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Friedrichs extensions for Sturm–Liouville operators with complex coefficients and their spectra

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In this paper, we study the Friedrichs extensions of Sturm–Liouville operators with complex coefficients according to the classification of B. M. Brown et al. [**3**]. We characterize the Friedrichs extensions both by boundary conditions at regular endpoint and asymptotic behaviours of elements in the maximal operator domains at singular endpoint. Some of spectral properties are also involved.

Keywords: Friedrichs extension; Sturm–Liouville operator; complex coefficient; sesquilinear form; spectrum

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1. Introduction

Consider the Sturm–Liouville differential equation

$$\tau y := \frac{1}{w}[-(py')' + qy] = \lambda y \text{ on } [a, b), \tag{1.1}$$

where $p(x), q(x)$ are complex functions with $p(x) \neq 0$ a.e. in $[a, b)$, and $w(x) > 0$ on $[a, b)$, $1/p, q, w$ are all locally integrable on $[a, b)$, $-\infty < a < b \leqslant +\infty$, λ is the so-called spectral parameter. The assumptions on p, q, w ensure that a is a regular endpoint of the equation $\tau y = \lambda y$, and b is a singular endpoint, i.e., at least one of $b = \infty$ or $\int_a^b \left(w + \frac{1}{|p|} + |q|\right) \mathrm{d}x = \infty$ holds. However, we note that the regular endpoint b is included in the analysis.

Spectral theory of differential operators is one of the hot research branches in differential equations (see the classical books [1, 5, 6, 9]). Among these researches, the Friedrichs extension plays an important role in the spectral analysis of differential operators. For a symmetric differential operator with real coefficients which is lower semi-bounded, the Friedrichs extension is a particular self-adjoint extension which preserves the lower bound of the given minimal differential operator. Suitable boundary value conditions are added on the endpoints to make the extension be the Friedrichs extension. Following this line, H. G. Kaper, M. K. Kwong and A. Zettl [13], M. Moller and A. Zettl [19], H. D. Niessen, A. Zettl [20] gave the characterization of the Friedrichs extensions for singular Sturm–Liouville differential operators, similar results are obtained by M. Moller and A. Zettl for singular $2n$ order differential operator, singular block operator matrices by A. Konstantinov and R. Mennicken [18], Schördinger operator by A. B. Keviczky, N. Saad and R. L. Hall [15], and for singular Hamiltonian operators with one singular endpoint by [10, 28–30] with the endpoint being limit-point case, limit-circle case and intermediate deficiency indices, respectively.

The theory for Sturm–Liouville theory for differential expressions of second order with complex potentials is currently of great interest as there are applied to so-called $\mathcal{PT}$ -symmetric quantum mechanics in theoretical physics (see [2, 3, 4, 21, 23] and references cited therein).

When $p(x) \equiv 1$ and $q(x)$ is a complex valued function with $\mathrm{Im}q(x)$ being semi-bounded, A. R. Sims in his seminal paper (see [23] for details) extended the famous limit point/limit circle classification of H. Weyl [27] to the case of complex coefficients. Since the restriction $\mathrm{Im}q(x)$ being semi-bounded, it is too sharp to reduce its applications. The restriction was relaxed in the paper of B. M. Brown, D. K. R. McCormack, W. D. Evans, M. Plum [3]. However, the classification in [3] formally depends on the choice of rotation angles (or the half-planes). A new classification of equation (1.1) which is independent of the rotation angles (or the half-planes) was given by J. Qi, Z. Zheng and H. Sun [21]. Moreover, the J-self-adjoint realization was characterized by boundary conditions in [21].

Unlike Sturm–Liouville operators with real coefficients, the spectral theories of Sturm–Liouville operators with complex coefficients are not fully investigated. In this paper, we will give the Friedrichs extension without the restriction on $q(x)$, and $p(x)$ is an arbitrarily complex function. More concise, we obtain the Friedrichs extension domain not by imposing boundary condition on the singular end-point, but by asymptotic behaviours of elements in the maximal operator domains at singular endpoint. The spectral properties of the Friedrichs extension are also given.

This paper is organized as follows. In §2, we give some preliminaries on differential operators with complex coefficients, the characterizations of Friedrichs extensions under case I and case II are given separately in §3 and 4. Section 5 deals with the spectral properties of the Friedrichs extensions.

