced effective divisor with strict normal crossings. In particular, the pair (X,D) is smooth and divisorially log terminal.

The goal is then to count the number of points on $\mathcal{U}$ of bounded log-anticanonical height (that is, with respect to $\omega_X(D)^\vee$), excluding any strict subvarieties (or, more generally, *thin* subsets) whose points would contribute to the main term. Setting $D = 0$ then recovers the setting of Manin's conjecture on rational points.

Remark 4. In its original form [16, 21], Manin's conjecture makes a prediction about the number of rational points on *smooth Fano varieties*: smooth projective varieties whose anticanonical bundle is ample. These conditions can be relaxed—for example, only requiring that the anticanonical be big and nef, viz. to *weak Fano varieties* and the two-dimensional varieties thereof, *weak del Pezzo surfaces*. Weak del Pezzo surfaces $\widetilde{S}$ are precisely the smooth del Pezzo surfaces $\widetilde{S} = S$ and the minimal desingularizations $\rho\colon \widetilde{S} \to S$ of del Pezzo surfaces with only **ADE**-singularities [10].

Since ρ is a crepant resolution—that is, $\omega_{\widetilde{S}} = \rho^*\omega_S$—counting points on S of bounded anticanonical height amounts to counting points on $\widetilde{S}$ of bounded anticanonical height after excluding points on the exceptional locus. By [2, 22], an asymptotic formula for the number of rational points on S should be interpreted in terms of its minimal desingularization $\widetilde{S}$; for example, the Picard rank ρ of $\widetilde{S}$ appears in the expected asymptotic formula. The number of rational points of bounded height has been shown to conform to the same prediction as in Manin's conjecture for many weak del Pezzo surfaces (see the references in [1, § 6.4.1]).

Generalizing the question even further, it suffices to assume that the anticanonical bundle is big to guarantee that the number of rational points of bounded anticanonical height outside a suitable divisor is finite. Adding some conditions that make Peyre's constant well-defined leads to the notion of an *almost Fano* variety [22, Définition 3.1], for which it makes sense to ask whether Manin's conjecture holds. While this is known to be the case for some of them, Lehmann, Sengupta, and Tanimoto showed that one cannot expect the conjecture to be true in general in this widest setting [19, Remark 1.1, Example 5.17].

To simplify the exposition, let $\widetilde{S}$ be a weak del Pezzo surface whose degree d is at most 7. Let $D = \sum_{\alpha \in \mathcal{A}} D_\alpha \subset \widetilde{S}$ be a reduced and effective divisor with strict normal crossings and irreducible components D_α.

Lemma 5. *The log-anticanonical bundle* $\omega_{\widetilde{S}}(D)^\vee$ *is nef if and only if all of the following conditions hold:*

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(i) *If E is a (-2)-curve and $D_\alpha.E > 0$ for some $\alpha \in \mathcal{A}$, then $E \subset D$.*

(ii) *If E is an arbitrary negative curve meeting two different D_α, D_β or one $D_\gamma \subset D$ with multiplicity $D_\gamma.E \geq 2$ (with $\alpha,\beta,\gamma \in \mathcal{A}$), then $E \subset D$.*

(iii) *If E is a negative curve, then*

$$\sum_{\substack{\alpha\in\mathcal{A}\\ D_\alpha\neq E}} D_\alpha.E \leq 2.$$

Proof. Recall that a divisor is nef if its intersection with all negative curves is nonnegative. If E is a (-2)-curve, then

$$(-K-D).E = -K.E - D.E = 0 + 2\delta_{E\subset D} - \sum_{\substack{\alpha\in\mathcal{A}\\ D_\alpha\neq E}} D_\alpha.E,$$

and this number is nonnegative if and only if (i) and (iii) hold for E. If E is a (-1)-curve, then

$$(-K-D).E = -K.E - D.E = 1 + \delta_{E\in D} - \sum_{\substack{\alpha\in\mathcal{A}\\ D_\alpha\neq E}} D_\alpha.E,$$

and this number is nonnegative if and only if (ii) and (iii) hold for E. □

Remark 6. If $\rho\colon \widetilde{S} \to S$ is the minimal desingularization of a singular del Pezzo surface, then Lemma 5 shows: If one of the (-2)-curves above a singularity $Q \in S$ is in D, then by Lemma 5 (i) all curves above this singularity must be in D. Similarly, if a (-1)-curve whose image in S contains a singularity Q is in D, then all (-2)-curves above Q must be in D. By Lemma 5 (iii), Q must be an **A**-singularity in both cases.

The surface $\widetilde{S}$ can be described by a sequence of $r = 9 - \deg \widetilde{S}$ blowups

$$\widetilde{S} = \widetilde{S}^{(r)} \xrightarrow{\pi_r} \widetilde{S}^{(r-1)} \to \cdots \to \widetilde{S}^{(1)} \xrightarrow{\pi_1} \widetilde{S}^{(0)} = \mathbb{P}^2,$$

where π_i is the blowup in a point p_i that does not lie on a (-2)-curve on $\widetilde{S}^{(i-1)}$. Let $\pi\colon \widetilde{S} \to \mathbb{P}^2$ be their composition. Let $\ell_0 = \pi^*\ell$, where ℓ is the class of a line on $\mathbb{P}^2$, and for $1 \leq i \leq r$, let $\ell_i = (\pi_{i+1}\cdots\pi_r)^* E^{(i)}$, where $E^{(i)}$ is the exceptional divisor of the ith blowup π_i. Then the Picard group of $\widetilde{S}$ is freely generated by the classes $\ell_0,\dots,\ell_r$. The intersection form is given by $\ell_i.\ell_j = 0$ for $i \neq j$, $\ell_0^2 = 1$, and $\ell_i^2 = -1$ for $i \geq 1$. Let P be the image of an exceptional divisor of one of the the blowups in $\mathbb{P}^2$ and n_P be the number of exceptional curves mapped to P. Then these negative curves form a chain, the first $n_P - 1$ of which are (-2)-curves whose classes have the form $\ell_{i_1} - \ell_{i_2}, \dots, \ell_{i_{s-1}} - \ell_{i_s}$ followed by a (-1)-curve whose class has the form ℓ_{i_s}. The anticanonical class is $3\ell_0 - \ell_1 - \cdots - \ell_r$, and we fix an anticanonical divisor $-K$. Denote by $[F]$ the class of a divisor or line bundle F in the Picard group. For $L_1, L_2 \in \operatorname{Pic}(\widetilde{S})_{\mathbb{R}}$, we write $L_1 \leq L_2$ if their difference $L_2 - L_1$ is in the effective cone.

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Lemma 7. *Let $L \in \operatorname{Pic}(\widetilde{S})$.*

(i) *If $L \le \sum_{1\le j\le r} a_j\ell_j$ for some $a_1,\dots,a_r \in \mathbb{Z}$, then L is not big.*

(ii) *If $L \le \ell_0 - \ell_i$ for some $i \ge 1$, then L is not nef or not big.*

Proof. For the first statement, we just have to note that $-\varepsilon\ell_0 + \sum_{1\le j\le k} a_j\ell_j$ is not effective for any $\varepsilon > 0$. Turning to the second statement, assume for contradiction that L is big and nef. Note that $\ell_0 - \ell_i$ has nonnegative intersection with all (-1)-curves.

If $\ell_0 - \ell_i$ has (strictly) negative intersection with a (-2)-curve E, then this curve needs to have class $[E] = \ell_i - \ell_j$ for some $j \ne i$ (cf. [20, Theorem 25.5.3]). Writing $\ell_0 - \ell_i = L + [F]$ with effective F, we get $F.E < 0$, so $E \subset F$, and $L \le \ell_0 - \ell_i - [E] = \ell_0 - \ell_j$. The only negative curves that could have negative intersection with $\ell_0 - \ell_j$ have class $\ell_j - \ell_k$. As curves of classes $\ell_i - \ell_j$, $\ell_j - \ell_k$, etc., are contracted to a single point by π, we can eventually find an $\ell_{i'}$ with $L \le \ell_0 - \ell_{i'}$ and such that $\ell_0 - \ell_{i'}$ has nonnegative intersection with all negative curves. Then $\ell_0 - \ell_{i'}$ is nef. But $(\ell_0 - \ell_{i'})^2 = 0$, whence it cannot be big. □

Proposition 8. *Assume that $\deg \widetilde{S} \le 4$. If $\omega_{\widetilde{S}}(D)^\vee$ is big and nef, then D is contained in the union of all negative curves.*

Proof. Assume for contradiction that D contains a nonnegative curve C, but that $-K - D$ is big and nef. In particular, $-K - D'$ is big for all $D' \subset D$. Since C is nonnegative, it is the strict transform of a curve C_0 on $\mathbb{P}^2$. Then

$$[C] = d\ell_0 - \sum_{i=1}^{r} a_i\ell_i,$$

where $d = \deg C_0$ and $a_i = C.\ell_i$.

We first reduce to the case of C_0 being a line. If $d \ge 3$, then $[-K - C] \le \sum_{1\le i\le r} a_i\ell_i$, which is not big by Lemma 7 (i). If C_0 is a nondegenerate conic, then $a_1,\dots,a_r \le 1$ since C_0 has multiplicity ≤ 1 in all images of the exceptional divisors. Moreover, since $C^2 \ge 0$, at most four of the a_i are nonzero. It follows that $[-K - C] \le \ell_0 - \ell_j$, so $-K - D$ is not big or not nef by Lemma 7 (ii).

Let C_0 be a line. As the self-intersection of C is nonnegative, $[C] = \ell_0 - \ell_j$ for some j or $[C] = \ell_0$. In the first case, C_0 contains the center $P = \pi_1 \cdots \pi_j(p_j)$ of a blowup. If $n_P > 1$, then $\pi^{-1}(P)$ contains (-2)-curves. Appealing to Lemma 5 (i), the first (-2)-curve must be contained in D, as must the remaining (-2)-curves by repeated applications. Let E_0 be the sum of these (-2)-curves. Then $C' = C + E_0 \subset D$ is of class $[C'] = \ell_0 - \ell_{j'}$, where $\ell_{j'}$ is the class of the final (-1)-curve in the chain. If $n_P = 1$ or $[C] = \ell_0$, set $E_0 = 0$ and $C' = C$; in the first case, set $j' = j$; in the latter case, fix an arbitrary j' and note that $[C] \le \ell_0 - \ell_{j'}$. Then C' satisfies the conditions in Lemma 5 for all negative curves in the preimage of P by this construction, and it does the same for all other curves contracted by π as it does not meet them. For what remains, we distinguish three cases.

Case 1. The curve C does not meet any of the remaining (-2)-curves, and $C.E \le 1$ for all remaining (-1)-curves. Then $(-K - C')$ is nef by Lemma 5. But $(-K - C')^2 \le 4 - (r-1) \le 0$, so it cannot be big.

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Case 2. The curve C meets one of the remaining (-2)-curves E. Then $E \subset D$ by Lemma 5 (i). Since E is the strict transform of a curve in $\mathbb{P}^2$, its class satisfies $[E] = l_0 - l_{i_1} - l_{i_2} - l_{i_3}$ for some pairwise different i_1, i_2, i_3 or $[E] \geq 2\ell_0 + \sum a_i \ell_i$ for some $a_i \in \mathbb{Z}$. In the first case, $[-K - C - E] \leq \ell_0 - \ell_k$ for $k \neq i_i, i_2, i_3, j$, and in the second case, $[-K - C - E] \leq \sum_{1 \leq i \leq r} a_i \ell_i$. In both cases, $-K - D$ is not big or not nef by Lemma 7 (ii) or (i), respectively.

Case 3. The curve C meets a (-1)-curve E with $C.E \geq 2$. By Lemma 5 (ii), $E \subset D$. As E is the strict transform of a curve on $\mathbb{P}^2$, its class verifies $[E] \geq [F]$ for a (-2)-class $[F]$ of the same shape as in the previous case; hence, $-K - D$ is not big or not nef. □

Remark 9. The assumption $\deg \widetilde{S} \leq 4$ in Proposition 8 is necessary: Let $\widetilde{S}$ be a smooth del Pezzo surface of degree at least 5 that is a blowup of $\mathbb{P}^2$ in at most 4 points in general position. Then the strict transform D of a line that meets precisely one of these points is an example of a nonnegative curve such that $\omega_{\widetilde{S}}(D)^\vee$ is big and nef.

Theorem 10. *Let $\widetilde{S}$ be a weak del Pezzo surface of degree $d \leq 4$. Precisely the following choices of a reduced effective divisor D make $(\widetilde{S}, D)$ a weak del Pezzo pair.*

(i) *The divisor D can be zero.*

(ii) *If $3 \leq d \leq 4$, then D can consist of all (-2)-curves corresponding to one* **A**-*singularity.*

(iii) *If $d = 4$, then D can consist of a (-1)-curve and all (-2)-curves corresponding to all singularities on its image in the anticanonical model, provided that those singularities are* **A**-*singularities and all curves in D form a chain.*

Proof. Let D be a reduced effective divisor such that $-K - D$ is big and nef. By Proposition 8, $D = \sum E_i$ has to be supported on negative curves. Consider the complete subgraph G of the Dynkin diagram on the vertices corresponding to components of D. By Lemma 5 (iii), each of its connected components is a path or a cycle. Let N_1 be the number of (-1)-curves in D, and N_2 be the number of (-2)-curves. Then $v = N_1 + N_2$ is the number of vertices of G, and denote by e its number of edges.

The self-intersection of the log-anticanonical divisor is

$$(-K - D)^2 = K^2 + \sum_i E_i^2 + 2\sum_i E_i.K + 2\sum_{i<j} E_i.E_j.$$

As $-K.E$ is zero for (-2)-curves and 1 for (-1)-curves, we get

$$(-K - D)^2 = d + 2(e - v) - N_1. \tag{14}$$

Since $-K - D$ is big and nef, this self-intersection must be positive.

