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COMPLETELY REDUCIBLE OPERATOR ALGEBRAS AND SPECTRAL SYNTHESIS

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1. Introduction. An algebra $\mathscr{A}$ of bounded operators on a Hilbert space H is said to be *reductive* if it is unital, weakly closed and has the property that if $M \subset H$ is a (closed) subspace invariant for every operator in $\mathscr{A}$, then so is $M^{\perp}$. Loginov and Šul'man [**6**] and Rosenthal [**9**] proved that if $\mathscr{A}$ is an abelian reductive algebra which commutes with a compact operator K having a dense range, then $\mathscr{A}$ is a von Neumann algebra. Note that in this case every invariant subspace of $\mathscr{A}$ is spanned by one-dimensional invariant subspaces. Indeed, the operator KK^* commutes with $\mathscr{A}$. Hence its eigenspaces are invariant for $\mathscr{A}$, so that H is an orthogonal sum of the finite-dimensional invariant subspaces of $\mathscr{A}$. From this our claim easily follows.

We shall generalize this result to operator algebras on Banach spaces. Let X be a Banach space. For (closed) subspaces M and N in X, we write $X = M \dotplus N$ if $M \cap N = 0$ and X is the algebraic direct sum of M and N. The algebra of all bounded operators on X will be denoted by $\mathscr{L}(X)$. For any subset S of $\mathscr{L}(X)$ we write Lat S for the collection of those subspaces which are invariant for every operator in S. We say that S *admits spectral synthesis* if every subspace of Lat S is spanned by one-dimensional elements of Lat S. An algebra $\mathscr{A}$ in $\mathscr{L}(X)$ is said to be *reflexive* if it contains all the operators which leave every subspace in Lat $\mathscr{A}$ invariant.

An operator algebra $\mathscr{A}$ is said to be *completely reducible* if it is weakly closed, contains the identity operator I and has the following property: for every subspace M in Lat $\mathscr{A}$ there is a subspace N in Lat $\mathscr{A}$ such that $M \dotplus N = X$. Equivalently, for every M in Lat $\mathscr{A}$ there is a projection E (continuous idempotent) commuting with $\mathscr{A}$ and such that $E(X) = M$. An operator $T \in \mathscr{L}(X)$ is said to be completely reducible if the weakly closed algebra generated by T and I is completely reducible. We say that a subset S of $\mathscr{L}(X)$ is *sufficient* if the closed subspace spanned by the ranges of all operators in S equals X.

In the present paper we shall show that if some additional conditions are satisfied, an abelian completely reducible algebra is reflexive and admits spectral synthesis. In particular, we shall prove that if $\mathscr{A}$ is an abelian completely reducible algebra commuting with a sufficient set of

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compact operators and such that every hyperinvariant subspace of $\mathscr{A}$ has a unique invariant complement, then $\mathscr{A}$ is reflexive and admits spectral synthesis. Our main tools are a theorem of Bochner and von Neumann [**2**] about almost periodic functions in a group and Lomonosov's theorem.

2. Almost periodic functions in a group. The result of this section (Theorem 2) seems to be known; however, for lack of the references, we give its proof here. For this purpose we need the following results of Bochner and von Neumann [**2**]. It is noteworthy that these results were stated in [**2**] for the more general case of a complete topological space X.

Let G be a group, and X a Banach space. Denote by $X(G)$ the Banach space of all bounded X-valued functions on G endowed with the supremum-norm. A function $F(g) \in X(G)$ is called *almost periodic* if the sets

$$R(F) = \{F(gh), h \in G\} \quad \text{and} \quad L(F) = \{F(hg), h \in G\}$$

are relatively compact in $X(G)$. Let $F(g)$ be an almost periodic function. Then the closure of the convex hull $\operatorname{Co}\{F(g, hg_2), h \in G\}$ in $X(G \times G)$ contains the unique constant function which is denoted by $MgF(g)$ and is called the mean of F. For every $h \in G$

$$MgF(gh) = MgF(hg) = MgF(g).$$

Now let $\rho(g)$ be a finite-dimensional irreducible unitary representation of G, and $\rho_{ij}(g)$ the entries of a matrix of $\rho(g)$ with respect to some orthonormal basis in the Hilbert space of this representation. For every almost periodic function $F(g)$ the functions $\rho_{ji}(g^{-1})F(g)$ are also almost periodic. The matrix

