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ON THE SOLVABILITY OF A NEUMANN BOUNDARY VALUE PROBLEM AT RESONANCE

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ABSTRACT. We study the existence of solutions of the semilinear equations (1) $\triangle u + g(x,u) = h$, $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ in which the non-linearity g may grow superlinearly in u in one of directions $u \longrightarrow \infty$ and $u \longrightarrow -\infty$, and (2) $-\triangle u + g(x,u) = h$, $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ in which the nonlinear term g may grow superlinearly in u as $|u| \longrightarrow \infty$. The purpose of this paper is to obtain solvability theorems for (1) and (2) when the Landesman-Lazer condition does not hold. More precisely, we require that h may satisfy $\int g_-^\delta(x)\,dx < \int h(x)\,dx = 0 < \int g_+^\gamma(x)\,dx$, where γ, δ are arbitrarily nonnegative constants, $g_+^\gamma(x) = \lim_{u\to\infty} \inf g(x,u)|u|^\gamma$ and $g_-^\delta(x) = \lim_{u\to-\infty} \sup g(x,u)|u|^\delta$. The proofs are based upon degree theoretic arguments.

1. **Introduction.** Let $\Omega \subset \mathbf{R}^N (N \geq 2)$ be a smooth bounded domain. In this paper we consider the Neumann problems

$$\triangle u + g(x,u) = h \text{ in } \Omega, \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \tag{1.1}$$

and

$$-\triangle u + g(x,u) = h \text{ in } \Omega, \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1.2}$$

where $\triangle$ denotes the Laplacian on $\mathbf{R}^N$, $h \in L^p(\Omega)$ ($p > N/2$) is given, $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$ and $g: \Omega \times \mathbf{R} \longrightarrow \mathbf{R}$ is a Caratheodory function, that is, $g(x,u)$ is continuous in $u \in \mathbf{R}$ for almost all $x \in \Omega$, is measurable in $x \in \Omega$ for all $u \in \mathbf{R}$ and satisfies for each $r > 0$ there exists $a_r \in L^p(\Omega)$ such that $|g(x,u)| \leq a_r(x)$ for a.e. $x \in \Omega$ and all $|u| \leq r$. The solvability of the problem (1.1) has been extensively studied when the nonlinearity g is assumed to have linear growth in u as $|u| \longrightarrow \infty$ (see [2, 3, 5, 8, 11]). When g is allowed to grow superlinearly in u in one of the directions $u \longrightarrow \infty$ and $u \longrightarrow -\infty$, and may grow sublinearly in the other, existence theorems for a solution to (1.1) were proved in [6, 7] if either

($\mathbf{F}_1$) for a.e. $x \in \Omega$ and all $|u| \geq r_0 \geq 0$, $g(x,u)u \geq 0$ and $\int_\Omega h(x)\,dx = 0$;

or

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$$(\mathbf{F}_2) \qquad \begin{gathered} g_-^0(x) = \lim_{u\to-\infty} \sup g(x,u),\, g_+^0(x) = \lim_{u\to\infty} \inf g(x,u) \\ \text{and } \int g_-^0(x)\,dx < \int h(x)\,dx < \int g_+^0(x)\,dx \end{gathered}$$

holds. The solvability of (1.2) has been extensively studied when g has no growth restriction in u as $|u| \longrightarrow \infty$ and $(\mathbf{F}_2)$ is satisfied (see [4, 10]). The purpose of this paper is to consider the problem (1.1), and to extend the main result of Kuo [7] when either $(\mathbf{F}_1)$ or $(\mathbf{F}_2)$ is not satisfied, and improve the main result of Robinson and Landesman [11] where it assumes that g has at most linear growth and satisfies the following condition $(\mathbf{G})$ with $\delta = \gamma = 1$ and $e = \tilde{e} = 0$ in $L^1(\Omega)$:

$$(\mathbf{G}) \qquad \begin{gathered} \text{There exist constants } k_0, \gamma, \delta \geq 0 \text{ and } e, \tilde{e} \in L^1(\Omega) \\ \text{such that for a.e. } x \in \Omega \text{ and } u \geq k_0 \end{gathered}$$