If G is connected and not a cycle, then $e = v - 1$, so $d - 2 - N_1 > 0$. In case $d = 4$, this leaves us with $N_1 \leq 1$, in case $d = 3$ with $N_1 = 0$, and in case $d \leq 2$ with an immediate contradiction. In each case, the resulting divisors satisfy the asserted description using Remark 6 and that the graph is a path.

It remains to prove that G has to be connected and not a cycle. If G is not connected and does not contain a cycle, then $(-K - D)^2 = d - 4 - N_1 \leq 0$, so $-K - D$ cannot be big, leaving only the case of graphs G containing a cycle, in which case $d-N_1>0$ for $-K-D$ to be big.

For $N_1=0$, we note that only Dynkin diagrams of type **A**, **D**, and **E** appear as intersection graphs of (-2)-curves, and these do not contain double edges nor more general cycles.

If $N_1=1$, then $d\geq 2$. The sum E_2 of the (-2)-curves in D forms a (-2)-class since $E_2^2=-2N_2+2(s-1)=-2$ and $-K.E_2=0$. As the Weyl group acts transitively on (-1)-curves and leaves the intersection pairing invariant, we can assume that $[E_1]=\ell_1$. Now $\ell_1.[E_2]\geq 2$, and so the (-2)-class needs to have the form $3\ell_0-2\ell_1-\ell_2-\cdots-\ell_8$. But such a class does not exist if $d\geq 2$ (cf. [20, Theorem 25.5.3]).

If $N_1=2$, then $d\geq 3$. In this case, the anticanonical model contracts (-2)-curves and maps (-1)-curves to lines. The resulting two lines then need to intersect with multiplicity 2, an impossibility.

Finally, if $N_1=3$, then $d=4$. In this case, the anticanonical model $\phi\colon \widetilde{S}\to S=Q_1\cap Q_2\subset\mathbb{P}^4$ is a (possibly singular) intersection of two quadrics, still contracting all (-2)-curves and mapping all (-1)-curves to lines. The resulting three lines need to intersect pairwise. If they were contained in a plane P, this plane would intersect Q_1 in three lines, an impossibility. So the three lines intersect in a point Q. The tangent space at Q needs to contain each plane containing two of these lines, whence Q is singular. Then $\widetilde{S}\to S$ factors through the blowup Y of $\mathbb{P}^4$ in Q. The strict transforms of the lines do not intersect on Y, and thus the (-1)-curves on $\widetilde{S}$ do not intersect. It follows that each of them intersects a (-2)-curve above Q. Hence, the (-2)-curves are contained in D, and at least one of them needs to intersect three other negative curves in D. Now, Lemma 5 (iii) implies that $-K-D$ cannot be nef.

Conversely, if D is one of the divisors in the statement, then it is nef by Lemma 5, and its self-intersection is positive by equation (14); hence, it is also big. □

3. Passage to a universal torsor

As in the introduction, let $S\subset\mathbb{P}^4_{\mathbb{Q}}$ be the singular quartic del Pezzo surface defined by the equations (1). By [12, 15] (but using the notation and numbering of [11, Section 8]), a Cox ring of its minimal desingularization $\widetilde{S}$ is

$$R=\mathbb{Q}[\eta_1,\ldots,\eta_9]/(\eta_1\eta_9+\eta_2\eta_8+\eta_4\eta_5^3\eta_6^2\eta_7) \tag{15}$$

with grading

$$\begin{aligned}
&\deg\eta_1=\ell_5,\quad \deg\eta_2=\ell_4,\quad \deg\eta_3=\ell_0-\ell_1-\ell_4-\ell_5,\\
&\deg\eta_4=\ell_1-\ell_2,\quad \deg\eta_5=\ell_3,\quad \deg\eta_6=\ell_2-\ell_3,\\
&\deg\eta_7=\ell_0-\ell_1-\ell_2-\ell_3,\quad \deg\eta_8=\ell_0-\ell_4,\quad \deg\eta_9=\ell_0-\ell_5
\end{aligned} \tag{16}$$

for a certain basis $\ell_0,\ldots,\ell_5$ of $\operatorname{Pic}\widetilde{S}$. See Figure 4 for the dual graph of the divisors E_i corresponding to η_i.

The minimal desingularization $\widetilde{S}$ can be described as a certain sequence of five iterated blowups of $\mathbb{P}^2_{\mathbb{Q}}$ in rational points [11]: first blow up three points P_1, P_2, P_4 on a line l_3,

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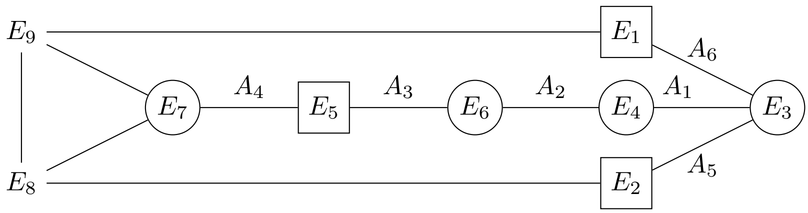

Figure 4. Configuration of the divisors E_i and the faces A_i of the Clemens complexes. The (-1)-curves are represented by squares and the (-2)-curves by circles.

resulting in exceptional curves E_1, E_2 and E_4'; then blow up the intersection of E_4' with a line l_7, resulting in an exceptional curve E_6'; then blow up the intersection of E_6' with the strict transform of l_7, resulting in an exceptional curve E_5. With this description, E_3 is the strict transform of l_3, E_4 that of E_4', E_6 that of E_6', E_7 that of l_7, E_8 that of a general line through P_2 and E_9 that of a line through P_1 such that E_7, E_8, E_9 meet in one point, recovering the above grading using a basis as before Lemma 7.

With a point of view coming from S, the divisor $D_1 = E_7$ is the (-2)-curve on $\widetilde{S}$ above the singularity Q_1 on S, the divisor $D_2 = E_3 + E_4 + E_6$ is the sum of the (-2)-curves above Q_2, the divisor $D_3 = D_1 + D_2 = E_3 + E_4 + E_6 + E_7$ is the sum of all (-2)-curves, and E_5, E_2, E_1 are the (-1)-curves that are the strict transforms of the three lines L_1, L_2, L_3 on S as in equation (4), respectively, while E_8 and E_9 correspond to the two further generators of the Cox ring. The divisors

$$D_4 = E_3 + \cdots + E_7, \quad D_5 = E_2 + E_3 + E_4 + E_6, \quad \text{and} \quad D_6 = E_1 + E_3 + E_4 + E_6$$

lie above the lines L_1, L_2 and L_3, respectively. Since $V \subset S$ is the complement of the lines, which contain the singularities, its preimage $\widetilde{V} \subset \widetilde{S}$ is the complement of the negative curves $E_1, \ldots, E_7$.

The irrelevant ideal of R is $I_{\mathrm{irr}} = \prod(\eta_i, \eta_j)$, where the product runs over all pairs $i < j$ such that there is no edge between E_i and E_j in Figure 4. The sections

$$L_0 = \{\eta_2\eta_3\eta_4\eta_5\eta_6\eta_7\eta_8,\ \eta_1^2\eta_2^2\eta_3^3\eta_4^2\eta_6,\ \eta_1\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2\eta_7,\ \eta_3\eta_4^2\eta_5^4\eta_6^3\eta_7^2,\ \eta_7\eta_8\eta_9\} \tag{17}$$

have anticanonical degree and define the morphism $\rho : \widetilde{S} \to S$.

As in [17, Proposition 4.1(i)], let $\widetilde{\mathcal{S}}$ be the integral model defined by the corresponding sequence of blowups of $\mathbb{P}^2_{\mathbb{Z}}$, and recall $\widetilde{\mathcal{U}}_i = \mathcal{S} - \overline{D_i}$. Consider the open subscheme $\mathcal{Y}$ of the spectrum of

$$R_{\mathbb{Z}} = \mathbb{Z}[\eta_1, \ldots, \eta_9]/(\eta_1\eta_9 + \eta_2\eta_8 + \eta_4\eta_5^3\eta_6^2\eta_7)$$

defined as the complement of $\mathbb{V}(I_{\mathrm{irr}} \cap R_{\mathbb{Z}})$. By [17, Proposition 4.1(ii)], $\mathcal{Y}$ is a $\mathbb{G}_{\mathrm{m},\mathbb{Z}}^6$-torsor over $\widetilde{\mathcal{S}}$ via a morphism $\pi : \mathcal{Y} \to \widetilde{\mathcal{S}}$; here, the action of $\mathbb{G}_{\mathrm{m},\mathbb{Z}}^6$ on $\mathcal{Y}$ is given by the degrees of the coordinates $(\eta_1, \ldots, \eta_9)$ in $\operatorname{Pic} \widetilde{S} \cong \mathbb{Z}^6$ in equation (16); see [17, Construction 3.1] for details. This torsor defines an explicit parametrization of integral points by lattice points.

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Lemma 11. *Let $i \in \{1,\dots,6\}$, and $\mathcal{Y}_i = \pi^{-1}(\widetilde{\mathcal{U}}_i) \subset \mathcal{Y}$. Then $\pi\colon \mathcal{Y}_i \to \widetilde{\mathcal{U}}_i$ is a $\mathbb{G}_{\mathrm{m},\mathbb{Z}}^6$-torsor. This morphism induces a 2^6-to-1-correspondence*

$$\mathcal{Y}_i(\mathbb{Z}) \cap \pi^{-1}(\widetilde{V})(\mathbb{Q}) \to \widetilde{\mathcal{U}}_i(\mathbb{Z}) \cap \widetilde{V}(\mathbb{Q}),$$

and we have

$$\mathcal{Y}_i(\mathbb{Z}) \cap \pi^{-1}(\widetilde{V})(\mathbb{Q}) = \{\boldsymbol{\eta} \in \mathbb{Z}^9 \mid (18),\ (19),\ (20)\ \text{hold},\ \eta_1 \cdots \eta_7 \neq 0,\},$$

where

$$\eta_1\eta_9 + \eta_2\eta_8 + \eta_4\eta_5^3\eta_6^2\eta_7 = 0, \tag{18}$$

$$\gcd(\eta_i,\eta_j) = 1 \qquad \text{if } E_i \text{ and } E_j \text{ do not share an edge in Figure 4, and} \tag{19}$$

$$|\eta_j| = 1 \qquad \text{if } E_j \subset |D_i|. \tag{20}$$

Proof. Since π is a $\mathbb{G}_{\mathrm{m},\mathbb{Z}}^6$-torsor so are its restrictions to the open subschemes $\widetilde{\mathcal{U}}_i$. Integral points $\boldsymbol{\eta} \in \operatorname{Spec} R_{\mathbb{Z}}$ are lattice points $(\eta_1,\dots,\eta_9) \in \mathbb{Z}^9$ satisfying the equation in the Cox ring. Such a point is integral on the complement of $\mathbb{V}(I_{\mathrm{irr}}) = \bigcup \mathbb{V}(\eta_i,\eta_j)$—the union running over all i,j which do not share an edge in Figure 4—if it does not reduce to any of the $\mathbb{V}(\eta_i,\eta_j)$ for any prime, that is, if the gcd-condition (19) holds.

Integral points on $\mathcal{Y}_1 \subset \mathcal{Y}$ are precisely those which do not reduce to $\pi^{-1}(E_7) = \mathbb{V}(\eta_7)$ at any place; that is, they are those points satisfying $\eta_7 \in \{\pm 1\}$. Analogously, integral points on $\mathcal{Y}_2$ are those satisfying $\eta_3,\eta_4,\eta_6 \in \{\pm 1\}$, integral points on $\mathcal{Y}_3$ are those satisfying $\eta_3,\eta_4,\eta_6,\eta_7 \in \{\pm 1\}$ and similarly for $\mathcal{Y}_4,\mathcal{Y}_5,\mathcal{Y}_6$. The preimage of $\widetilde{V}$ in the universal torsor is the complement of $\eta_1 \cdots \eta_7 = 0$. $\square$

We now turn to studying the log-anticanonical bundles and the height functions associated with them. Recall that the case D_6 can be reduced to D_5 by symmetry as in (8).

Lemma 12. *The only nonzero reduced effective divisors $D \subset \widetilde{S}$ such that $\omega_{\widetilde{S}}(D)^\vee$ is big and nef are D_i for $i \in \{1,2,4,5,6\}$. Consider the sets*

$$\begin{aligned}
M_1 &= \{\eta_2\eta_3\eta_4\eta_5\eta_6\eta_8, \eta_1\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2, \eta_3\eta_4^2\eta_5^4\eta_6^3\eta_7, \eta_8\eta_9\},\\
M_2 &= \{\eta_2\eta_5\eta_7\eta_8, \eta_1^2\eta_2^2\eta_3^2\eta_4, \eta_4\eta_5^4\eta_6^2\eta_7^2\},\\
M_4 &= \{\eta_2\eta_8, \eta_1\eta_2\eta_3\eta_4\eta_5\eta_6, \eta_4\eta_5^3\eta_6^2\eta_7\}, \quad \text{and}\\
M_5 &= \{\eta_5\eta_7\eta_8, \eta_1^2\eta_2\eta_3^2\eta_4, \eta_1\eta_3\eta_4\eta_5^2\eta_6\eta_7\}
\end{aligned}$$

of monomials in the Cox ring R of degree $\omega_{\widetilde{S}}(D_i)^\vee$ for $i = 1,2,4,5$, respectively. For $\boldsymbol{\eta} \in \mathbb{Z}^9$ satisfying equation (19), none of these sets can vanish simultaneously modulo a prime p. The respective log-anticanonical bundles are base point free.