$$(f_{ij}(\rho)) = (Mg(\rho_{ji}(g^{-1})F(g)))$$

is called an expansion matrix of F. For every almost periodic function F there exists an at most countable sequence $\rho_1, \rho_2, \ldots, \rho_n, \ldots$ of the pairwise inequivalent irreducible finite-dimensional unitary representations ρ such that an expansion matrix $(f_{ij}(\rho))$ is not identically zero. Let d_n be the dimension of ρ_n and

$$f_{ij}^{(n)} = Mg(\rho_{n,ji}(g^{-1})F(g)).$$

The formal series

$$\sum_{n=1}^{\infty} d_n \sum_{i,j=1}^{d_n} f_{ij}^{(n)} \rho_{n,ij}$$

is called the *Fourier expansion* of F.

THEOREM 1. *Let ρ_n, $n = 1, 2, \ldots$ denote a sequence of pairwise inequivalent irreducible finite-dimensional unitary representations of G, and the d_n*

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their respective dimensions. Then there exists a square array of numbers $r_n^{(m)}$, $m, n = 1, 2, \ldots$ *with the following properties:*

(i) *For each* m *only a finite number of* $r_n^{(m)}$ *are non-zero.*

(ii) *If an almost periodic function* F *has a Fourier expansion*

$$\sum_{n=1}^{\infty} d_n \sum_{i,j=1}^{d_n} f_{ij}^{(n)} \rho_{n,ij},$$

then F *is the limit,* $m \to \infty$, *of finite aggregates*

$$F_m = \sum_{n=1}^{\infty} r_n^{(m)} d_n \sum_{i,j=1}^{d_n} f_{ij}^{(n)} \rho_{n,ij}.$$

For the proof see [**2**, Theorem 30, p. 38].

THEOREM 2. *Let* X *be a Banach space,* G *an abelian group of bounded linear operators in* X. *If, for every* $x \in X$, *the orbits* $\{gx, g \in G\}$ *are relatively compact in* X, *then* G *admits spectral synthesis.*

Proof. Choose $x \in X$, and let $F^x(g) = gx$. We claim that $F^x(g)$ is almost periodic. Indeed, by the Banach-Steinhaus theorem, there is a positive constant M such that $\|g\| < M$ for all $g \in G$. Let $\{h_i x\}_{i=1}^n$ be an ϵ/M-net for the set $\{hx, h \in G\}$. Then for every $h \in G$

$$\begin{aligned} \inf_{1 \leqq i \leqq n} \sup_{g \in G} \|F^x(gh) - F^x(gh_i)\| &= \inf_{1 \leqq i \leqq n} \sup_{g \in G} \|ghx - gh_i x\| \\ &< M \inf_i \|hx - h_i x\| < \epsilon, \end{aligned}$$

and $\{F^x(gh_i)\}_{i=1}^n$ is an ϵ-net for $R(F^x) = L(F^x)$.

It is well known that every irreducible unitary representation of an abelian group is one-dimensional. Hence every term of the Fourier expansion of $F^x(g)$ is an X-valued function on G of the form

$$\chi(g) Mg(\chi(g^{-1})gx),$$

where χ is an abelian character of G. Let y be a constant function in a closure of

$$\mathrm{Co}\,\{\chi(g_1^{-1}ag_2^{-1})g_1 a g_2 x, a \in G\}$$

in $X(G \times G)$. It is easy to see that, for every $h \in G$, hy lies in the closure of

$$\mathrm{Co}\,\{\chi(g_1^{-1}ag_2^{-1})g_1 a g_2 hx, a \in G\}.$$

From the uniqueness of the mean it follows that

$$hMg(\chi(g^{-1})F^x(g)) = Mg(\chi(g^{-1})F^x(gh)).$$

Now for every $h \in G$

$$\begin{aligned} hMg(\chi(g^{-1})gx) &= Mg(\chi(g^{-1})hgx) = Mg(\chi(h)\chi(g^{-1}h^{-1})hgx) \\ &= \chi(h)Mg(\chi((gh)^{-1})ghx) = \chi(h)Mg(\chi(g^{-1})gx), \end{aligned}$$

that is, every non-zero value of a Fourier term of $F^x(g)$ is an eigenvector

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for all operators in G. By Theorem 1, gx can be approximated by the vectors of the form

$$\sum_{K=1}^{N} \alpha_K \chi_K(g) M_g(\chi_K(g^{-1})gx).$$

In particular, x can be approximated in X by the vectors of the form

$$\sum_{K=1}^{N} \alpha_K M_g(\chi_K(g^{-1})gx).$$

Since x has been chosen arbitrarily, it follows that the eigenvectors of G span X. Repeating the same argument for an arbitrary invariant subspace of G, we complete the proof.