$$g(x,u)u \geq e(x)|u|^{1-\gamma}, \tag{1.3}$$

and for a.e. $x \in \Omega$ and all $u \leq -k_0$

$$g(x,u)u \geq \tilde{e}(x)|u|^{1-\delta}. \tag{1.4}$$

Moreover, we obtain some new existence theorems of (1.2) in which the nonlinearity $g(x,u) \in O(|u|^{p/p-1})$ as $|u| \longrightarrow \infty$ and $(\mathbf{F}_2)$ is not satisfied, and hence cannot be obtained from Hirano [4], and McKenna and Rauch [10] in which g can have arbitrary growth in u as $|u| \longrightarrow \infty$ and satisfies $(\mathbf{F}_2)$. Concerning the growth condition of the nonlinear term g, we assume that:

$$(\mathbf{H}) \qquad \begin{gathered} \text{There exist constants } a, \tilde{k}_0, \tilde{\tilde{k}}_0, \alpha, \beta \geq 0 \text{ and } b, c, d \in L^p(\Omega) b \geq 0 \text{ in } \Omega \\ \text{such that for a.e. } x \in \Omega, u \geq \tilde{k}_0. \end{gathered}$$

$$c(x) \leq g(x,u) \leq a|u|^{\alpha} + b(x), \tag{1.5}$$

and for a.e. $x \in \Omega$, $u \leq -\tilde{\tilde{k}}_0$

$$-a|u|^{\beta} - b(x) \leq g(x,u) \leq d(x); \tag{1.6}$$

and h may satisfy

$$(\mathbf{F}_3) \qquad \int_\Omega g_-^{\delta}(x)\,dx < \int_\Omega h(x)\,dx = 0 < \int_\Omega g_+^{\gamma}\,dx,$$

where $g_-^{\gamma}(x) = \lim_{u\to-\infty} \sup g(x,u)|u|^{\delta}$ and $g_+^{\gamma}(x) = \lim_{u\to\infty} \inf g(x,u)|u|^{\gamma}$. Based on the Leray-Schauder degree theory (see [9]), we obtain solvability theorems of (1.1) and (1.2) under assumptions with $(\mathbf{F}_3)$ may be satisfied. Moreover, we combine either $(\mathbf{F}_1)$ or $(\mathbf{F}_2)$

with ($\mathbf{F}_3$) to obtain some new solvability conditions which are given in Theorem 2.6 of Section 2.

In the following we shall make use of the real Banach spaces $L^p(\Omega)$, $C(\bar{\Omega})$, and the Sobolev spaces $W^{2,p}(\Omega)$. The norms of $L^p(\Omega)$, $C(\bar{\Omega})$, $W^{2,p}(\Omega)$ are denoted by $\|u\|_{L^p}$, $\|u\|_C$, $\|u\|_{W^{2,p}}$, respectively, the compact embedding $W^{2,p}(\Omega) \longrightarrow C(\bar{\Omega})$, for $p > N/2$ has been noted below. By a solution of (1.1) (or (1.2)), we mean a function $u \in W^{2,p}(\Omega)$ satisfies the differential equation in (1.1) for a.e. $x \in \Omega$.

Finally we note that (see [1]) for each $p > 1$, there exists $K(p) > 0$ such that for all $u \in W^{2,p}(\Omega)$, $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$

$$\|u - Pu\|_{W^{2,p}} \leq K(p)\|\triangle u\|_{L^p}, \tag{1.7}$$

and there exists $K > 0$ such that for all $u \in W^{2,2}(\Omega)$, $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$

$$\|u - Pu\|_{L^2}^2 \leq K\langle -\triangle u, u\rangle, \tag{1.8}$$

where $P: L^2(\Omega) \longrightarrow L^2(\Omega)$, $Pu = \int u/|\Omega|$ for $u \in L^2(\Omega)$.

2. **Existence theorems.** The following Theorem 1 is an existence theorem for a solution of (1.1) when $p > N/2$ ($N \geq 2$), h satisfies ($\mathbf{F}_3$) and g satisfies (**G**), (**H**) with $\frac{\alpha(p-1)+\beta}{p} \leq 1$ and $\beta < 1$.