The log-anticanonical bundle $\omega_{\widetilde{S}}(D_3)^\vee$ is big, but not nef, whence not base point free. It has a representation $\omega_{\widetilde{S}}(D_3)^\vee \cong \mathcal{L}_1 \otimes \mathcal{L}_2^\vee$ as a quotient of the nef bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ whose sections are elements of degree $4\ell_0 - \ell_1 - \ell_2$ and $3\ell_0 - \ell_1 - \ell_2 - \ell_3$ in the Cox ring,

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respectively. Consider the sets of Cox ring elements

$$\begin{aligned} L_2 &= \eta_1\eta_2 L_0 \cup \{\eta_4^2\eta_5^6\eta_6^4\eta_7^3\} && \text{of degree } \mathcal{L}_2,\\ L_3 &= \{\eta_2\eta_5\eta_8, \eta_1\eta_2\eta_3\eta_4\eta_5^2\eta_6, \eta_4\eta_5^4\eta_6^2\eta_7\} && \text{of degree } \omega_{\widetilde{S}}(D_3)^\vee, \text{ and}\\ L_1 &= L_2L_3 \cup \{\eta_1^3\eta_2^3\eta_3^2\eta_4\eta_8\eta_9\} && \text{of degree } \mathcal{L}_1. \end{aligned}$$

Then neither L_1 nor L_2 can vanish simultaneously modulo a prime p.

Proof. The first statement is a special case of Theorem 10.

For the first set, assume that $p \mid \eta_8\eta_9$ for a prime p. Then $p \nmid \eta_3\cdots\eta_6$, since the corresponding divisors $E_3,\dots,E_6$ share an edge with neither E_8 nor E_9 in Figure 4, while at most one of η_1,η_2,η_7 can be divisible by p. Hence, the second or third section is not divisible by p.

For the second set, assume that $p \mid \eta_2\eta_5\eta_7\eta_8$. If $p \mid \eta_5\eta_7$, then $p \nmid \eta_1^2\eta_2^2\eta_3^2\eta_4$; if $p \mid \eta_2\eta_8$ and $p \nmid \eta_7$, then $p \nmid \eta_4\eta_5^3\eta_6^2\eta_7$. Regarding the case $i=4$, if $p \mid \eta_2\eta_8$, then $p \nmid \eta_4\eta_5^3\eta_6^2\eta_7$. For $i=5$, assume $p \mid \eta_5\eta_7\eta_8$. If $p \mid \eta_5\eta_7$, then $p \nmid \eta_1^2\eta_2\eta_3^2\eta_4$; if $p \mid \eta_8$ and $p \nmid \eta_7$, then $p \nmid \eta_1\eta_3\eta_4\eta_5^2\eta_6\eta_7$.

Turning to L_2, we know that the anticanonical sections in equation (17) cannot be divisible by p simultaneously, so for all sections in L_2 to be be divisible by p simultaneously, $p \mid \eta_1\eta_2$. But then $p \nmid \eta_4^2\eta_5^6\eta_6^4\eta_7^3$.

Lastly, assume that the monomials in L_1 are divisible by p so that p in particular divides all monomials in L_2L_3. Since p cannot divide all monomials in L_2, it has to divide all monomials in L_3. If follows that $p \mid \eta_2\eta_5\eta_8$. If $p \mid \eta_2$, then the last monomial in L_3 cannot be zero modulo p. If $p \mid \eta_8$, then p can divide only one of η_2 or η_7 but none of the remaining variables in the latter two sections of L_3, so one of those two sections is nonzero modulo p. So $p \mid \eta_5$. Now p can divide at most one of η_6 and η_7 but none of the remaining variables, and so the last section of L_1 is nonzero modulo p, a contradiction.

By the same arguments (replacing vanishing modulo p by vanishing over $\overline{\mathbb{Q}}$), the log-anticanonical bundles $\omega_{\widetilde{S}}(D_i)^\vee$ for $i=1,2,4,5$ and the bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ are base point free, whence nef. On the other hand, $\omega_{\widetilde{S}}(D_3)^\vee$ is not nef since its intersection number with E_5 is -1. $\square$

Lemma 13. *For $i \in \{1,\dots,5\}$, the morphism $\rho\colon \widetilde{S} \to S$ induces bijections*

$$\widetilde{\mathcal{U}}_i(\mathbb{Z}) \cap \widetilde{V}(\mathbb{Q}) \to \mathcal{U}_i(\mathbb{Z}) \cap V(\mathbb{Q}).$$

Proof. Consider the morphism $f\colon \mathcal{Y} \to \mathbb{P}^4_{\mathbb{Z}}$ defined by $\boldsymbol{\eta} \mapsto (s_0(\boldsymbol{\eta}) : \cdots : s_4(\boldsymbol{\eta}))$ with the anticanonical sections s_j in equation (17). We only have to show that equation (19) holds for $\boldsymbol{\eta}$ if and only if the corresponding gcd-condition in (2), resp. in (7), holds for $f(\boldsymbol{\eta})$. To this end, we note that $\eta_1\eta_2\eta_3\eta_4\eta_5^2\eta_6\eta_7$ is in the radical of the ideal generated by M_2 as in Lemma 12, so the gcd-condition (2) can be rewritten as

$$|\eta_7|\gcd\{m(\boldsymbol{\eta}) \mid m \in M_1\} = 1 \quad \text{and} \quad |\eta_3\eta_4\eta_6|\gcd\{m(\boldsymbol{\eta}) \mid m \in M_2\} = 1.$$

By Lemma 12, these gcds are one, and so the claim follows for $i \in \{1,2\}$. Since $\mathcal{U}_3$ and $\widetilde{\mathcal{U}}_3$ are the intersections of the respective open subschemes within the first two cases, the assertion follows for $i=3$. The cases $i=4$ and $i=5$ can be proved using an analogous reformulation of the first two conditions in equation (7). $\square$

These sets of sections define adelic metrics on line bundles isomorphic to $\omega_{\widetilde{S}}(D_i)^\vee$ for $i \in \{1,2,4,5\}$ and the line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$, and the latter two metrics induce one on $\mathcal{L}_1 \otimes \mathcal{L}_2^\vee \cong \omega_{\widetilde{S}}(D_3)^\vee$. The metrics on the bundles isomorphic to the log-anticanonical bundles then induce log-anticanonical height functions $\widetilde{H}_i$ for $i \in \{1,\dots,5\}$.

Lemma 14. *For $\boldsymbol{\eta} = (\eta_1,\dots,\eta_9) \in \mathbb{R}^9$ satisfying (18) and (20), let*

$$\mathcal{H}_i(\boldsymbol{\eta}) = \begin{cases} \max\{|\eta_2\eta_3\eta_4\eta_5\eta_6\eta_8|, |\eta_1\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|, |\eta_3\eta_4^2\eta_5^4\eta_6^3|, |\eta_8\eta_9|\}, & i=1; \\ \max\{|\eta_2\eta_5\eta_7\eta_8|, |\eta_1^2\eta_2^2|, |\eta_5^4\eta_7^2|\}, & i=2; \\ \max\{|\eta_2\eta_5\eta_8|, |\eta_1\eta_2\eta_5^2|, |\eta_5^4|, \min\{\left|\eta_1^2\eta_2^2\right|, |\eta_8\eta_9|\}\}, & i=3; \\ \max\{|\eta_2\eta_8|, |\eta_1\eta_2|, 1\}, & i=4; \\ \max\{|\eta_1^2|, |\eta_1\eta_5^2\eta_7|, |\eta_5\eta_7\eta_8|\}, & i=5. \end{cases}$$

For $\boldsymbol{\eta} \in \mathcal{Y}_i(\mathbb{Z}) \cap \pi^{-1}(\widetilde{V})(\mathbb{Q})$, we have $\mathcal{H}_i(\boldsymbol{\eta}) = \widetilde{H}_i(\pi(\boldsymbol{\eta})) = H_i(\rho(\pi(\boldsymbol{\eta})))$, where H_i is the height in equation (3) and $\widetilde{H}_i$ is the log-anticanonical height on $\widetilde{S}(\mathbb{Q})$ induced by the sections in Lemma 12.

Proof. For $i \in \{1,2,4,5\}$, the metrics induce height functions $\widetilde{H}_i(x) = H_{\mathbb{P}^{N_i}}(f_i(x))$ verifying $f_i(\pi(\boldsymbol{\eta})) = (m_0(\boldsymbol{\eta}) : \dots : m_{N_i}(\boldsymbol{\eta}))$ for the sections $m_0,\dots,m_{N_i} \in M_i$ constructed in Lemma 12, so

$$H_i(\pi(\boldsymbol{\eta})) = \prod_v \max_{m \in M_i} \{|m(\boldsymbol{\eta})|_v\}.$$

By the same lemma, the p-adic contributions to this product are 1, whence $\widetilde{H}_i(\pi(\boldsymbol{\eta})) = \mathcal{H}_i(\boldsymbol{\eta})$.

To check that these height functions coincide with the ones defined in the introduction, we note that, for example, for a point in $\mathcal{Y}_1(\mathbb{Z})$ we have $\eta_7 \in \{\pm 1\}$, and thus $|\eta_2\eta_3\eta_4\eta_5\eta_6\eta_8| = |\eta_2\eta_3\eta_4\eta_5\eta_6\eta_7\eta_8| = |x_0|$. We get analogous identities for the other coordinates and cases. There is no section corresponding to x_2 in the second height function, but, for integral points on $\mathcal{Y}_2$, we have $|x_2| = \sqrt{|\eta_1^2\eta_2^2||\eta_5^4\eta_7^2|} = \sqrt{|x_1x_3|}$; hence, it can never contribute to the maximum.

The case $i=3$ is more complicated. The log-anticanonical height function $\widetilde{H}_3$ induced by the metrics on $\mathcal{L}_1$ and $\mathcal{L}_2$ satisfies

$$\widetilde{H}_3(\pi(\boldsymbol{\eta})) = \frac{\max_{s \in L_1} |s(\boldsymbol{\eta})|}{\max_{s \in L_2} |s(\boldsymbol{\eta})|} = \max\left\{\max_{s \in L_3} |s(\boldsymbol{\eta})|, \frac{\left|\eta_1^3\eta_2^3\eta_3^2\eta_4\eta_8\eta_9\right|}{\max_{s \in L_2} |s(\boldsymbol{\eta})|}\right\}$$

by an analogous argument as in the previous cases. Now note that, for $\boldsymbol{\eta} \in \mathcal{Y}_3(\mathbb{Z}) \cap \pi^{-1}(\widetilde{V})(\mathbb{Q})$, we can simplify this to

$$\widetilde{H}_3(\pi(\boldsymbol{\eta})) = \max\{|\eta_2\eta_5\eta_8|, \left|\eta_1\eta_2\eta_5^2\right|, \left|\eta_5^4\right|, \min\{\left|\eta_1^2\eta_2^2\right|, |\eta_8\eta_9|\}\}. \tag{21}$$

Indeed, $\eta_3,\eta_4,\eta_6,\eta_7$ have absolute value 1 and the remaining variables absolute value at least one. Then

$$\frac{\left|\eta_1^3\eta_2^3\eta_3^2\eta_4\eta_8\eta_9\right|}{\max_{s \in L_2} |s(\boldsymbol{\eta})|} \le \min\{\left|\eta_1^2\eta_2^2\right|, |\eta_8\eta_9|\}$$

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follows from $\eta_1^3\eta_2^3\eta_3^4\eta_4^2\eta_6, \eta_1\eta_2\eta_7\eta_8\eta_9 \in L_2$. Now assume that $\eta_1^2\eta_2^2$ is larger than the other three terms in the maximum in equation (21); then the maximum over L_2 can only be attained by $\eta_1^3\eta_2^3\eta_3^4\eta_4^2\eta_6$ or $\eta_1\eta_2\eta_7\eta_8\eta_9$, and the inverse inequality follows.

Finally, we note that $|\eta_3| = |\eta_4| = |\eta_6| = |\eta_7| = 1$ implies $\left|\eta_1^2\eta_2^2\right| = |x_1|$ and $|\eta_8\eta_9| = |x_4|$. □

Lemma 15. *For $i \in \{1,\dots,5\}$, we have*

$$N_i(B) = \frac{1}{2^{6-\#D_i}} \# \left\{ (\eta_1 \dots, \eta_9) \in \mathbb{Z}^9 \;\middle|\; \begin{array}{l} (18),\ (19) \text{ hold, } \eta_j = 1 \text{ if } E_j \subset D_i, \\ \eta_1 \cdots \eta_7 \neq 0,\ \mathcal{H}_i(\eta_1, \dots, \eta_9) \leq B \end{array} \right\},$$

where $\#D_i$ denotes the number of irreducible components of D_i.

Proof. We combine Lemma 11 and Lemma 14. The $\#D_i$ coordinates η_j belonging to irreducible components $E_j \subset D_i$ satisfy $|\eta_j| = 1$. By symmetry, we can further assume that $\eta_j = 1$, making the 2^6-to-1-correspondence from Lemma 11 a $2^{6-\#D_i}$-to-1-correspondence. □

4. Counting

In our counting process, we treat η_9 as a dependent variable using the torsor equation from (15), which we regard as a congruence modulo the coefficient η_1 of η_9. First, we sum over η_8 and then over the remaining variables. Since this is similar to the case of rational points in [11], we shall be brief.

In this section, we use the notation

$$\boldsymbol{\eta}^{(i)} = (\eta_j)_{j \in J_i} = \begin{cases} (\eta_1, \dots, \eta_6), & i = 1; \\ (\eta_1, \eta_2, \eta_5, \eta_7), & i = 2; \\ (\eta_1, \eta_2, \eta_5), & i = 3; \\ (\eta_1, \eta_2), & i = 4; \\ (\eta_1, \eta_5, \eta_7), & i = 5 \end{cases}$$

for $(7 - \#D_i)$-uples indexed by

$$J_i = \{ j \in \{1, \dots, 7\} \mid E_j \not\subset D_i \}. \tag{22}$$

We write $\mathcal{H}_i(\boldsymbol{\eta}^{(i)}, \eta_8)$ for $\mathcal{H}_i(\eta_1, \dots, \eta_9)$, where $\eta_j = 1$ whenever $E_j \subset D_i$ and where η_9 is expressed in terms of $\eta_1, \dots, \eta_8$ using the torsor equation (18), assuming $\eta_1 \neq 0$.