3. Main results. For an algebra $\mathscr{A}$ in $\mathscr{L}(X)$ we denote by $\mathscr{A}'$ the commutant of $\mathscr{A}$ and by $\mathscr{A}'' = (\mathscr{A}')'$ the double commutant of $\mathscr{A}$. If U is a subset of $\mathscr{L}(X)$ and $x \in X$, we denote by Ux the set of all vectors Tx with $T \in U$. For $M \subset X$, M^- means the (strong) closure of M.

LEMMA 3. *Suppose $\mathscr{A}$ is a uniformly closed subalgebra of $\mathscr{L}(X)$ and $S \subset \mathscr{L}(X)$ is a sufficient family of injective compact operators. If the range of every operator in S is invariant for $\mathscr{A}$ and U is a (norm) bounded subset in $\mathscr{A}$, then $(Ux)^-$ is compact for every $x \in X$.*

Proof. Applying the Closed Graph Theorem twice, we obtain that for every $K \in S$ there exists a linear bounded mapping $\pi_K: \mathscr{A} \to \mathscr{L}(X)$ such that

$$TK = K\pi_K(T)$$

for every $T \in \mathscr{A}$.

Suppose U is a bounded subset of $\mathscr{A}$; without any loss of generality one can assume that U is a subset of the unit ball in $\mathscr{A}$. For $K_1, K_2, \ldots,$ K_n in S and $x_1, x_2, \ldots, x_n$ in X we can write:

$$U\left(\sum_{i=1}^{n} K_i x_i\right) \subset \sum_{i=1}^{n} K_i \pi_{K_i}(U) x_i.$$

Since every set $\pi_{K_i}(U)$ is bounded, the closure of the right side of this inclusion is compact. Thus the closure of the left side is also compact. Since S is a sufficient family, there is a dense manifold M in X such that $(Uy)^-$ is compact for every $y \in M$. Fix $\epsilon > 0$ and $x \in X$. Suppose that $y \in M$ and $\|x - y\| < \epsilon/4$, and let $\{T_j y\}_{j=1}^{m}$ be an $\epsilon/2$-net in Uy. Then for every $T \in U$ there is an index j such that $\|T_j y - Ty\| < \epsilon/2$. Consequently,

$$\begin{aligned}\|Tx - T_j x\| &\leqq \|(T - T_j)(x - y)\| + \|Ty - T_j y\| \\ &\leqq \|T - T_j\| - \|x - y\| + \|Ty - T_j y\| < \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

Thus $\{T_j x\}_{j=1}^{n}$ is an ϵ-net for Ux, and the result follows.

Remark. If $S \subset \mathscr{A}'$, we need not suppose that the operators in S are injective, because we can assume in this case that for every $K \in S$, π_K is the identity mapping on $\mathscr{A}$.

LEMMA 4. *Let $\mathscr{B}$ be a complete Boolean algebra of projections in $\mathscr{L}(X)$ which admits spectral synthesis. Then $\mathscr{B}$ is totally atomic. Moreover, if X is a common eigenvector for all elements of $\mathscr{B}$, then there exists an atom $E \in \mathscr{B}$ such that $Ex = x$.*

Proof. Suppose x is an eigenvector for every $E \in \mathscr{B}$; then $Ex = x$ or $Ex = 0$ for $E \in \mathscr{B}$. Let

$$E_0 = \inf \{E \in \mathscr{B} : Ex = x\}.$$

We claim that E_0 is an atom in $\mathscr{B}$. If not, there are non-zero E_1 and E_2 in $\mathscr{B}$ such that $E_0 = E_1 + E_2$. Since E_1x and E_2x can equal only 0 or x, $E_ix = x$ for $i = 1$ or $i = 2$. However, $E_0 > E_i$, which contradicts the definition of E_0. This means that E_0 is actually an atom in $\mathscr{B}$. Now fix $E \in \mathscr{B}$. Then $E(X) \in \text{Lat}\,\mathscr{B}$ and, because $\mathscr{B}$ admits spectral synthesis, there exists a family $\{x_\alpha\}_{\alpha \in \mathscr{T}}$ of common eigenvectors which span $E(X)$. As we have seen, for every $\alpha \in \mathscr{T}$ there is an atom $E_\alpha \in \mathscr{B}$ with $E_\alpha x_\alpha = x_\alpha$. Hence $E = \sup_{\alpha \in \mathscr{T}} E_\alpha$, and the proof is complete.