2. Preliminaries

In this paper, we consider the second order Sturm–Liouville equation

$$\tau y := \frac{1}{w}[-(py')' + qy] = \lambda y \text{ on } [a, b), \tag{2.1}$$

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where $p(x), q(x)$ are complex functions with $p(x) \neq 0$ a.e. in $[a, b)$, and $w(x) > 0$ on $[a, b)$, $1/p, q, w$ are all locally integrable on $[a, b)$, $-\infty < a < b \leqslant +\infty$, λ is the spectral parameter.

By $L^2_w[a, b)$, we mean the Hilbert space defined by

$$L^2_w[a, b) := \{y : \ (a, b) \to \mathbb{C} \, \text{is measurable} : \int_a^b w(x)|y(x)|^2 \, \mathrm{d}x < \infty\}$$

with inner product $\langle y, z \rangle := \int_a^b \bar{z}(x) w(x) y(x) \, \mathrm{d}x$ and the norm $\|y\| = (\langle y, y \rangle)^{1/2}$ for $y, z \in L^2_w[a, b)$. Here $w(x)$ is called the weight function. Similar Hilbert space can be defined by replacing $w(x)$ with other positive weight functions, such as $L^2_{|p|}[a, b)$ in §4.

2.1. Classification of (1.1) and corresponding operators

In paper [3], the hypothesis

$$\Omega = \overline{\text{co}} \left\{ \frac{q(x)}{w(x)} + r p(x) : \ r > 0, x \in (a, b) \right\} \neq \mathbb{C} \tag{2.2}$$

is introduced, where $\overline{\text{co}}$ denotes the closed convex hull (i.e., the smallest closed convex set containing the exhibited set). Then for each point on the boundary $\partial\Omega$, there exists a line through this point such that each point of Ω either lies in the same side of this line or is on it. That is, there exists a supporting line through this point. Let K be a point on $\partial\Omega$. Denote by L an arbitrary supporting line touching Ω at K, which may be the tangent to Ω at K if it exists. We then perform a transformation of the complex plane $z \mapsto z - K$ and a rotation through an appropriate angle $\theta \in (-\pi, \pi]$, so that the image of L now coincides with the new imaginary axis and the set Ω lies in the new right nonnegative half-plane. Therefore, for all $x \in (a, b)$ and $0 < r < \infty$,

$$\mathrm{Re} \left\{ \mathrm{e}^{i\theta} \left[\frac{q(x)}{w(x)} + r p(x) - K \right] \right\} \geqslant 0. \tag{2.3}$$

$$\mathrm{Re} \left[\mathrm{e}^{i\theta} (\lambda - K) \right] \geqslant 0. \tag{2.4}$$

For convenience, we define all such admissible values of K and θ by Π, i.e.,

$$\Pi = \left\{ (\theta, K) : \ \theta \in (-\pi, \pi], K \in \partial\Omega, \mathrm{Re} \left\{ \mathrm{e}^{i\theta} (\mu - K) \right\} \geqslant 0 \text{ for all} \mu \in \Omega \right\},$$

and we define

$$E = \{\theta \in (-\pi, \pi] : \exists K \in \partial\Omega, (\theta, K) \in \Pi\}. \tag{2.5}$$

Note that for fixed $\theta_0 \in E$, the K such that $(\theta_0, K) \in \Pi$ may be not unique.

Since $\Omega \neq \mathbb{C}$ is convex and closed space, one sees that $\Pi, E \neq \emptyset$, and if we define

$$\Lambda_{\theta,\,K} = \left\{\mu \in \mathbb{C} :\ \mathrm{Re}\left\{\mathrm{e}^{i\theta}(\mu - K)\right\} < 0\right\}, \tag{2.6}$$

then,

$$\mathbb{C}\backslash\Omega = \bigcup_{(\theta,K)\in\Pi} \Lambda_{\theta,\,K}.$$

Note that each $\Lambda_{\theta,\,K}$ is a half plane. Then $\Lambda_{\theta_1,\,K_1} \cap \Lambda_{\theta_2,\,K_2} \neq \emptyset$ for $\theta_1 \neq \theta_2 (\mathrm{mod}\,\pi)$.

Let $r \to 0$ and $r \to \infty$ in (2.3), respectively, we have the following lemma.

Lemma 2.1. *For each* $(\theta, K) \in \Pi$ *and* $\lambda \in \Lambda_{\theta,K}$ *there exists* $\delta_\lambda > 0$ *such that*

$$\mathrm{Re}\left\{\mathrm{e}^{i\theta}(q - K\,w))\right\} \geqslant 0,\ \mathrm{Re}\left\{\mathrm{e}^{i\theta}(q - \lambda\,w)\right\} \geqslant \delta_\lambda w,\ \mathrm{Re}\left\{\mathrm{e}^{i\theta}p\right\} \geqslant 0 \tag{2.7}$$

on $[a, b)$.