Lemma 16. *For $i \in \{1,\dots,5\}$, we have*

$$N_i(B) = \frac{1}{2^{6-\#D_i}} \sum_{\boldsymbol{\eta}^{(i)} \in \mathbb{Z}_{\neq 0}^{J_i}} \theta_1(\boldsymbol{\eta}^{(i)}) V_{i,1}(\boldsymbol{\eta}^{(i)}; B) + O(B \log B)$$

with

$$V_{i,1}(\boldsymbol{\eta}^{(i)}; B) = \int_{\mathcal{H}_i(\boldsymbol{\eta}^{(i)}, \eta_8) \leq B} \frac{\mathrm{d}\eta_8}{|\eta_1|}$$

and

$$\theta_1(\boldsymbol{\eta}^{(i)}) = \prod_p \theta_{1,p}(I_p(\boldsymbol{\eta}^{(i)})),$$

where $I_p(\boldsymbol{\eta}^{(i)}) = \{j \in J_i \mid p \mid \eta_j\}$ *and*

$$\theta_{1,p}(I) = \begin{cases} 1, & I = \emptyset, \{1\}, \{2\}, \{7\}; \\ 1 - \frac{1}{p}, & I = \{4\}, \{5\}, \{6\}, \{1,3\}, \{2,3\}, \{3,4\}; \{4,6\}, \{5,6\}, \{5,7\}; \\ 1 - \frac{2}{p}, & I = \{3\}; \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. The proof is as in [11, Lemma 8.4], with slightly different height functions and some $\eta_i = 1$, which leads to different error terms. In the first case, using the second height condition, the error term is

$$\ll \sum_{\eta_1,\dots,\eta_6} 2^{\omega(\eta_3)+\omega(\eta_3\eta_4\eta_5\eta_6)} \ll \sum_{\eta_2,\dots,\eta_6} \frac{2^{\omega(\eta_3)+\omega(\eta_3\eta_4\eta_5\eta_6)}B}{|\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|} \ll B\log B.$$

In the second case, using the second and the third height conditions, it is

$$\ll \sum_{\eta_1,\eta_2,\eta_5,\eta_7} 2^{\omega(\eta_5)} \ll \sum_{\eta_1,\eta_5} \frac{2^{\omega(\eta_5)}B}{|\eta_1\eta_5^2|} \ll B\log B.$$

In the third case, using the second height condition, it is

$$\ll \sum_{\eta_1,\eta_2,\eta_5} 2^{\omega(\eta_5)} \ll \sum_{\eta_1,\eta_5} \frac{2^{\omega(\eta_5)}B}{|\eta_1\eta_5^2|} \ll B\log B.$$

The remaining cases are very similar. □

Lemma 17. *For* $i \in \{1,\dots,5\}$, *we have*

$$N_i(B) = \frac{1}{2^{6-\#D_i}}\left(\prod_p \omega_{i,p}\right) V_{i,0}(B) + O(B(\log B)^{b_i-2}\log\log B)$$

with

$$V_{i,0}(B) = \int_{|\eta_j|\geq 1 \ \forall j\in J_i} V_{i,1}(\boldsymbol{\eta}^{(i)};B)\,\mathrm{d}\boldsymbol{\eta}^{(i)}$$

and

$$\omega_{i,p} = \begin{cases} \left(1-\frac{1}{p}\right)^{6-\#D_i}\left(1+\frac{6-\#D_i}{p}\right), & i \in \{1,2,4,5,6\}; \\ \left(1-\frac{1}{p}\right)^2\left(1+\frac{2}{p}-\frac{1}{p^2}\right), & i = 3. \end{cases}$$

Proof. In the first case, by equation (18), the last height condition is

$$|(\eta_2\eta_8^2 + \eta_4\eta_5^3\eta_6^2\eta_8)/\eta_1| \leq B. \tag{23}$$

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Hence, by [11, Lemma 5.1(4)],

$$V_{1,1}(\eta_1,\dots,\eta_6;B) \ll \frac{B^{1/2}}{|\eta_1\eta_2|^{1/2}} = \frac{B}{|\eta_1\eta_2\eta_3\eta_4\eta_5\eta_6|}\left(\frac{B}{|\eta_1\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|}\right)^{-1/2}.$$

In the second case, we use

$$V_{2,1}(\eta_1,\eta_2,\eta_5,\eta_7;B) \ll \frac{B}{|\eta_1\eta_2\eta_5\eta_7|}.$$

In the third case, we use

$$V_{3,1}(\eta_1,\eta_2,\eta_5;B) \ll \frac{B}{|\eta_1\eta_2\eta_5|}.$$

Therefore, [11, Proposition 4.3, Corollary 7.10] gives the result in the first three cases. The final cases are similar to the second and third cases. □

5. Volume asymptotics

We must show that the real integrals $V_{i,0}(B)$ in Lemma 17 grow of order $B(\log B)^{b_i-1}$. In the first and third case, this is more subtle than for rational points.

Lemma 18. *We have* $|V'_{1,0}(B)-V_{1,0}(B)| \ll B(\log B)^4$, *where*

$$V'_{1,0}(B) = \int_{\substack{|\eta_2|,\dots,|\eta_6|\geq 1\\ \mathcal{H}'_1(\eta_1,\dots,\eta_6,\eta_8)\leq B}} \frac{\mathrm{d}\eta_1\dots\mathrm{d}\eta_6\,\mathrm{d}\eta_8}{|\eta_1|}$$

with

$$\mathcal{H}'_1(\eta_1,\dots,\eta_6,\eta_8) = \max\left\{\begin{matrix}|\eta_2\eta_3\eta_4\eta_5\eta_6\eta_8|, |\eta_1\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|,\\ |\eta_3\eta_4^2\eta_5^4\eta_6^3|, |\eta_2\eta_8^2/\eta_1|, |\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|\end{matrix}\right\}.$$

Proof. We must show that adding the condition $|\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2| \leq B$, removing the condition $|\eta_1|\geq 1$, and replacing equation (23) by $|\eta_2\eta_8^2/\eta_1| \leq B$ in the integration domain changes the integral by $\ll B(\log B)^4$.

Adding the condition $|\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2| \leq B$ does not change $V_{1,0}(B)$ since this inequality follows from $|\eta_1| \geq B$ and the second height condition. Afterwards, we can remove the condition $|\eta_1|\geq 1$ from $V_{1,0}(B)$ since this changes the integral by

$$\int_{\substack{|\eta_1|\leq 1,\ |\eta_2|,\dots,|\eta_6|\geq 1\\ \mathcal{H}_1(\eta_1,\dots,\eta_6,\eta_8)\leq B\\ |\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|\leq B}} \frac{\mathrm{d}\eta_1\dots\mathrm{d}\eta_6\,\mathrm{d}\eta_8}{|\eta_1|} \ll \int_{\substack{|\eta_1|\leq 1,\ |\eta_2|,\dots,|\eta_6|\geq 1\\ |\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|\leq B}} \frac{B^{1/2}\,\mathrm{d}\eta_1\dots\mathrm{d}\eta_6}{|\eta_1\eta_2|^{1/2}},$$

where we estimate the integral over η_8 as in the proof of Lemma 17. Now we observe that the new condition $|\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2| \leq B$ together with $|\eta_2|,\dots,|\eta_6|\geq 1$ implies $|\eta_2|,\dots,|\eta_6| \leq B$; all these conditions and $|\eta_1|\leq 1$ allow us to bound the error as required.

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Finally, we must replace equation (23) by $|\eta_2\eta_8^2/\eta_1| \le B$. A comparison of $\mathcal{H}_1$ in Cox coordinates (Lemma 14) with H_1 as in equation (3) motivates the transformation

$$\eta_8 = \frac{B}{\eta_2\eta_3\eta_4\eta_5\eta_6}x_0, \quad \eta_1 = \frac{B}{\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2}x_2, \tag{24}$$

which turns $\frac{\mathrm{d}\eta_1\,\mathrm{d}\eta_8}{|\eta_1|}$ into $\frac{B\,\mathrm{d}x_0\,\mathrm{d}x_2}{|x_2\eta_2\eta_3\eta_4\eta_5\eta_6|}$, and the transformation $\eta_3 = \frac{B}{\eta_4^2\eta_5^4\eta_6^3}x_3$, which turns $\frac{\mathrm{d}\eta_3}{|\eta_3|}$ into $\frac{\mathrm{d}x_3}{|x_3|}$. These transformations turn $\mathcal{H}_1(\eta_1,\dots,\eta_6,\eta_8) \le B$ into

$$|x_0|,|x_2|,|x_3|,|x_0(x_0+x_3)/x_2| \le 1.$$

Furthermore, they turn $|\eta_3| \ge 1$ and $|\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2| \le B$ (which imply $|\eta_2\eta_4^2\eta_5^2\eta_6^2| \le B$) into a condition $X_3 \le |x_3| \le X_3'$ for certain X_3 and X_3', whose values (depending on $\eta_2,\eta_4,\eta_5,\eta_5,B$) will not matter to us. Altogether, this shows that

$$V_{1,0}(B) = \int_{\substack{|\eta_2|,|\eta_4|,|\eta_5|,|\eta_6|\ge 1\\ |\eta_2\eta_4^2\eta_5^2\eta_6^2|\le B}} W(\eta_2,\eta_4,\eta_5,\eta_6,B)\frac{B\,\mathrm{d}\eta_2\,\mathrm{d}\eta_4\,\mathrm{d}\eta_5\,\mathrm{d}\eta_6}{|\eta_2\eta_4\eta_5\eta_6|} + O(B(\log B)^4)$$

with

$$W(\eta_2,\eta_4,\eta_5,\eta_6,B) = \int_{\substack{|x_0|,|x_2|,|x_3|,|x_0(x_0+x_3)/x_2|\le 1\\ X_3\le|x_3|\le X_3'}} \frac{\mathrm{d}x_0\,\mathrm{d}x_2\,\mathrm{d}x_3}{|x_2x_3|}.$$

Now the following Lemma 19 shows that we can replace $|x_0(x_0+x_3)/x_2| \le 1$ by $|x_0^2/x_2| \le 1$ with an error of $O(1)$. We plug this back into $V_{1,0}(B)$ and observe that the integral of $O(1)\cdot B/|\eta_2\eta_4\eta_5\eta_6|$ is $\ll B(\log B)^4$, while the inverse of our previous transformations turn the main term into $V_{1,0}'(B)$ since they turn $|x_0^2/x_2| \le 1$ into $|\eta_2\eta_8^2/\eta_1| \le B$ and since we can remove the condition $|\eta_2\eta_4^2\eta_5^2\eta_6^2| \le B$, which is implied by the others. □

To complete the proof of Lemma 18, we show:

Lemma 19. *We have* $W(\eta_2,\eta_4,\eta_5,\eta_6,B) = W'(\eta_2,\eta_4,\eta_5,\eta_6,B) + O(1)$, *where*

$$W'(\eta_2,\eta_4,\eta_5,\eta_6,B) = \int_{\substack{|x_0|,|x_2|,|x_3|,|x_0^2/x_2|\le 1\\ X_3\le|x_3|\le X_3'}} \frac{\mathrm{d}x_0\,\mathrm{d}x_2\,\mathrm{d}x_3}{|x_2x_3|}.$$

Proof. As a first step, we integrate over x_2 to get

$$W(\eta_2,\eta_4,\eta_5,\eta_6,B) = \int_{\substack{|x_0(x_0+x_3)|\le 1\\ |x_0|,|x_3|\le 1\\ X_3\le|x_3|\le X_3'}} (-2\log|x_0| - 2\log|x_0+x_3|)\frac{\mathrm{d}x_0\,\mathrm{d}x_3}{|x_3|} \tag{25}$$

and shall integrate the two terms individually.