THEOREM 5. *Let $\mathscr{A} \subset \mathscr{L}(X)$ be a completely reducible algebra. Suppose the following conditions are satisfied:*

(i) *For every $M \in \text{Lat}\,\mathscr{A}$ with $\dim M > 1$ there is non-zero $N \in \text{Lat}\,\mathscr{A}$ properly contained in M;*

(ii) *$\mathscr{A}$ contains a sufficient family of compact operators;*

(iii) *There is $C > 0$ such that for every projection E in $\mathscr{A}'$ the inequality $\|E\| < C$ holds.*

Then

(i′) *$\mathscr{A}$ admits spectral synthesis;*

(ii′) *$\mathscr{A}$ is contained in a uniformly closed algebra generated by a totally atomic Boolean algebra of projections;*

(iii′) *Every unital weakly closed subalgebra of $\mathscr{A}$ is reflexive.*

Proof. By Zorn's lemma, $\mathscr{A}'$ contains a maximal abelian family of projections $\mathscr{B}$. Clearly $\mathscr{B}$ is a strongly closed Boolean algebra. We claim that $\mathscr{B}$ is complete. Let $\{E_n\}_{n=1}^{\infty}$ be an increasing sequence in $\mathscr{B}$. We shall show that this sequence strongly converges. Suppose not. Then for some $x \in X$, $\lim_n E_nX$ does not exist. From (ii) and the remark after Lemma 3 it follows that the closure of the sequence $\{E_nx\}_{n=1}^{\infty}$ is compact. So we can assert that there are two subsequences $\{E_{n_K}\}_{K=1}^{\infty}$ and $\{E_{m_K}\}_{K=1}^{\infty}$ such that

$$\lim_K E_{n_K}x = x_1 \quad \text{and} \quad \lim_K E_{m_K}x = x_2 \quad \text{with } x_1 \neq x_2.$$

Passing to subsequences, one can assume that $n_K \leqq m_K \leqq n_{K+1}$ for

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$K = 0, 1, 2, \ldots$. Let $F_K = E_{n_K+1} - E_{m_K}$. Then

$$\lim_K F_K x = x_1 - x_2 = x_0 \neq 0.$$

Since $\|F_K\| < C$ for all K and $F_{K+1}F_K = 0$,

$$\|F_{K+1}x_0\| = \|F_{K+1}(F_K x - x_0)\| \to 0.$$

Also,

$$\|F_K x - F_K x_0\| = \|F_K(F_K x - x_0)\| \to 0.$$

Therefore

$$\lim_K F_K x_0 = \lim_K F_{K+1} x_0 = 0 \quad \text{and} \quad x_0 = \lim_K F_K x = 0.$$

The obtained contradiction shows that in fact every increasing sequence is strongly convergent. By [**3**, Lemma XVII.3.4], $\mathscr{B}$ is σ-complete. Since $\mathscr{B}$ is strongly closed, by [**3**, Lemma XVII.3.23], $\mathscr{B}$ is complete.

Now let E be in $\mathscr{B}$ with $\dim E(X) > 1$. We claim that there are non-zero E_1 and E_2 in $\mathscr{B}$ such that $E = E_1 + E_2$. Indeed, from (i) it follows that there are non-zero E_1 and E_2 in $\mathscr{A}'$ such that $E_1E_2 = E_2E_1 = 0$ and $E = E_1 + E_2$. If E_1 or E_2 are not in $\mathscr{B}$, then the family $\mathscr{B} \cup \{E_1, E_2\}$ would be an abelian family of projections larger than $\mathscr{B}$. So these projections are actually in $\mathscr{B}$, and our claim is proved.