THEOREM 1. *Let $p > N/2$, $g: \Omega \times \mathbf{R} \longrightarrow \mathbf{R}$ be a Caratheodory function satisfying* (**G**) *and* (**H**) *with* $\frac{\alpha(p-1)+\beta}{p} \leq 1$ *and* $\beta < 1$, *then the problem (1.1) is solvable for any $h \in L^p(\Omega)$ provided that* ($\mathbf{F}_3$) *holds.*

PROOF. We may assume that $k_0 = \tilde{k}_0 = \tilde{\tilde{k}}_0$, the Lebesgue measure $|\Omega| = 1$ and let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be a continuous function defined by

$$f(u) = \begin{cases} u & \text{if } |u| \leq 1 \\ \frac{u}{|u|} & \text{if } |u| > 1. \end{cases}$$

We consider the boundary value problems

$$\begin{aligned} \triangle u + (1-t) f\Big(\int u\Big) + t g(x,u) &= th \text{ in } \Omega, \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2.1}$$

for $0 \leq t \leq 1$. The problem (2.1) has only a trivial solution when $t = 0$, and becomes the original problem (1.1) when $t = 1$. To apply the Leray-Schauder degree theory, it suffices to show that there exists $R_0 > 0$ such that for all $0 < t < 1$, $\|u\|_C < R_0$ for all possible solutions u of (2.1). We note first that there exist $\tilde{\tilde{e}} \in L^p(\Omega)$ and Caratheodory functions $g_1, g_2: \Omega \times \mathbf{R} \longrightarrow \mathbf{R}$ (see Kuo [7]) such that $g = g_1 + g_2$, $0 \leq g_1(x,u) \leq a|u|^\alpha$, $|g_2(x,u)| \leq a|u|^\beta + \tilde{\tilde{e}}(x)$ for a.e. $x \in \Omega$ and all $u \in \mathbf{R}$, and there exist constants $C_0, C_1 \geq 0$ such that for $0 < t < 1$ and all possible solutions u of (2.1)

$$t\|g_1(x,u)\|_{L^p} \leq C_1\Big(\|u\|_C^{\frac{\alpha(p-1)}{p}} + \|u\|_C^{\frac{\alpha(p-1)+\beta}{p}}\Big) \tag{2.2}$$

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and

$$\|\triangle u\|_{L^p} = \|th - tg(x,u) - (1-t)f\Big(\int u\Big)\|_{L^p} \le C_0\Big(1 + \|u\|_C^{\frac{\alpha(p-1)}{p}} + \|u\|_C^{\frac{\alpha(p-1)+\beta}{p}} + \|u\|_C^{\beta}\Big) \tag{2.3}$$

hold. Next we shall show that solutions of (2.1) for all $0 < t < 1$ have an *a priori* bound in $C(\bar{\Omega})$. If this is not true, then there exist a sequence $\{u_n\}$ in $W^{2,p}(\Omega)$ and a corresponding sequence $\{t_n\}$ in $(0,1)$ such that u_n satisfies (2.1) with $t = t_n$ and $\|u_n\|_C \ge n$ for all n. Let $v_n = u_n/\|u_n\|_C$, then $\|v_n\|_C = 1$ and by (2.3), we have

$$\|\triangle v_n\|_{L^p} \le C_0\Big(1 + \|u_n\|_C^{\frac{\alpha(p-1)}{p}} + \|u_n\|_C^{\frac{\alpha(p-1)+\beta}{p}} + \|u_n\|_C^{\beta}\Big)/\|u_n\|_C. \tag{2.4}$$

By hypothesis $\frac{\alpha(p-1)+\beta}{p} \le 1$ and $\beta < 1$, the right hand side of (2.4) is bounded by a constant independent of n. Hence $\{v_n - Pv_n\}$ has a subsequence convergent in $C(\bar{\Omega})$. Because $\{\int v_n\}$ is bounded in $\mathbf{R}$, we may assume without loss of generality that $\{v_n\}$ converges to w weakly in $W^{2,p}(\Omega)$ and strongly in $C(\bar{\Omega})$ for some $w \not\equiv 0$ because $\|v_n\|_C = 1$. By (2.2), the sequence $t_n g_1(x,u_n)/\|u_n\|_C$ has a subsequence convergent weakly in $L^p(\Omega)$, we say to m. Clearly $|t_n h - t_n g_2(x,u_n) - (1-t_n)f(\int u_n)|/\|u_n\|_C \longrightarrow 0$ in $L^p(\Omega)$ as $n \longrightarrow \infty$, and $m(x) \ge 0$ for a.e. $x \in \Omega$. We may assume that $[(1-t_n)f(\int u_n) + t_n g(x,u_n) - t_n h]/\|u_n\|_C \longrightarrow m$ weakly in $L^p(\Omega)$. Since $\triangle\colon D(\triangle) \subset W^{2,p}(\Omega) \longrightarrow L^p(\Omega)$ is also weakly closed, it follows that $w \in W^{2,p}(\Omega)$ and

$$\triangle w + m = 0 \text{ in } \Omega,\ \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega. \tag{2.5}$$