To determine the integral over the first one, we remove the condition $|x_0(x_0+x_3)| \le 1$, introducing an error of at most

$$|R_1(\eta_2,\eta_4,\eta_5,\eta_6,B)| \le 4\int_{\substack{x_0,|x_3|\le 1,x_0\ge 0\\ |x_0(x_0+x_3)|\ge 1}} -\log x_0\frac{\mathrm{d}x_0\,\mathrm{d}x_3}{|x_3|}$$

by using the symmetry in the signs of x_0 and x_3. The last inequality implies that x_3 has a distance of at least $1/|x_0|$ (which is ≥ 1) from $-x_0$. Since $x_0 > 0$ and $x_3 > -1$, it cannot

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be smaller, and thus $-x_0+1/x_0 \le x_3 \le 1$ holds. We thus get

$$|R_1(\eta_2,\eta_4,\eta_5,\eta_6,B)| \ll \int_{0\le x_0\le 1} -\log x_0 \left(\int_{-x_0+\frac{1}{x_0}\le x_3\le 1} \frac{\mathrm{d}x_3}{|x_3|}\right)\mathrm{d}x_0$$
$$\ll \int_{0\le x_0\le \frac{\sqrt{5}-1}{2}} \left|\log x_0 \log\left(-x_0+\frac{1}{x_0}\right)\right| \mathrm{d}x_0 \ll 1.$$

We can now integrate the first term in equation (25) over x_0 and get

$$\int_{\substack{|x_0|,|x_3|,|x_0(x_0+x_3)|\le 1\\ X_3\le |x_3|\le X_3'}} -2\log|x_0| \frac{\mathrm{d}x_0\,\mathrm{d}x_3}{|x_3|} = \int_{\substack{|x_3|\le 1\\ X_3\le|x_3|\le X_3'}} 4\frac{\mathrm{d}x_3}{|x_3|} + O(1). \tag{26}$$

To treat the second term, we begin with a change of variables $x_0' = x_0+x_3$ and add the condition $|x_0'|\le 1$, introducing an error of at most

$$|R_2(\eta_2,\eta_4,\eta_5,\eta_6,B)| \le \int_{\substack{|x_0'-x_3|,|x_3|,|x_0'(x_0'-x_3)|\le 1\\ x_0'>1}} 4\log x_0' \frac{\mathrm{d}x_0'\,\mathrm{d}x_3}{|x_3|},$$

again using the symmetry of the integral. The third condition implies $|x_3-x_0'|\le 1/|x_0'| < 1$—that is, $x_0'-1/x_0' < x_3$—and thus we get

$$|R_2(\eta_2,\eta_4,\eta_5,\eta_6,B)| \ll \int_{x_0'>1} \log x_0' \left(\int_{x_0'-\frac{1}{x_0'}}^{1} \frac{\mathrm{d}x_3}{|x_3|}\right)\mathrm{d}x_0'$$
$$\ll \int_{1<x_0'\le 2} \log x_0' \left|\log\left(x_0'-\frac{1}{x_0'}\right)\right| \mathrm{d}x_0' \ll 1.$$

(For the second inequality, note that $x_0'-1/x_0'\le 1$ implies $x_0'\le 2$.) Thus, the second term of equation (25) is

$$\int_{\substack{|x_0'(x_0'-x_3)|,|x_0'|,|x_0'-x_3|,|x_3|\le 1\\ X_3\le|x_3|\le X_3'}} -2\log|x_0'| \frac{\mathrm{d}x_0'\,\mathrm{d}x_3}{|x_3|} + O(1).$$

The condition $|x_0'(x_0'-x_3)|\le 1$ is implied by the second and third condition, so we can remove it. Removing $|x_0'-x_3|\le 1$ introduces an error of at most

$$|R_3(\eta_2,\eta_4,\eta_5,\eta_6,B)| \le \int_{\substack{x_0',|x_3|\le 1\\ |x_0'-x_3|>1\\ x_0'\ge 0}} -2\log x_0' \frac{\mathrm{d}x_0'\,\mathrm{d}x_3}{|x_3|}$$

by the symmetry of the integral. The conditions imply $-1\le x_3\le x_0'-1$ and thus

$$|R_3(\eta_2,\eta_4,\eta_5,\eta_6,B)| \ll \int_{0\le x_0'\le 1} -\log x_0' \left(\int_{-1}^{x_0'-1} \frac{\mathrm{d}x_3}{|x_3|}\right)\mathrm{d}x_0'$$
$$\ll \int_{0\le x_0'\le 1} \log x_0' \log|x_0'-1|\,\mathrm{d}x_0' \ll 1.$$

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Thus, the integral of the second summand of equation (25) is

$$\int_{\substack{|x_0'|,|x_3|\le 1\\ X_3\le|x_3|\le X_3'}} -2\log|x_0'|\frac{\mathrm{d}x_0'\,\mathrm{d}x_3}{|x_3|}+O(1)=\int_{\substack{|x_3|\le 1\\ X_3\le|x_3|\le X_3'}} 4\frac{\mathrm{d}x_3}{|x_3|}+O(1). \tag{27}$$

Since

$$\int_{|x_0|,|x_2|,|x_0^2/x_2|\le 1}\frac{\mathrm{d}x_0\,\mathrm{d}x_2}{|x_2|}=8, \tag{28}$$

adding equations (26) and (27) yields the desired result. □

Lemma 20. *We have* $|V_{3,0}'(B)-V_{3,0}(B)|\ll B(\log B)^2$, *where*

$$V_{3,0}'(B)=\int_{\substack{|\eta_1|,|\eta_2|,|\eta_5|\ge 1,\\ \mathcal{H}_3'(\eta_1,\eta_2,\eta_5,\eta_8)\le B}}\frac{\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_5\,\mathrm{d}\eta_8}{|\eta_1|}$$

with

$$\mathcal{H}_3'(\eta_1,\eta_2,\eta_5,\eta_8)=\max\{|\eta_2\eta_5\eta_8|,|\eta_1\eta_2\eta_5^2|,|\eta_5^4|,\,|\eta_1^2\eta_2^2|\}.$$

Proof. The difference that we must estimate is the integral over

$$|\eta_2\eta_5\eta_8|,\,|\eta_1\eta_2\eta_5^2|,\,|\eta_5^4|,\,|(\eta_2\eta_8^2+\eta_5^3\eta_8)/\eta_1|\le B\le|\eta_1^2\eta_2^2|.$$

Using the condition $|(\eta_2\eta_8^2+\eta_5^3\eta_8)/\eta_1|\le B$ in [11, Lemma 5.1(4)], we have

$$|V_{3,0}'(B)-V_{3,0}(B)|\ll\int_{\substack{|\eta_1|,|\eta_2|,|\eta_5|\ge 1\\ |\eta_1\eta_2\eta_5^2|\le B}}\frac{B^{1/2}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_5}{|\eta_1\eta_2|^{1/2}}.$$

The remaining conditions imply $|\eta_1|,|\eta_2|\le B$. Now the result follows by integrating first over $|\eta_5|\le(B/|\eta_1\eta_2|)^{1/2}$ and then over $1\le|\eta_1|,|\eta_2|\le B$. □

Lemma 21. *We have*

$$V_{1,0}'(B)=2^5C_1B(\log B)^5,\quad V_{2,0}(B)=2^3C_2B(\log B)^4,\quad V_{3,0}'(B)=2^2C_3B(\log B)^3,$$
$$V_{4,0}(B)=2C_4B(\log B)^2,\quad \textit{and}\quad V_{5,0}(B)=2^2C_5B(\log B)^3$$

with

$$C_1=8\operatorname{vol}\left\{(t_2,\dots,t_6)\in\mathbb{R}_{\ge 0}^5\;\middle|\;\begin{array}{l}t_2+2t_3+2t_4+2t_5+2t_6\le 1,\\ t_3+2t_4+4t_5+3t_6\le 1\end{array}\right\}=\frac{13}{4320},$$

$$C_2=4\operatorname{vol}\{(t_1,t_2,t_5,t_7)\in\mathbb{R}_{\ge 0}^4\mid 2t_1+2t_2\le 1,\ 4t_5+2t_7\le 1\}=\frac{1}{32},$$

$$C_3=4\operatorname{vol}\{(t_1,t_2,t_5)\in\mathbb{R}_{\ge 0}^3\mid 2t_1+2t_2\le 1,\ 4t_5\le 1\}=\frac{1}{8},$$

$$C_4=4\operatorname{vol}\{(t_1,t_2)\in\mathbb{R}_{\ge 0}^2\mid t_1+t_2\le 1\}=2,\quad \textit{and}$$

$$C_5=4\operatorname{vol}\{(t_1,t_5,t_7)\in\mathbb{R}_{\ge 0}^3\mid 2t_1\le 1,\ t_1+2t_5+t_7\le 1\}=\frac{7}{24}.$$

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Proof. Again, we apply the coordinate change (24), which shows that

$$V'_{1,0}(B) = \int_{\substack{|\eta_2|,|\eta_3|,|\eta_4|,|\eta_5|,|\eta_6|\geq 1\\ |\eta_3\eta_4^2\eta_5^4\eta_6^3|,|\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|\leq B}} \frac{B\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3\,\mathrm{d}\eta_4\,\mathrm{d}\eta_5\,\mathrm{d}\eta_6}{|\eta_2\eta_3\eta_4\eta_5\eta_6|} \cdot \int_{|x_0|,|x_2|,|x_0^2/x_2|\leq 1} \frac{\mathrm{d}x_0\,\mathrm{d}x_2}{|x_2|}.$$

The integral over x_0, x_2 is 8 by equation (28). Restricting to positive η_i introduces a factor of 2^5. Substituting $\eta_i = B^{t_i}$ turns $\mathrm{d}\eta_i/\eta_i$ into $\log B\,\mathrm{d}t_i$, and we thus arrive at

$$V'_{1,0}(B) = 2^5 \int_{\substack{t_2,t_3,t_4,t_5,t_6\geq 0\\ t_3+2t_4+4t_5+3t_6\leq 1\\ t_2+2t_3+2t_4+2t_5+2t_6\leq 1}} 8B(\log B)^5\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4\,\mathrm{d}t_5\,\mathrm{d}t_6.$$

This integral can be interpreted as the volume of a polytope, which we compute using Magma.

For the second case, using the first height condition yields

$$V_{2,0}(B) = 2\int_{|\eta_1|,|\eta_2|,|\eta_5|,|\eta_7|\geq 1,\ |\eta_1^2\eta_2^2|,|\eta_5^4\eta_7^2|\leq B} \frac{B\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_5\,\mathrm{d}\eta_7}{|\eta_1\eta_2\eta_5\eta_7|}.$$

We proceed as in the first case; here, we can compute the volume by hand. The final two cases are analogous.

For the third case, we observe that $|\eta_1\eta_2\eta_5^2|$ can be ignored in the definition of $\mathcal{H}'_3$ since it is the geometric average of $|\eta_1^2\eta_2^2|$ and $|\eta_5^4|$. Now the computation is very similar to the second case. □

Plugging this into Lemma 17 (after applying Lemma 18 and Lemma 20 in the first and third cases) completes the proof of Theorem 1.

6. The leading constant

We show that Theorem 1 can be abstractly formulated as Theorem 2. Part of the leading constants (9) are p-adic Tamagawa volumes $\tau_{(\widetilde{S},D_i),p}(\widetilde{\mathcal{U}}_i(\mathbb{Z}_p))$ as defined in [7, §§ 2.1.10, 2.4.3]. These measures are similar to the usual Tamagawa volumes studied in the context of rational points, except for factors $\|1_{D_i}\|_p$ that are constant and equal to 1 on the set of p-adic integral points at almost all places (in fact, at all finite places in our cases). Over the reals, the analogous volumes, when evaluated on the full space of real points, would be infinite. Instead, *residue measures* $\tau_{i,D_A,\infty}$ supported on minimal strata $D_A(\mathbb{R})$ of the boundary divisors appear in the leading constant (10), cf. [7, § 2.1.12]. These can be interpreted as a density function for the set of integral points (100% of which are in arbitrarily small real-analytic neighborhoods of the boundary; hence, a density function has to be supported on the boundary), cf. [8, 3.5.8], or the leading constant of an asymptotic expansion of the volume of *height balls* with respect to $\tau_{(\widetilde{S},D_i),\infty}$, cf. [7, Theorem 4.7].

In addition, we have to compute factors $\alpha_{i,A}$ as in equation (11) (cf. [25]), similar to Peyre's in the case of rational points [22]. Again, there is one of these factors associated with any minimal stratum A of the boundary.

In order to compute the Tamagawa volumes, we work with the chart

$$f\colon V' = \widetilde{S} \setminus \mathbb{V}(\eta_1\eta_2\eta_3\eta_4\eta_5\eta_6) \to \mathbb{A}^2_{\mathbb{Q}},$$
$$(\eta_1 : \eta_2 : \eta_3 : \eta_4 : \eta_5 : \eta_6 : \eta_7 : \eta_8 : \eta_9) \mapsto \left(\eta_7 \cdot \frac{\eta_5^2\eta_6}{\eta_1\eta_2\eta_3}, \eta_8 \cdot \frac{1}{\eta_1\eta_3\eta_4\eta_5\eta_6}\right)$$

and its inverse $g\colon \mathbb{A}^2_{\mathbb{Q}} \to \widetilde{S}$,

$$(x,y) \mapsto (1:1:1:1:1:1:x:y:-x-y).$$

Note that the two elements

$$\eta_7 \cdot \frac{\eta_5^2\eta_6}{\eta_1\eta_2\eta_3} \quad \text{and} \quad \eta_8 \cdot \frac{1}{\eta_1\eta_3\eta_4\eta_5\eta_6}$$

have degree 0 in the field of fractions of the Cox ring. The rational map they define is thus invariant under the torus action and descends to $\widetilde{S}$.

Lemma 22. *The images of the sets of p-adic integral points are*

$$\begin{aligned}
f(\widetilde{\mathcal{U}}_1(\mathbb{Z}_p) \cap V'(\mathbb{Q}_p)) &= \{(x,y) \in \mathbb{Q}_p^2 \mid |x| \geq 1 \textit{ or } |xy^2| \geq 1\},\\
f(\widetilde{\mathcal{U}}_2(\mathbb{Z}_p) \cap V'(\mathbb{Q}_p)) &= \{(x,y) \in \mathbb{Q}_p^2 \mid |y| \leq 1 \textit{ or } |xy^2| \leq 1 \textit{ or } |x+y| \leq 1\}\\
&= \{|y| \leq 1\} \cup \{|y| > 1,\ |xy^2| \leq 1\} \cup \{|y| > 1,\ |x+y| \leq 1\},\\
f(\widetilde{\mathcal{U}}_3(\mathbb{Z}_p) \cap V'(\mathbb{Q}_p)) &= f(\widetilde{\mathcal{U}}_1(\mathbb{Z}_p) \cap V'(\mathbb{Q}_p)) \cap f(\widetilde{\mathcal{U}}_2(\mathbb{Z}_p) \cap V'(\mathbb{Q}_p))\\
&= \{|y| \leq 1,\ |x| \geq 1\} \cup \{|y| > 1,\ |xy^2| = 1\} \cup \{|y| > 1,\ |x+y| \leq 1\},\\
f(\widetilde{\mathcal{U}}_4(\mathbb{Z}_p) \cap V'(\mathbb{Q}_p)) &= \{|x| \geq 1, |y| \leq 1\} \cup \{|y| > 1, |x+y| \leq 1\}, \qquad \textit{and}\\
f(\widetilde{\mathcal{U}}_5(\mathbb{Z}_p) \cap V'(\mathbb{Q}_p)) &= \{|x|, |y| \leq 1\} \cup \{|y| > 1, \left|xy^2\right| \leq 1\} \cup \{|y| > 1, |x+y| \leq 1\}.
\end{aligned}$$

Here, the unions are disjoint.