Now let G denote the collection of all operators of the form $2E - I$ with $E \in \mathscr{B}$. It is easy to verify (see, for example, [**3**, Lemma XV.6.2]) that G is an abelian group. From (ii), (iii) and the remark to Lemma 3 it follows that $(Gx)^-$ is compact for every $x \in X$. By Theorem 2, $\mathscr{B}$ admits spectral synthesis. Now, by Lemma 4, $\mathscr{B}$ is totally atomic. But, as was observed above, every atom in $\mathscr{B}$ is a one-dimensional projection. Therefore X is spanned by one-dimensional subspaces of Lat $\mathscr{A}$; in particular, $\mathscr{A}$ is abelian. Choose M in Lat $\mathscr{A}$. We shall show that for the weak closure $\mathscr{A}_0$ of the restriction $\mathscr{A}|M$ the conditions of the present theorem are satisfied. Clearly $\mathscr{A}_0$ is completely reducible and the conditions (i) and (iii) are satisfied. Denote by L the closed subspace spanned by the ranges of all compact operators in $\mathscr{A}_0$. Then $L \in \text{Lat}\,\mathscr{A}_0$. Suppose $L \neq M$. Let K be a complement for L in Lat $\mathscr{A}_0$ and N be a complement for M in Lat $\mathscr{A}$. Then $L \dotplus K \dotplus N = X$ and $L \dotplus N \neq X$, since $K \neq 0$. On the other hand, the range of every compact operator in $\mathscr{A}$ is contained in $L \dotplus N$, which contradicts (ii). So (ii) is satisfied for $\mathscr{A}_0$. Applying the assertion which was proved for $\mathscr{A}$ to $\mathscr{A}_0$, we conclude that M is spanned by one-dimensional elements of Lat $\mathscr{A}_0$. Thus $\mathscr{A}$ admits spectral synthesis and (i′) is proved.

By Lemma 4, $\mathscr{B}$ contains the projection onto every one-dimensional subspace of Lat $\mathscr{B}$. Thus every one-dimensional subspace in Lat $\mathscr{B}$ is also in Lat $\mathscr{A}$. Since $\mathscr{B}$ admits spectral synthesis, Lat $\mathscr{A} \supset$ Lat $\mathscr{B}$.

Denote by $\mathscr{U}$ the uniformly closed algebra generated by $\mathscr{B}$. It is known that $\mathscr{U}$ is reflexive [**3**, Theorem XVII.3.16], so that $\mathscr{A} \subset \mathscr{U}$ and (ii′) is proved. By [**5**] every unital weakly closed subalgebra of $\mathscr{U}$ is reflexive. This completes the proof.

THEOREM 6. *Suppose $\mathscr{A} \subset \mathscr{L}(X)$ is an abelian completely reducible algebra generated (as a unital weakly closed algebra) by a family of compact operators. If the norms of all the projections in $\mathscr{A}'$ are bounded above by a common constant, then $\mathscr{A}$ is an algebra of scalar type spectral operators and $\mathscr{A}$ admits spectral synthesis.*

Proof. Let M be a closed subspace spanned by ranges of all compact operators in $\mathscr{A}$. Then $M \in \operatorname{Lat} \mathscr{A}$. Let $N \in \operatorname{Lat} \mathscr{A}$, $M \dotplus N = X$. Suppose $T \in \mathscr{A}$ and $T = \lim_\alpha K_\alpha$ (in the weak topology), where $\{K_\alpha\}$ is a net of compact operators in $\mathscr{A}$. Then, for every $x \in N$, $Tx \in N \cap M = 0$. This means that $\mathscr{A}|N = \{\lambda|_N\}$, so that $\mathscr{A}_0 = \mathscr{A}|M$ is weakly closed. Suppose $L \in \operatorname{Lat}\mathscr{A}_0$. If $\dim L < \infty$ and L does not contain a subspace in $\operatorname{Lat}\mathscr{A}_0$ other than 0 and L, then clearly $\dim L = 1$. If $\dim L = \infty$ and all compact operators in $\mathscr{A}_0$ vanish on L, then $\mathscr{A}_0|L$ contains only multiples of the identity and therefore has a non-trivial invariant subspace. On the other hand, if $\mathscr{A}_0|L$ contains a non-zero compact operator, then by [**7**] $\mathscr{A}_0|L$ has a non-trivial invariant subspace. We see that for $\mathscr{A}_0$ the conditions of Theorem 5 are satisfied. From Theorem 5 and [**3**, XVII.3.25] it follows that $\mathscr{A}_0$ (and also $\mathscr{A}$) is an algebra of scalar type spectral operators which admits spectral synthesis.

COROLLARY 7. *Let $K \in \mathscr{L}(X)$ be a compact completely reducible operator. If the norms of all projections commuting with K are bounded above by a common constant, then K is a scalar type spectral operator. If, in addition, X is a Hilbert space, then K is similar to a normal operator.*

It is known [**1**] that every compact normal operator in a Hilbert space is reductive. However, not every compact scalar type spectral operator in a Banach space is completely reducible. Indeed, let X be a Banach space which is not isomorphic to a Hilbert space and let P be a finite-rank projection in X with a kernel N. Since $\dim X/N < \infty$, N is also non-isomorphic to a Hilbert space and, by Lindenstrauss-Zafriri's theorem, N contains a subspace M having no complement in N. Since $M \in \operatorname{Lat} P$, P is not completely reducible.