By (2.5) we have $\int m = 0$, so that $m(x) = 0$ for a.e. $x \in \Omega$. Consequently, $w \equiv 1$ or $w \equiv -1$. We consider only the case $w \equiv 1$, for the case $w \equiv -1$ can be treated similarly. By the properties of $N(\triangle)$ that there exists an $n_0 \in N$ such that $v_n(x) \ge \frac{1}{2} \ge \frac{k_0}{n}$ in $\bar{\Omega}$ for all $n \ge n_0$, and hence $u_n(x) \longrightarrow \infty$ for each $x \in \Omega$. Integrating (2.1) when $u = u_n$ and $t = t_n$, we have

$$t_n \int g(x,u_n) < (1-t_n)f\Big(\int u_n\Big) + t_n \int g(x,u_n) = t_n \int h = 0. \tag{2.6}$$

Since $t_n \ne 0$, using (1.3) and the fact that $\frac{k_0}{n} \le \frac{1}{2} \le v_n(x) \le 1$ for all $n \ge n_0$ and $x \in \bar{\Omega}$, we have

$$\|u_n\|_C^{\gamma} g(x,u_n) = \frac{g(x,u_n)u_n}{|u_n|^{1-\gamma}}|v_n|^{-\gamma} \ge e|v_n|^{-\gamma} \ge -|e|2^{\gamma} \tag{2.7}$$

for all $n \ge n_0$ and $x \in \bar{\Omega}$ with $u_n(x) \ne 0$.

Applying the Fatou lemma to the inequality

$$\|u_n\|_C^{\gamma} \int g(x,u_n) < 0, \tag{2.8}$$

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we have

$$\int g_+^{\gamma}(x)\,dx \le 0, \tag{2.9}$$

which contradicts the second inequality of $(\mathbf{F}_3)$, and the proof is complete.

By slightly modifying the proof and the solvability condition of Theorem 1, we obtain the following theorem.

THEOREM 2. *Under assumptions of Theorem 1, the problem (1.1) is solvable for any $h \in L^p(\Omega)$ provided that either*

$$(\mathbf{F}_4) \qquad \int g_-^{\delta}(x)\,dx < 0 = \int h(x)\,dx \le \int c(x)\,dx$$

or

$$(\mathbf{F}_5) \qquad \int d(x)\,dx \le \int h(x)\,dx = 0 < \int g_+^{\gamma}(x)\,dx$$

holds.

THEOREM 3. *Let $p > N/2$ $(N \ge 2)$, $p \ge 2$, $g\colon \Omega \times \mathbf{R} \longrightarrow \mathbf{R}$ be a Caratheodory function satisfying $(\mathbf{G})$ and $(\mathbf{H})$ with $\alpha, \beta \le \frac{p}{p-1}$, then the problem (1.2) is solvable provided that one of $\mathbf{F}_j$, $j = 3, 4, 5$ holds.*

PROOF. We may assume that $\alpha = \beta \ge \frac{1}{p-1}$ and consider the boundary value problems

$$-\triangle u + (1-t) f\Big(\int u\Big) + t g(x,u) = th \text{ in } \Omega,\ \frac{\partial u}{\partial n} = 0 \text{ on } \Omega \tag{2.10}$$

for $0 \le t \le 1$, where f is defined as in the proof of Theorem 1. To show that all possible solutions of (2.10) and $0 < t < 1$ have an *a priori* bound in $C(\bar{\Omega})$, it suffices to show that there exists a constant $C_0' > 0$ such that for all possible solutions u of (2.10) and $0 < t < 1$

$$\begin{aligned} \|\triangle u\|_{L^p} &= \Big\| th - tg(x,u) - (1-t) f\Big(\int u\Big)\Big\|_{L^p} \\ &\le C_0'\big(1 + \|u\|_C^{\frac{\alpha(p-1)}{p}}\big) \end{aligned} \tag{2.11}$$

hold. We may then use (2.11) as (2.3) to show the existence $\|u\|_C < R_0$ for some constant $R_0 > 0$ independent of u. We note first that there exist Caratheodory functions $g_1, g_2\colon \Omega \times \mathbf{R} \longrightarrow \mathbf{R}$ and $\tilde{\tilde{e}} \in L^p(\Omega)$ such that for a.e. $x \in \Omega$ and all $u \in \mathbf{R}$

$$0 \le |g_1(x,u)| \le a|u|^{\alpha}, \quad 0 \le g_1(x,u)u, \quad |g_2(x,u)| \le \tilde{\tilde{e}}(x) \quad \text{and } g = g_1 + g_2. \tag{2.12}$$