Proof. Consider the image (x,y) of an integral point $\pi(\eta_1,\dots,\eta_9) \in \widetilde{\mathcal{U}}_1(\mathbb{Z}_p)$. Assume $|x| < 1$. Then $\eta_5 \notin \mathbb{Z}_p^\times$ or $\eta_6 \notin \mathbb{Z}_p^\times$ (since $\eta_7 \in \mathbb{Z}_p^\times$). In both cases, the coprimality conditions imply $\eta_8 \in \mathbb{Z}_p^\times$, and thus $|xy^2| = |\eta_7\eta_8^2/\eta_1^3\eta_2\eta_3^3\eta_4^2\eta_6| \geq 1$.

On the other hand, let us consider a point (x,y) in the above set and construct an integral point $(\eta_1,\dots,\eta_9)$ on the torsor with $f(\pi(\eta_1,\dots,\eta_9)) = (x,y)$. If $|x| < 1$, we distinguish two cases for $|y|$:

(i) If $1/|x|^{1/2} \leq |y| < 1/|x|$, let $\eta_5 = xy$, $\eta_6 = 1/xy^2$, $\eta_9 = -1 - x/y$ and the remaining coordinates be 1. Then $\eta_9 \in -1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ since $|x/y| \leq |x|^{1/2} < 1$, and thus the coprimality conditions are satisfied.

(ii) If $1/|x| \leq |y|$, let $\eta_4 = 1/xy$, $\eta_6 = x$, $\eta_9 = -1 - x/y$, and let all the other coordinates be 1. Since $|x/y| \leq |x|^2 < 1$, we again have $\eta_9 \in -1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$, and thus the coprimality conditions hold.

If $|x| \geq 1$, we distinguish three cases for $|y|$.

(i) If $|y| < 1$, let $\eta_2 = 1/x$, $\eta_8 = y$, $\eta_9 = -1 - y/x$ and the remaining coordinates be 1. Then $\eta_9 \in -1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ since $|y/x| < 1$.

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(ii) If $1 \le |y| < |x|$, let $\eta_3 = 1/y$, $\eta_2 = y/x$, $\eta_9 = -1 - y/x$ and the remaining coordinates be 1. Again, we have $|y/x| < 1$ so that $\eta_9 \in -1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$.

(iii) Finally, if $|x| \le |y|$, let $\eta_3 = 1/x$, $\eta_4 = x/y$, $\eta_1 = -1 - x/y$ and the remaining coordinates be 1. If $|y| > |x|$, we have $\eta_1 \in -1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$; if $|x| = |y|$, we have $\eta_4 \in \mathbb{Z}_p^\times$. In both cases, the coprimality conditions on the torsor are satisfied.

We now turn to $\widetilde{\mathcal{U}}_2$. Let (x,y) be in the image of the set of integral points. If $|y| > 1$, we have either $|\eta_5| < 1$ or $|\eta_1| < 1$. In the first case, we get $|xy^2| = |\eta_7\eta_8^2/\eta_1^3\eta_2| = |\eta_7| \le 1$ (since all other variables have to be units); for the second case, we note that

$$x + y = \frac{\eta_4\eta_5^3\eta_6^2\eta_7 + \eta_2\eta_8}{\eta_1\eta_2\eta_3\eta_4\eta_5\eta_6} = -\frac{\eta_1\eta_9}{\eta_1\eta_2\eta_3\eta_4\eta_6},$$

and thus $|x+y| = |\eta_9| \le 1$ (since all other variables have to be units).

On the other hand, let (x,y) be in the set on the right-hand side in the statement of the lemma. We want to construct an integral point on the torsor lying above (x,y). If $|y| \le 1$ and $|x| \le 1$, let $\eta_8 = y$, $\eta_7 = x$, $\eta_9 = -x-y$ and the remaining variables be 1, which satisfies the coprimality conditions. If $|y| \le 1$ and $|x| > 1$, let $\eta_8 = y$, $\eta_2 = 1/x$, $\eta_9 = -1 - y/x$ and the remaining variables be 1. Then $\eta_9 \in -1 - p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$, so $(\eta_1, \dots, \eta_9)$ is integral. Let now $|y| > 1$. If $|xy^2| \le 1$, let $\eta_5 = 1/y$, $\eta_7 = xy^2$, $\eta_9 = -1 - xy$ and the remaining variables be 1; again, $\eta_9 \in \mathbb{Z}_p^\times$. Finally, if $|x+y| \le 1$, let $\eta_1 = 1/x$, $\eta_9 = -x-y$, $\eta_8 = -\eta_1\eta_9 - 1$ and the remaining variables be 1. Then $\eta_8 \in \mathbb{Z}_p^\times$, so $(\eta_1, \dots, \eta_9)$ is integral, and, since $\eta_8/\eta_1 = (-\eta_1\eta_9 - 1)/\eta_1 = x + y - x = y$, it indeed lies above (x,y). For the disjoint union description of $\widetilde{\mathcal{U}}_2$, we just have to observe that $|y| > 1$ and $|xy^2| \le 1$ implies $|x| = |y|^{-2} < 1$, while $|y| > 1$ and $|x+y| \le 1$ implies $|x| = |y| > 1$.

The third set consists of points that are integral with respect to both Q_1 and Q_2. Therefore, we obtain it as the intersection of the previous two sets. For the description of $\widetilde{\mathcal{U}}_3$ as a disjoint union, we start with the one of $\widetilde{\mathcal{U}}_2$ and intersect each set with $\widetilde{\mathcal{U}}_1$. Here, $|y| \le 1$ implies $|x| \ge 1$ since otherwise $|xy^2| < 1$. Furthermore, $|y| > 1$ and $|xy^2| \le 1$ implies $|x| < 1$; hence, $|xy^2| \ge 1$ must hold. Finally, $|y| > 1$ and $|x+y| \le 1$ implies $|x| = |y| > 1$.

The final two cases are analogous. $\square$

Lemma 23. *Let v be a place of $\mathbb{Q}$. For the measures $\tau_{(\widetilde{S},D_i),v}$ defined in [7, § 2.4.3], we have*

$$\mathrm{d}f_*\tau_{(\widetilde{S},D_1),v} = \frac{1}{|x|\max\{|y|,1,|x|,|y(y+x)|\}}\,\mathrm{d}x\,\mathrm{d}y,$$

$$\mathrm{d}f_*\tau_{(\widetilde{S},D_2),v} = \frac{1}{\max\{|xy|,1,|x^2|\}}\,\mathrm{d}x\,\mathrm{d}y,$$

$$\mathrm{d}f_*\tau_{(\widetilde{S},D_3),v} = \frac{1}{|x|\max\{|y|,1,|x|,M(x,y)\}}\,\mathrm{d}x\,\mathrm{d}y,$$

$$\mathrm{d}f_*\tau_{(\widetilde{S},D_4),v} = \frac{1}{|x|\max\{|y|,1,|x|\}}\,\mathrm{d}x\,\mathrm{d}y, \qquad \textit{and}$$

$$\mathrm{d}f_*\tau_{(\widetilde{S},D_5),v} = \frac{1}{\max\{|xy|,1,|x|\}}\,\mathrm{d}x\,\mathrm{d}y,$$

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where

$$M(x,y) = \min\left\{\frac{|y(x+y)|}{|x^3|}, \frac{|x+y|}{|x|}, |y(x+y)|, \left|x^{-1}\right|\right\},$$

and all absolute values are $|\cdot| = |\cdot|_v$.

Proof. In the first case, we have

$$\mathrm{d}f_*\tau_{(\widetilde{S},D_1),v} = \|(\mathrm{d}x \wedge \mathrm{d}y) \otimes 1_{E_7}\|^{-1}_{\omega_{\widetilde{S}}(D),v} \mathrm{d}x\,\mathrm{d}y. \tag{29}$$

To make sense of this, we need a metric on the log-canonical bundle, not just on a line bundle isomorphic to it. To this end, we consider the isomorphism between the canonical bundle $\omega_{\widetilde{S}}$ and the line bundle whose meromorphic sections are elements of degree $\omega_{\widetilde{S}}$ of the field of fractions of the Cox ring that maps $\mathrm{d}x \wedge \mathrm{d}y$ to $1/\eta_1^2\eta_2^2\eta_3^3\eta_4^2\eta_6$; in addition, we consider the isomorphisms between $\mathcal{O}(E_i)$ and the line bundles whose sections are elements of the Cox ring mapping 1_{E_i} to η_i. Together, these induce an isomorphism from each $\omega_{\widetilde{S}}(D_1)$ to the line bundle whose sections are functions of the Cox ring of degree $\omega_{\widetilde{S}}(D_1)$, and we can pull back the adelic metric we constructed along this isomorphism (and similarly for the log-canonical bundles in the remaining cases). In Cox coordinates, the norm in equation (29) at a point $\boldsymbol{\eta}$ is

$$\frac{|\eta_1^2\eta_2^2\eta_3^3\eta_4^2\eta_6|}{|\eta_7|\max\{|\eta_2\eta_3\eta_4\eta_5\eta_6\eta_8|, |\eta_1\eta_2\eta_3^2\eta_4^2\eta_5^2\eta_6^2|, |\eta_3\eta_4^2\eta_5^4\eta_6^3\eta_7|, |\eta_8\eta_9|\}}. \tag{30}$$

In the second case, we can analogously determine the norm

$$\|(\mathrm{d}x \wedge \mathrm{d}y) \otimes 1_{E_3} \otimes 1_{E_4} \otimes 1_{E_6}\|^{-1}_{\omega_{\widetilde{S}}(D_2),v}$$

in Cox coordinates:

$$\frac{1}{|\eta_3\eta_4\eta_6|}\frac{|\eta_1^2\eta_2^2\eta_3^3\eta_4^2\eta_6|}{\max\{|\eta_2\eta_5\eta_7\eta_8|, |\eta_1^2\eta_2^2\eta_3^2\eta_4|, |\eta_4\eta_5^4\eta_6^2\eta_7^2|\}}. \tag{31}$$

In the third case, the norm

$$\|(\mathrm{d}x \wedge \mathrm{d}y) \otimes 1_{E_3} \otimes 1_{E_4} \otimes 1_{E_6} \otimes 1_{E_7}\|^{-1}_{\omega_{\widetilde{S}}(D_3),v}$$

at a point $\boldsymbol{\eta}$ in Cox coordinates is

$$\frac{1}{|\eta_3\eta_4\eta_6\eta_7|}\frac{|\eta_1^2\eta_2^2\eta_3^3\eta_4^2\eta_6|}{\max\{|\eta_2\eta_5\eta_8|, |\eta_1\eta_2\eta_3\eta_4\eta_5^2\eta_6|, |\eta_4\eta_5^4\eta_6^2\eta_7|, M_0(\boldsymbol{\eta})\}} \tag{32}$$

with

$$M_0(\boldsymbol{\eta}) = \frac{\left|\eta_1^3\eta_2^3\eta_3^2\eta_4\eta_8\eta_9\right|}{\max_{s\in B}\{|s(\boldsymbol{\eta})|\}}.$$

Then $M_0(g(x,y)) = M(x,y)$ as above after removing terms that can never contribute to the minimum.

In the remaining two cases, the norms of interest are

$$\frac{1}{|\eta_3\eta_4\eta_5\eta_6\eta_7|}\frac{|\eta_1^2\eta_2^2\eta_3^3\eta_4^2\eta_6|}{\max\{\eta_2\eta_8,\eta_1\eta_2\eta_3\eta_4\eta_5\eta_6,\eta_4\eta_5^3\eta_6^2\eta_7\}} \quad \text{and}$$
$$\frac{1}{|\eta_2\eta_3\eta_4\eta_6|}\frac{|\eta_1^2\eta_2^2\eta_3^3\eta_4^2\eta_6|}{\max\{|\eta_5\eta_7\eta_8|,|\eta_1^2\eta_2\eta_3^2\eta_4|,|\eta_1\eta_3\eta_4\eta_5^2\eta_6\eta_7|\}},$$

respectively. □

Lemma 24. *Let p be a finite prime. Then*

$$\tau_{(\widetilde{S},D_i),p}(\widetilde{\mathcal{U}}_i(\mathbb{Z}_p)) = \begin{cases} 1+\frac{6-\#D_i}{p}, & i=1,2,4,5; \\ 1+\frac{2}{p}-\frac{1}{p^2}, & i=3. \end{cases}$$

Proof. We compute

$$\tau_{(\widetilde{S},D_i),p}(\widetilde{\mathcal{U}}_i(\mathbb{Z}_p)) = \int_{f(\widetilde{\mathcal{U}}_i(\mathbb{Z}_p)\cap V'(\mathbb{Q}_p))} \mathrm{d}f_*\tau_{(\widetilde{S},D_i),p} \tag{33}$$

for $i\in\{1,\dots,5\}$. For $i=1$, the previous two lemmas transform this into

$$\int_{\substack{x,y\in\mathbb{Q}_p\\ |x|\geq 1 \text{ or } |xy^2|\geq 1}} \frac{1}{|x|\max\{|y|,1,|x|,|y(y+x)|\}}\,\mathrm{d}x\,\mathrm{d}y.$$

Subdividing the domain of integration into the regions with $|x|>|y|$, $|x|=|y|$, and $|x|<|y|$ in order to simplify the denominator, we get

$$\begin{aligned} &\int_{\substack{|y|<|x|\\ |x|\geq 1}} \frac{1}{|x|\max\{|x|,|xy|\}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\substack{|y|=|x|\\ |x|\geq 1}} \frac{1}{|x|\max\{|x|,|y(y+x)|\}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{\substack{|x|<|y|\\ |xy^2|\geq 1}} \frac{1}{|xy^2|}\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{34}$$

after simplifying the description of the domains ($|x|<1$ would imply $|xy^2|\leq|x|^3<1$ in the first two cases; $|y|^2<1/|x|$ would imply $|y|^2<1/|x|\leq 1\leq|x|^2<|y|^2$ in the third case).