LEMMA 8. *Suppose $\mathscr{A} \subset \mathscr{L}(X)$ is a completely reducible algebra.*

(i) *If $M \in \operatorname{Lat}\mathscr{A}$ has a complement in $(\operatorname{Lat}\mathscr{A}) \cap (\operatorname{Lat}\mathscr{A}')$, then M has no other complement in $\operatorname{Lat}\mathscr{A}$.*

(ii) *If $M \in (\operatorname{Lat}\mathscr{A}) \cap (\operatorname{Lat}\mathscr{A}')$ and M has a unique complement in $\operatorname{Lat}\mathscr{A}$, then this unique complement in $\operatorname{Lat}\mathscr{A}$ is also in $\operatorname{Lat}\mathscr{A}'$.*

For the proof see [**4**, Proposition 17].

THEOREM 9. *Let $\mathscr{A}$ be an abelian completely reducible algebra in $\mathscr{L}(X)$ such that $\mathscr{A}'$ contains a sufficient set of compact operators. Suppose one of the following conditions satisfied:*

(i) *$\mathscr{A}'$ is abelian;*

(ii) *For every $M \in \operatorname{Lat} \mathscr{A}'$ there exists a projection with range M which belongs to $\mathscr{A}$;*

(iii) *Every subspace in $\operatorname{Lat} \mathscr{A}'$ has a unique complement in $\operatorname{Lat} \mathscr{A}$;*

(iv) *$\mathscr{A}'$ is completely reducible.*

Then $\mathscr{A}$ admits spectral synthesis. Moreover, $\mathscr{A}$ is generated, as a uniformly closed algebra, by a totally atomic Boolean algebra of projections.

Proof. We show that each of conditions (i)–(iii) implies (iv). Let (i) be satisfied. Assume that $M \in \operatorname{Lat} \mathscr{A}'$ and that N and N_1 are two complements to M in $\operatorname{Lat} \mathscr{A}$. Denote by E the projection onto M along N and by E_1 the projection onto M along N_1. Since $EE_1 = E_1E$, we conclude that $N = N_1$, and therefore (i) $\Rightarrow$ (iii). Suppose (ii) satisfied and assume M, N, N_1 and E are as above. Now $E \in \mathscr{A}$. Since $N_1 \in \operatorname{Lat} E$ and $N_1 \cap M = 0$, N_1 must be contained in N. But from the equalities $M \dotplus N = M \dotplus N_1 = X$ it follows that $N_1 = N$. Thus (ii) $\Rightarrow$ (iii). Finally, (iii) $\Rightarrow$ (iv) by Lemma 8 (ii).

Now suppose $\mathscr{A}'$ is completely reducible. Denote by $\mathscr{B}$ the family of all projections in $\mathscr{A}''$. Since $\mathscr{A}$ is abelian, so are $\mathscr{A}''$ and $\mathscr{B}$. Let $\{E_\alpha\}$ be an arbitrary subset of $\mathscr{B}$. Since the subspaces

$$M = \vee_\alpha E_\alpha(X) \quad \text{and} \quad N = \cap_\alpha E_\alpha(X)$$

are in $\operatorname{Lat} \mathscr{A}'$ and $\mathscr{A}'$ is completely reducible, one can find projections F and G in $\mathscr{B}$ such that $M = F(X)$ and $N = G(X)$. But then

$$F = \sup_\alpha E_\alpha \quad \text{and} \quad G = \inf_\alpha E_\alpha,$$

so that $\mathscr{B}$ is a complete Boolean algebra. By Bade's theorem (see [**3**, Lemma XVII.3.3]), $\mathscr{B}$ is bounded.