This may be done by defining $\tilde{\tilde{e}}(x) = \max\{|c(x)|, |d(x)|, b(x), a_{k_0}(x)\}$,

$$g_1(x,u) = \begin{cases} \min\{g(x,u) + \tilde{\tilde{e}}(x), a|u|^{\alpha}\}\theta(u) \\ \max\{g(x,u) - \tilde{\tilde{e}}(x), -a|u|^{\alpha}\}\theta(u) \end{cases} \tag{2.13}$$

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and $g_2 = g - g_1$, where $\theta: \mathbf{R} \longrightarrow \mathbf{R}$ is a continuous function such that for $u \in \mathbf{R}$, $0 \leq \theta(u) \leq 1$, $\theta(u) = 0$ for $|u| \leq k_0$ and $\theta(u) = 1$ for $|u| \geq 2k_0$. Taking the inner product of (2.10) with u in $L^2(\Omega)$, we have

$$\begin{aligned} -\int (\triangle u)u &\leq -\int (\triangle u)u + (1-t) f\Big(\int u\Big) \int u + t \int g_1(x,u)u \\ &= t \int \big[h - g_2(x,u)\big] u \\ &\leq \big(\|h\|_{L^1} + \|\tilde{\tilde{e}}\|_{L^1}\big) \|u\|_C. \end{aligned} \tag{2.14}$$

Similarly we also have

$$t \int g_1(x,u)u \leq \big[\|h\|_{L^1} + \|\tilde{\tilde{e}}\|_{L^1}\big] \|u\|_C.$$

Hence by (1.8) and (2.14), we have

$$\|u - Pu\|_{L^2}^2 \leq C_1' \|u\|_C$$

and

$$\begin{aligned} t^p \int |g_1(x,u)|^p &\leq t \int |g_1(x,u)|\, |g_1(x,u)|^{p-1} \\ &\leq t \int |g_1(x,u)| (a|u|^\alpha)^{p-1} \\ &\leq \|u\|_C^{\alpha(p-1)-1} a^{p-1} C_1' \|u\|_C \\ &\leq C_2' \|u\|_C^{\alpha(p-1)} \end{aligned} \tag{2.15}$$

for some constants $C_1', C_2' \geq 0$ independent of u. It follows from (2.15) that

$$\|t g_1(x,u)\|_{L^p} \leq C_3' \|u\|_C^{\frac{\alpha(p-1)}{p}} \tag{2.16}$$

for some constant $C_3' > 0$ independent of u. Therefore, by (1.7), (2.14) and (2.16) that there exists a constant $C_0' > 0$ such that (2.11) holds for all possible solutions u of (2.15) and $0 < t < 1$, and the proof is complete.

THEOREM 4. *Let $p > N/2$ ($N \geq 2$), $p \geq 2$ and $g: \Omega \times \mathbf{R} \longrightarrow \mathbf{R}$ be a Caratheodory function satisfying* (**G**) *and* (**H**) *with $\alpha, \beta \leq \frac{p}{p-1}$. Then the problem (1.2) is solvable for any $h \in L^p(\Omega)$ provided that*

$$\int d(x) \leq \int h(x) \leq \int c(x) \tag{$\mathbf{F}_6$}$$

holds.

COROLLARY 5. *Let $p > N/2$ ($N \geq 2$), $p \geq 2$ and $g: \Omega \times \mathbf{R} \longrightarrow \mathbf{R}$ be a Caratheodory function satisfying* (**G**) *and* (**H**) *with $\alpha, \beta \leq \frac{p}{p-1}$. Then the problem (1.2) is solvable for any $h \in L^p(\Omega)$ provided that* ($\mathbf{F}_1$) *holds.*

If either $\delta = 0$ or $\gamma = 0$, then conditions ($\mathbf{F}_4$) and ($\mathbf{F}_5$) can be respectively replaced by

$$\int g_-^0(x)\, dx < \int h(x)\, dx \leq \int c(x)\, dx \tag{$\mathbf{F}_7$}$$

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and

$$\int d(x)\,dx \le \int h(x)\,dx < \int g_+^0(x)\,dx. \qquad (\mathbf{F}_8)$$

THEOREM 6. *Under assumptions of Theorem 4, the problem (1.2) is solvable for any $h \in L^p(\Omega)$ provided that either $(\mathbf{F}_7)$ or $(\mathbf{F}_8)$ holds.*

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