The first of the integrals in equation (34) is

$$\begin{aligned} \int_{|x|\geq 1}\frac{1}{|x|^2}\int_{|y|<|x|}\frac{1}{\max\{1,|y|\}}\,\mathrm{d}y\,\mathrm{d}x &= \int_{|x|\geq 1}\frac{1}{|x|^2}\left(\frac{1}{p}+\int_{1\leq|y|<|x|}\frac{1}{|y|}\,\mathrm{d}y\right)\mathrm{d}x \\ = \int_{|x|\geq 1}\frac{1}{|x|^2}\left(\frac{1}{p}+\left(1-\frac{1}{p}\right)|v(x)|\right)\mathrm{d}x &= \frac{1}{p}+\sum_{\delta\geq 0}\left(1-\frac{1}{p}\right)^2\frac{\delta}{p^\delta} = \frac{2}{p}, \end{aligned} \tag{35}$$

while the second integral is

$$\int_{\substack{|y+x|\leq\frac{1}{p}\\ |x|\geq 1}}\frac{1}{|x|^2} + \int_{\substack{|y+x|\geq 1\\ |x|\geq 1, |y|=|x|}}\frac{1}{|xy(x+y)|}. \tag{36}$$

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The first integral in equation (36) is $\frac{1}{p}\int_{|x|\geq 1}\frac{1}{|x|^2}\,\mathrm{d}x=\frac{1}{p}$. Turning to the second one, we note that $|x|=|y|$ is implied by the ultrametric triangle inequality if $|x+y|<|x|$. The set of $y\in\mathbb{Q}_p$ with $|x+y|=|x|$ and $|y|=|x|$ has volume $|x|-2|x|/p$ since the two sets $\{y\mid |y-0|<|x|\}$ and $\{y\mid |y+x|<|x|\}$ have volume $|x|/p$ and are disjoint (because $|y|<|x|$ implies $|y+x|=|x|$). We thus get

$$\begin{aligned}&\int_{|x|\geq 1}\frac{1}{|x|^2}\left(\sum_{0\leq\delta<|v(x)|}\left(1-\frac{1}{p}\right)\frac{p^\delta}{p^\delta}+\left(1-\frac{2}{p}\right)\frac{|x|}{|x|}\right)\mathrm{d}x\\&\quad=\int_{|x|\geq 1}\frac{1}{|x|^2}\left(\left(1-\frac{1}{p}\right)|v(x)|+\left(1-\frac{2}{p}\right)\right)\mathrm{d}x=\frac{1}{p}+1-\frac{2}{p}=1-\frac{1}{p},\end{aligned}$$

computing the integral over x similarly as in equation (35). The second integral in equation (34) thus evaluates to 1. Finally, the third integral in equation (34) is

$$\begin{aligned}&\int\frac{1}{|y|^2}\int_{1/|y|^2\leq|x|<|y|}\frac{1}{|x|}\,\mathrm{d}x\,\mathrm{d}y=\int_{|y|\geq 1}\frac{1}{|y|^2}\sum_{-2|v(y)|\leq\delta<|v(y)|}\left(1-\frac{1}{p}\right)\mathrm{d}y\\&\quad=\int_{|y|\geq 1}\left(1-\frac{1}{p}\right)\frac{3|v(y)|}{|y|^2}=\frac{3}{p},\end{aligned}$$

again computed analogously to the previous ones. Adding the three terms in equation (34), we arrive at our claim for $i=1$.

For $i=2$, we get

$$\begin{aligned}&\int_{|y|\leq 1,\ |xy^2|\leq 1,\ \text{or}\ |x+y|\leq 1}\frac{1}{|x|\max\{|y|,|x^{-1}|,|x|\}}\,\mathrm{d}x\,\mathrm{d}y\\&\quad=\int_{|y|\leq 1}\frac{1}{\max\{1,|x^2|\}}\,\mathrm{d}x\,\mathrm{d}y+\int_{\substack{|y|>1\\|x+y|\leq 1}}\frac{1}{|y^2|}\,\mathrm{d}x\,\mathrm{d}y+\int_{\substack{|y|>1\\|x|\leq 1/|y|^2}}\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for the integral (33) (since $|x|=|y|$ in the second case). The first integral is then

$$1+\int_{|x|>1}\frac{1}{|x^2|}\,\mathrm{d}x=1+\frac{1}{p},$$

while the other two integrals are

$$\int_{|y|>1}\frac{1}{|y^2|}\,\mathrm{d}y=\frac{1}{p}.$$

For $i=3$, we compute the integral on the right hand side of equation (33) to be

$$\begin{aligned}&\int_{f(\widetilde{\mathcal{U}}_3(\mathbb{Z}_p)\cap V'(\mathbb{Q}_p))}\frac{1}{|x|\max\{|y|,1,|x|,M(g(x,y))\}}\,\mathrm{d}x\,\mathrm{d}y\\&\quad=\int_{|y|\leq 1,\ |x|\geq 1}\frac{1}{|x^2|}\,\mathrm{d}x\,\mathrm{d}y+\int_{\substack{|y|>1\\|x+y|\leq 1}}\frac{1}{|y^2|}\,\mathrm{d}x\,\mathrm{d}y+\int_{\substack{|y|>1\\|x|=1/|y|^2}}\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

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(again using that $|x| = |y|$ in the second case). The first integral is then

$$\int_{|x|\geq 1} \frac{1}{|x^2|}\,\mathrm{d}x = 1,$$

while the second one is

$$\int_{|y|>1} \frac{1}{|y^2|}\,\mathrm{d}y = \frac{1}{p},$$

and the third one is

$$\left(1-\frac{1}{p}\right)\int_{|y|>1} \frac{1}{|y^2|}\,\mathrm{d}y = \frac{1}{p} - \frac{1}{p^2}.$$

The final two cases are similar: for $i = 4$, the integrals over the two disjoint sets in Lemma 22 are 1 and p^{-1}, respectively, while, for $i = 5$, the integrals over the three disjoint sets are 1, p^{-1} and p^{-1}, respectively. $\square$

The remaining parts of the constant are associated with maximal faces of the Clemens complex. Recall from Section 1.2 and Figure 1 that the Clemens complex of the geometrically irreducible divisor D_1 consists of just one vertex E_7. For D_2, we have three vertices corresponding to its components, and two 1-simplices $A_1 = \{E_3, E_4\}$ and $A_2 = \{E_4, E_6\}$ between the intersecting exceptional curves (Figures 1 and 4). The Clemens complex for $D_3 = D_1 + D_2$ is the disjoint union of the previous two cases; its maximal-dimensional faces are again A_1 and A_2. For D_4, they are $A_1, \ldots, A_4$, and for D_5, they are A_1, A_2, A_5 (Figure 4).

For a face A of the Clemens complexes associated with D_i, we set $D_A = \bigcap_{E\in A} E$ and $\Delta_{i,A} = D_i - \sum_{E\in A} E$. For a maximal-dimensional face A of a Clemens complex, the adjunction isomorphism and a metric on the log-canonical bundle $\omega_{\widetilde{S}}(D_i)$ induce a metric on the bundle $\omega_{D_A} \otimes \mathcal{O}_{\widetilde{S}}(\Delta_{i,A})|_{D_A}$ on D_A. Since A is maximal, the canonical section $1_{\Delta_{i,A}}$ does not have a pole on D_A, so since $D_A(\mathbb{R})$ is compact, the norm $\left\|1_{\Delta_{i,A}}|_{D_A}\right\|_{\mathcal{O}_{\widetilde{S}}(\Delta_{i,A})|_{D_A},\infty}$ is bounded on $D_A(\mathbb{R})$ for any metric. Hence,

$$\left\|\omega \otimes 1_{\Delta_{i,A}}|_{\mathcal{O}_{\widetilde{S}}}\right\|^{-1}_{\omega_{D_A}\otimes\mathcal{O}_{\widetilde{S}}(\Delta_{i,A})|_{D_A},\infty} |\omega| = \left\|1_{\Delta_{i,A}}|_{D_A}\right\|_{\mathcal{O}_{\widetilde{S}}(\Delta_{i,A})|_{D_A},\infty} \tau_{D_A,\infty}$$

(where the equality holds for any choice of metrics on ω_{D_A} and $\mathcal{O}_{\widetilde{S}}(\Delta_{i,A})|_{D_A}$ compatible with the one on their tensor product) defines a finite measure on $D_A(\mathbb{R})$, independent of the choice of a form $\omega \in \omega_{D_A}$. We further normalize this measure with a factor $c_{\mathbb{R}}^{\#A} = 2^{\#A}$, call it *residue measure* and denote it by $\tau_{i,D_A,\infty}$. See [7, §§ 2.1.12, 4.1] for details on this construction.

Lemma 25. *We have*

$$\tau_{1,E_7,\infty}(E_7(\mathbb{R})) = 8 \quad \textit{and} \quad \tau_{i,D_A,\infty}(D_A(\mathbb{R})) = 4$$

for $i \in \{2, \ldots, 5\}$ and every maximal-dimensional face A of the Clemens complex for D_i.

Proof. Following [7, 2.1.12], we can compute the unnormalized Tamagawa volume of E_7 by integrating

$$\|\mathrm{d}y\|^{-1}_{\omega_{E_7},\infty} = \lim_{x\to 0}\left(|x|\|(\mathrm{d}x\wedge\mathrm{d}y)\otimes 1_{E_7}\|^{-1}_{\omega_{\widetilde{S}}(E_7),\infty}\right).$$

Again evaluating equation (30) in the image of (x,y), we get the volume

$$\begin{aligned}\tau'_{1,E_7,\infty}(E_7(\mathbb{R})) &= \int_{\mathbb{R}} \lim_{x\to 0}\frac{|x|}{|x|\max\{1,|x|,|y|,|y(y+x)|\}}\,\mathrm{d}y\\ &= \int_{\mathbb{R}}\frac{1}{\max\{1,|y^2|\}}\,\mathrm{d}y = 4,\end{aligned}$$

which we normalize by multiplying with $c_{\mathbb{R}} = 2$.

For the second and third cases, we work in neighborhoods of the two intersection points $D_{A_1} = E_3\cap E_4$ and $D_{A_2} = E_4\cap E_6$. The Tamagawa measures on these points are simply real numbers. In order to compute them, we consider the charts

$$\begin{aligned} g' &: \mathbb{A}^2_{\mathbb{Q}} \to \widetilde{S},\ (a,b)\mapsto (1:1:a:b:1:1:1:1:-1-b) \quad \text{and}\\ g'' &: \mathbb{A}^2_{\mathbb{Q}} \to \widetilde{S},\ (c,d)\mapsto (1:1:1:c:1:d:1:1:-1-cd).\end{aligned}$$

We have $x = 1/a = d$, $y = 1/ab = 1/cd$ for these charts. Since

$$\|\mathrm{d}x\wedge\mathrm{d}y\| = |\det(J_{f\circ g'})|\|\mathrm{d}a\wedge\mathrm{d}b\|,$$

we can use equation (31) to compute the norms

$$\begin{aligned}\|(\mathrm{d}a\wedge\mathrm{d}b)\otimes 1_{E_3}\otimes 1_{E_4}\otimes 1_{E_6}\|_{\omega_{\widetilde{S}}(D_2),\infty} &= \max\{|a^3b^2|,|ab|,|ab^2|\} \quad \text{and}\\ \|(\mathrm{d}c\wedge\mathrm{d}d)\otimes 1_{E_3}\otimes 1_{E_4}\otimes 1_{E_6}\|_{\omega_{\widetilde{S}}(D_2),\infty} &= \max\{|c^2d|,|cd|,|c^2d^3|\}.\end{aligned}$$

Analogously to the first case, we now arrive at

$$\tau'_{2,D_{A_1},\infty} = \lim_{(a,b)\to(0,0)}\frac{|ab|}{\max\{|a^3b^2|,|ab|,|ab^2|\}} = 1,$$

and, similarly, $\tau'_{2,D_{A_2},\infty} = 1$ for the unnormalized measures on the points $D_{A_i}(\mathbb{R})$, which we multiply with $c^2_{\mathbb{R}} = 4$.

In the third case, using the same change of variables and the description (32) of the metric in Cox coordinates, we get

$$\|(\mathrm{d}a\wedge\mathrm{d}b)\otimes 1_{E_3}\otimes 1_{E_4}\otimes 1_{E_6}\otimes 1_{E_7}\|_{\omega_{\widetilde{S}}(D_3),\infty} = |ab|\max\{1,|ab|,|b|,M_0(g'(a,b))\}$$

with $M_0(g'(ab))\to 0$, as $(a,b)\to(0,0)$, whence $\tau'_{3,D_{A_1},\infty} = 1$ for the unnormalized measure. Finally,

$$\|(\mathrm{d}c\wedge\mathrm{d}d)\otimes 1_{E_3}\otimes 1_{E_4}\otimes 1_{E_6}\otimes 1_{E_7}\|_{\omega_{\widetilde{S}}(D_3),\infty} = |cd|\max\{1,|cd|,\left|cd^2\right|,M_0(g''(c,d))\},$$

where again $M_0(g''(c,d))\to 0$, and we get $\tau'_{3,D_{A_2},\infty} = 1$ for the unnormalized measure. Again, we multiply both measures with $c^2_{\mathbb{R}} = 4$.