Now, just as in the proof of Theorem 5, we conclude that $\mathscr{B}$ admits spectral synthesis and is totally atomic. Suppose $x \in X$ is a common eigenvector for all elements of $\mathscr{B}$. By Lemma 4, there is an atom $E_0 \in \mathscr{B}$ such that $E_0x = x$. Denote by M the range of E_0. If every compact operator $K \in \mathscr{A}'$ vanishes on M, we have $E_0K = KE_0 = 0$ and, because the set of all compact operators in $\mathscr{A}'$ is sufficient, $E_0 = 0$, which is impossible since $E_0x = x \neq 0$. Therefore the algebra $\mathscr{A}_0 = \mathscr{A}'|M$ contains a non-zero compact operator. On the other hand, $\mathscr{A}_0$ is transitive, because M is an atom in $\operatorname{Lat} \mathscr{A}'$. By Lomonosov's theorem [**7**] the algebra $\mathscr{A}_0'$ (which obviously contains $\mathscr{A}|M$) consists only of the multiples of the identity on M. It follows that the subspace, spanned by x, is in $\operatorname{Lat} \mathscr{A}$. Since $\mathscr{B}$ admits spectral synthesis, we can write $\operatorname{Lat} \mathscr{A} \supset \operatorname{Lat} \mathscr{B}$. Therefore, if $\mathscr{U}$ is a uniformly closed algebra generated by $\mathscr{B}$, then $\mathscr{A} \subset \mathscr{U}$.

Consequently, $\mathscr{A}$ is reflexive. Since $\mathscr{A}$ is completely reducible, $\mathscr{A} = \mathscr{A}''$. On the other hand, from the definition of $\mathscr{B}$ it follows that $\mathscr{U} \subset \mathscr{A}'' = \mathscr{A}$. Hence $\mathscr{A} = \mathscr{U}$ and the proof is finished.

COROLLARY 10. *Let $T \in \mathscr{L}(X)$ be a completely reducible operator commuting with a compact operator having a dense range. If every hyperinvariant subspace of T has a unique complement in* Lat T, *then T is a scalar type spectral operator which admits spectral synthesis.*

One can easily see that the proof of Theorem 9 actually leads to the following corollary:

COROLLARY 11. *Suppose $T \in \mathscr{L}(X)$ commutes with a compact operator having a dense range. If every hyperinvariant subspace of T has a complement which is also hyperinvariant for T, then T is a spectral operator of scalar type and spectral measure associated with T is totally atomic.*

The following proposition is an obvious generalization of [**4**, Proposition 18].

PROPOSITION 12. *Let $\mathscr{A}$ be a completely reducible algebra in $\mathscr{L}(X)$. If for every $M \in$* Lat $\mathscr{A}$ *having in* Lat $\mathscr{A}$ *more than one complement, every complement to M in* Lat $\mathscr{A}$ *also has more than one complement in* Lat $\mathscr{A}$, *then the weakly closed algebra generated by $\mathscr{A}$ and $\mathscr{A}'$ is completely reducible.*

Proof. Let M be in $(\text{Lat}\,\mathscr{A}) \cap (\text{Lat}\,\mathscr{A}')$. By Lemma 8 (ii) it suffices to show that M has a unique complement in Lat $\mathscr{A}$. Suppose this is not so. Then there is $N \in \text{Lat}\,\mathscr{A}$ such that $M \dotplus N = X$. Hence N has in Lat $\mathscr{A}$ a complement other than M. However, this contradicts Lemma 8 (i).

Now Theorem 9 and Proposition 12 imply

COROLLARY 13. ([**6**], [**9**]) *An abelian reductive algebra that commutes with a sufficient set of compact operators must be a totally atomic von Neumann algebra.*

PROPOSITION 14. *Suppose $\mathscr{A} \subset \mathscr{L}(X)$ is a completely reducible algebra. If $M + N$ is closed for every M and N in* Lat $\mathscr{A}$, *then the weakly closed algebra generated by $\mathscr{A} \cup \mathscr{A}'$ is completely reducible.*

Proof. By Lemma 8 (ii) it suffices to show that every $M \in (\text{Lat}\,\mathscr{A}) \cap (\text{Lat}\,\mathscr{A}')$ has a unique complement in Lat $\mathscr{A}$. Suppose

$$M \in (\text{Lat}\,\mathscr{A}) \cap (\text{Lat}\,\mathscr{A}'),$$

N and N_1 are in Lat $\mathscr{A}$ with $M \dotplus N = M \dotplus N_1 = X$ and $N \neq N_1$. Clearly N does not contain N_1. Hence there is a non-zero $L \in \text{Lat}\,\mathscr{A}$ such that $N_1 = L \dotplus N_1 \cap N$. Let E denote the projection onto M along

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N and $D = E(L)$. We claim that $D \in \operatorname{Lat}\mathscr{A}$. For every $x \in L$ and $y \in N$ the vector $Z = (E - I)x - y \in N$, so that

$$x + y + z = Ex = E(x + y) \in L + N.$$

Since $L + N$ is closed, it is in Lat E, so that $D = E(L + N)$ is also closed and therefore is in $\operatorname{Lat}\mathscr{A}$. Suppose $D_1 \in \operatorname{Lat}\mathscr{A}$ with $D_1 \dotplus D = M$. The subspace D is contained in $L \dotplus N \dotplus D_1$; hence the latter subspace contains M and N; that is, $L \dotplus N \dotplus D_1 = X$. By Lemma 8 (i) N has no other complement in $\operatorname{Lat}\mathscr{A}$ than M. So $L \dotplus D_1 = M$, but this is impossible, because $L \cap M = 0$.

4. Reductive algebras. In Theorem 15 below we generalize slightly some results of [**8**].

THEOREM 15. *Suppose $\mathscr{A}$ is a reductive algebra in a separable Hilbert space H such that $\mathscr{A}|M$ has a non-trivial invariant subspace whenever M is in* $\operatorname{Lat}\mathscr{A}$ *with* $\dim M > 1$. *If the von Neumann algebra generated by $\mathscr{A}$ contains either a sufficient set of compact operators or a set of compact operators such that the intersection of their kernels is zero, then $\mathscr{A}$ is a totally atomic abelian von Neumann algebra.*

Proof. Obviously the assumption about the ranges of compact operators in the von Neumann algebra is equivalent to the assumption about their kernels. So we suppose that the assumption about kernels is satisfied. Denote by H_1 the subspace of H spanned by all subspaces in $\operatorname{Lat}\mathscr{A}$ of dimension $\leqq 1$. Then $H_1 \in \operatorname{Lat}\mathscr{A}$ and $H_0 = H_1^{\perp} \in \operatorname{Lat}\mathscr{A}$. We claim that $H_0 = 0$. Suppose not. Let $\mathscr{U}$ denote the von Neumann algebra generated by $\mathscr{A}$. By our assumption there is a compact $K \in \mathscr{U}$ such that $K_0 = K|H_0 \neq 0$. Denote by $\mathscr{A}_0$ and $\mathscr{U}_0$ the respective restrictions of $\mathscr{A}$ and $\mathscr{U}$ to H_0. Suppose $F \subset H_0$ is an eigenspace of $K_0{}^*K_0$ which corresponds to some non-zero eigenvalue. Clearly F is finite-dimensional and the orthogonal projection P onto F lies in $\mathscr{U}_0$. Since $\operatorname{Lat}\mathscr{A}_0 = \operatorname{Lat}\mathscr{U}_0$, for every $M \in \operatorname{Lat}\mathscr{A}_0$ either $M \cap F \neq 0$ or $M \perp F$. Denote by M_0 the intersection of all subspaces in $\operatorname{Lat}\mathscr{A}_0$ containing F. Clearly $M_0 \in \operatorname{Lat}\mathscr{A}_0$. We claim that every non-zero subspace in $\operatorname{Lat}\mathscr{A}_0$ contained in M_0 has a non-zero intersection with F. Indeed, if for some non-zero $L \in \operatorname{Lat}\mathscr{A}_0$, $L \subset M$ and $L \cap F = 0$, then $L \perp F$ and F is contained in the subspace $L^{\perp} \cap M_0$ which belongs to $\operatorname{Lat}\mathscr{A}_0$ and is smaller than M_0. Now suppose $n = \dim F$. Since $\operatorname{Lat}\mathscr{A}_0$ has no non-zero atoms, there are non-zero $M_1, M_2, \ldots, M_{n+1}$ in $\operatorname{Lat}\mathscr{A}_0$ such that $M_0 = M_1 \oplus M_2 \oplus \ldots \oplus M_{n+1}$. Since $M_j \cap F \neq 0$ for $j = 1, 2, \ldots, n + 1$, we conclude that the dimension of F is at least $n + 1$. This contradiction shows that in fact $H_0 = 0$ and $H_1 = H$. It is easy to see now that $\mathscr{U}$ is an abelian totally atomic von Neumann algebra. An application of Sarason's result [**10**] leads to the desired conclusion.

COROLLARY 16. *Suppose* $T \in \mathscr{L}(H)$ *is a reductive operator such that for every* $M \in \operatorname{Lat} T$ *with* $\dim M > 1$ *the operator* $T|M$ *has a non-trivial invariant subspace and suppose* $P(\cdot, \cdot)$ *is a polynomial in two variables. If* $P(T, T^*)$ *is a compact operator with a dense range or zero kernel, then* T *is normal* (*in fact, diagonal*).

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