The computations in the cases $i = 4,5$ are analogous. □

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These Tamagawa numbers are multiplied with rational numbers $\alpha_{i,A}$, where A is a maximal-dimensional face of the Clemens complex for D_i, depending on the geometry of certain effective cones, as in equation (11). Following [25, § 2.2],

$$\widetilde{U}_{i,A} = X \setminus \bigcup_{\substack{E_j \subset D_i,\\ E_j \not\in A}} E_j \tag{37}$$

is the complement of all boundary components not belonging to A, and $\operatorname{Eff}\widetilde{U}_{i,A} \subset (\operatorname{Pic}\widetilde{U}_{i,A})_{\mathbb{R}}$ is its effective cone; all volume functions are normalized as in [25, Remark 2.2.9 (iv)].

Lemma 26. *We have*

$$\begin{aligned}
\alpha_{1,E_7} &= \operatorname{vol}\left\{(t_2,\dots,t_6) \in \mathbb{R}^5_{\geq 0} \;\middle|\; \begin{aligned} &t_2 + 2t_3 + 2t_4 + 2t_5 + 2t_6 \leq 1 \\ &t_3 + 2t_4 + 4t_5 + 3t_6 \leq 1 \end{aligned}\right\} = \frac{13}{34560},\\
\alpha_{2,A_1} &= \operatorname{vol}\{(t_1,t_2,t_5,t_7) \in \mathbb{R}^4_{\geq 0} \mid t_1 + t_2 \leq 2t_5 + t_7,\ 4t_5 + 2t_7 \leq 1\} = 1/256,\\
\alpha_{2,A_2} &= \operatorname{vol}\{(t_1,t_2,t_5,t_7) \in \mathbb{R}^4_{\geq 0} \mid t_1 + t_2 \geq 2t_5 + t_7,\ 2t_1 + 2t_2 \leq 1\} = 1/256,\\
\alpha_{3,A_1} &= \operatorname{vol}\{(t_1,t_2,t_5) \in \mathbb{R}^3_{\geq 0} \mid t_1 + t_2 \leq 2t_5,\ 4t_5 \leq 1\} = 1/96,\\
\alpha_{3,A_2} &= \operatorname{vol}\{(t_1,t_2,t_5) \in \mathbb{R}^3_{\geq 0} \mid t_1 + t_2 \geq 2t_5,\ 2t_1 + 2t_2 \leq 1\} = 1/48,\\
\alpha_{4,A_1} &= 0,\\
\alpha_{4,A_2} &= \operatorname{vol}\{(t_1,t_2) \in \mathbb{R}^2_{\geq 0} \mid t_1 + t_2 \leq 1/2\} = 1/8,\\
\alpha_{4,A_3} &= \operatorname{vol}\{(t_1,t_2) \in \mathbb{R}^2_{\geq 0} \mid 1/2 \leq t_1 + t_2 \leq 2/3\} = 7/72,\\
\alpha_{4,A_4} &= \operatorname{vol}\{(t_1,t_2) \in \mathbb{R}^2_{\geq 0} \mid 2/3 \leq t_1 + t_2 \leq 1\} = 5/18,\\
\alpha_{5,A_1} &= \operatorname{vol}\{(t_1,t_5,t_7) \in \mathbb{R}^3_{\geq 0} \mid t_1 \leq 2t_5 + t_7,\ 4t_5 + 2t_7 \leq 1\} = 1/48,\\
\alpha_{5,A_2} &= \operatorname{vol}\{(t_1,t_5,t_7) \in \mathbb{R}^3_{\geq 0} \mid t_1 \geq 2t_5 + t_7,\ 2t_1 \leq 1\} = 1/96, \quad \textit{and}\\
\alpha_{5,A_5} &= \operatorname{vol}\{(t_1,t_5,t_7) \in \mathbb{R}^3_{\geq 0} \mid t_1 + 2t_5 + t_7 \leq 1,\ 4t_5 + 2t_7 \geq 1\} = 1/24.
\end{aligned}$$

Proof. To compute $\alpha_{i,A}$, we choose $j_0 \in \{1,\dots,7\}$ such that $E_{j_0} \in A$ and such that the classes of E_j for $j \in \{1,\dots,7\} \setminus \{j_0\}$ form a basis of $\operatorname{Pic}\widetilde{S}$. The latter holds for $j_0 \in \{1,2,3,6,7\}$ since the data in [12] show that $\operatorname{Pic}\widetilde{S}$ has rank 6 and is generated by the classes of the negative curves $E_1,\dots,E_7$, where

$$E_1 + E_2 + E_3 - 2E_5 - E_6 - E_7 \tag{38}$$

is a principal divisor. An inspection of Figure 4 shows that $E_{j_0} \in A$ for some $j_0 \in \{3,6,7\}$.

Hence, there are unique linear combinations $\sum_{j \neq j_0} a_j E_j$ of class $\omega_{\widetilde{S}}(D_i)^\vee$ and $\sum_{j \neq j_0} b_j E_j$ of the same class as E_{j_0}; the coefficients $a_j, b_j \in \mathbb{Z}$ can be computed using equation (38) and the fact that $2E_1 + 2E_2 + 3E_3 + 2E_4 + E_6$ has anticanonical class by equation (17). For the following computations, it is useful to know that

$$2E_4 + 4E_5 + 3E_6 + 2E_7, \quad 2E_1 + 2E_2 + 3E_3 + 2E_4, \quad E_1 + E_2 + 2E_3 + 2E_4 + 2E_5 + 2E_6 \tag{39}$$

have class $\omega_{\widetilde{S}}(E_{j_0})^\vee$ for $j_0 = 3,6,7$, respectively (expressed without using E_{j_0}).

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Let $J = \{j \in \{1,\dots,7\} \mid E_j \subset D_i,\ E_j \notin A\}$ and $J' = \{1,\dots,7\} \setminus (J \cup \{j_0\})$. By definition (37),

$$\operatorname{Pic} \widetilde{U}_{i,A} = (\operatorname{Pic} \widetilde{S})/\langle E_j \mid j \in J\rangle;$$

hence, a basis is given by the classes of E_j for $j \in J'$ modulo the classes of E_j for $j \in J$, and its effective cone is generated by the classes of E_j for $j \in J' \cup \{j_0\}$ modulo the classes of E_j for $j \in J$. Working with the dual basis, we obtain

$$\alpha_{i,A} = \operatorname{vol}\left\{(t_j) \in \mathbb{R}^{J'}_{\geq 0} \mid \sum_{j \in J'} a_j t_j = 1,\ \sum_{j \in J'} b_j t_j \geq 0\right\}.$$

If $A = \{E_{j_0}, E_{j_1}\}$ is a 1-simplex, then $j_1 \in J$, and the next step is to eliminate the variable t_{j_1} using the equation, which gives a description of $\alpha_{i,A}$ as the volume of a polytope in $\mathbb{R}^{J_i}_{\geq 0}$ with J_i as in equation (22) defined by two inequalities.

In the first case, we have $\widetilde{U}_{1,E_7} = \widetilde{S}$ and corresponding effective cone $\operatorname{Eff} \widetilde{S}$, whose dual is the nef cone of $\widetilde{S}$. Working with the dual basis of the classes of $E_1,\dots,E_6$ and using descriptions (38) and (39) for E_7 and $\omega_{\widetilde{S}}(D_1)^\vee$, we obtain

$$\alpha_1 = \operatorname{vol}\left\{(t_1,\dots,t_6) \in \mathbb{R}^6_{\geq 0} \;\middle|\; \begin{array}{l} t_1 + t_2 + t_3 - 2t_5 - t_6 \geq 0 \\ t_1 + t_2 + 2t_3 + 2t_4 + 2t_5 + 2t_6 = 1 \end{array}\right\}$$

and eliminate t_1.

In the second case, there are two constants α_{2,A_i} associated with the maximal faces $A_1 = \{E_3, E_4\}$ and $A_2 = \{E_4, E_6\}$ of the Clemens complex. The subvarieties used in their definition are $\widetilde{U}_{2,A_1} = \widetilde{S} \setminus E_6$ and $\widetilde{U}_{2,A_2} = \widetilde{S} \setminus E_3$. In the first case, we have $J = \{6\}$, choose $j_0 = 3$ and obtain $J' = \{1,2,4,5,7\}$. Therefore, the Picard group of $\widetilde{U}_{A_1}$ is $(\operatorname{Pic} \widetilde{S})/\langle E_6\rangle$ with a basis is given by the classes of E_1, E_2, E_4, E_5, E_7 modulo E_6, and its effective cone is generated by the classes of $E_1,\dots,E_5,E_7$ modulo E_6. Since E_3 has the same class as $-E_1 - E_2 + 2E_5 + E_6 + E_7$ in $\operatorname{Pic} \widetilde{S}$ by equation (38), while $E_4 + 4E_5 + 2E_6 + 2E_7$ has class $\omega_{\widetilde{S}}(D_2)^\vee$ by equation (39), we obtain (working modulo E_6)

$$\alpha_{2,A_1} = \operatorname{vol}\left\{(t_1,t_2,t_4,t_5,t_7) \in \mathbb{R}^5_{\geq 0} \;\middle|\; \begin{array}{l} -t_1 - t_2 + 2t_5 + t_7 \geq 0 \\ t_4 + 4t_5 + 2t_7 = 1 \end{array}\right\}$$

and eliminate t_4.

The computation of α_{2,A_2} is similar. Here, we choose $j_0 = 6$, and our basis is given by the classes of E_1, E_2, E_4, E_5, E_7 modulo E_3. The divisor E_6 has the same class as $E_1 + E_2 + E_3 - 2E_5 - E_7$, while $2E_1 + 2E_2 + 2E_3 + E_4$ has class $\omega_{\widetilde{S}}(D_2)^\vee$. Therefore,

$$\alpha_{2,A_2} = \operatorname{vol}\left\{(t_1,t_2,t_4,t_5,t_7) \in \mathbb{R}^5_{\geq 0} \;\middle|\; \begin{array}{l} t_1 + t_2 - 2t_5 - t_7 \geq 0 \\ 2t_1 + 2t_2 + t_4 = 1 \end{array}\right\};$$

again, we eliminate t_4.

The further cases are analogous. The only exceptional case is the computation of α_{4,A_1}. Working with $J = \{5,6,7\}$, $j_0 = 3$ and $J' = \{1,2,4\}$, a similar computation as for α_{2,A_1}

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shows

$$\alpha_{4,A_1} = \mathrm{vol}\left\{ (t_1,t_2,t_4) \in \mathbb{R}^5_{\geq 0} \;\middle|\; \begin{array}{l} -t_1 - t_2 \geq 0, \\ t_4 = 1 \end{array} \right\},$$

which clearly has volume 0 in the hyperplane $t_4 = 1$. □

Remark 27. This last phenomenon $\alpha_{4,A_1} = 0$ is an instance of the obstruction described in [25, Theorem 2.4.1 (i)]: The regular function

$$s = \frac{\eta_1\eta_2\eta_3}{\eta_5\eta_6\eta_7}$$

on $\widetilde{\mathcal{U}}_4$ is also regular on $\widetilde{U}_{4,A_1}$. On the one hand, this regular function induces the relation $[E_1]+[E_2] = 0$ in $\mathrm{Pic}(\widetilde{U}_{i,A_4})$, while both classes on the left are nonzero; this makes the pseudo-effective cone fail to be strictly convex, and the resulting polytope has volume 0. On the other hand, s vanishes on $D_{4,A_1} = \{\eta_3 = \eta_4 = 0\}$ so that $|s| < 1$ on a sufficiently small real-analytic neighborhood W of D_{4,A_1}; but W is integral on $\widetilde{\mathcal{U}}_4(\mathbb{Z})$ and nonzero on $V = \widetilde{S} \setminus (E_1 + \cdots + E_7)$, so $|s| \geq 1$ on the set $V(\mathbb{Q}) \cap \widetilde{\mathcal{U}}_4(\mathbb{Z})$ counted by N_4. It follows that any sufficiently small analytic neighborhood of $D_{4,A_1}(\mathbb{R})$ cannot contribute to N_4, which is reflected by the vanishing of the corresponding part of the expected leading constant.

Lemma 28. *For $i \in \{1,\ldots,5\}$, the Archimedean contributions to the expected constants are*

$$c_{i,\infty} = \sum_A \alpha_{i,A}\tau_{i,D_A,\infty}(D_A(\mathbb{R})) = C_i,$$

where the sum runs through the maximal faces A of the Clemens complex, with C_i as in Lemma 21.

Proof. This follows from Lemma 25 and Lemma 26. For $i = 2,\ldots,6$, we observe that the polytopes of volumes $\alpha_{i,A}$ in Lemma 26 fit together to the one appearing in the description of C_i in Lemma 21. □

We conclude by noting that the classes of E_3, E_4, E_6, E_7 in $\mathrm{Pic}\,\widetilde{S}$ are linearly independent; hence, $\mathrm{rk}\,\mathrm{Pic}\,\widetilde{U}_i = \mathrm{rk}\,\mathrm{Pic}\,\widetilde{S} - \#D_i$ (with $\#D_i$ as in Lemma 15). This observation, Lemma 24 and Lemma 28 allow us to reformulate Theorem 1 as Theorem 2 for $i \in \{1,\ldots,5\}$. For the final case, we equip the log-anticanonical bundle $\omega_{\widetilde{S}}(D_6)^\vee$ with the metric pulled back from $\omega_{\widetilde{S}}(D_5)^\vee$ along the isomorphism 8; since all constructions in this section are invariant under metric-preserving isomorphisms, the theorem follows for $i = 6$.

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