With these definitions and the similar line of H. Weyl's in [**27**], B. M. Brown et al. [**3**] divided (1.1) into three cases with respect to the corresponding half-planes $\Lambda_{\theta,\,K}$ as follows.

Theorem 2.2 (see [[**3**], theorem 2.1]). *For* $\lambda \in \Lambda_{\theta,\,K}$, $(\theta, K) \in \Pi(\alpha)$ *(where* α *is related to the boundary conditions on regular endpoint), the Weyl circles converge either to a limit-point* $m(\lambda)$ *or a limit-circle* $C_b(\lambda)$*. The following distinct cases are possible, the first two cases being sub-cases of the limit-point case.*

(I) *there exists a unique solution* y *of equation* (1.1) *satisfying*

$$\int_a^b \left[\mathrm{Re}\left\{\mathrm{e}^{i\theta}p\right\}|y'|^2 + \mathrm{Re}\left\{\mathrm{e}^{i\theta}(q - K\,w)\right\}|y|^2\right] + \int_a^b w\,|y|^2 < \infty \tag{2.8}$$

and this is the only one satisfying $y \in L^2_w[a, b)$*;*

(II) *there exists a unique solution of equation* (1.1) *satisfying* (2.8)*, but all solutions of (1.1) belong to* $L^2_w[a, b)$*.*

(III) *all solutions of* (1.1) *satisfy* (2.8)*.*

Remark 2.3. If $q(x)$ and $p(x)$ are real-valued, then $\Omega \subset \mathbb{R}$ and $(\theta, K) = (\pi/2, 0) \in \Pi$, and hence $\mathrm{Re}\left\{\mathrm{e}^{i\theta}p(x)\right\} = \mathrm{Re}\left\{\mathrm{e}^{i\theta}(q(x) - K\,w)\right\} \equiv 0$. So case II is vacuous. This means that the classification mentioned above reduces to Weyl's limit-point, limit-circle classification.

Since E is the set of rotation angles, in what follows, by a point $\theta \in E$, we denote the collection of points in the sense of θ module π. If E has only one point, then the classification of Brown et al. in theorem 2.2 is independent of the choice of $(\theta, K) \in \Pi$. Using variation of parameters formula, we can verify that if all solutions of (1.1) belong to $L^2_w[a, b)$ for some $\lambda_0 \in \mathbb{C}$, then it is true for all $\lambda \in \mathbb{C}$. This also means that case I is independent of the choice of $(\theta, K) \in \Pi$, too. However, if E has more than one point, cases II and III depend on the choice of $(\theta, K) \in \Pi$ in general since the rotation angle θ lies in (2.8). The exact dependence of cases II and III on (θ, K) is given in [**24**].

THEOREM 2.4. *If there exists a* $(\theta_0, K_0) \in \Pi$ *such that* (1.1) *is in case* II *w.r.t.* Λ_{θ_0,K_0}*, then* (1.1) *is in case* II *w.r.t.* $\Lambda_{\theta,K}$ *for all* $(\theta, K) \in \Pi$ *except for at most one* $\theta_1 \in E$ *(in the sense of mod* π*) such that* (1.1) *is in case* III *w.r.t.* Λ_{θ_1,K_1}*, where* $(\theta_1, K_1) \in \Pi$*.*

Theorem 2.4 indicates that if there exist $\theta_1, \theta_2 \in E$ such that $\theta_1 \neq \theta_2 \pmod\pi$ and (1.1) is in case III w.r.t. Λ_{θ_j,K_j} for $j = 1, 2$, then (1.1) is in case III w.r.t. $\Lambda_{\theta,K}$ for all $(\theta, K) \in \Pi$.

In what follows, we always assume that E has more than one point. Firstly, we prepare some properties of the set E.

LEMMA 2.5. *Let* E *be defined as in* (2.5)*.*

(i) *If* E *has more than one point, then* E *is a sub-interval of* $(-\pi, \pi]$*.*

(ii) *If* E *has more than one point, then for each* $\lambda \in \mathbb{C} \setminus \Omega$*, there exist* $\theta_1, \theta_2 \in E$ *with* $\theta_1 < \theta_2$ *such that for* $\theta \in (\theta_1, \theta_2) \subset E$*,* $\lambda \in \Lambda_{\theta,\,K}$*, where* $(\theta, K) \in \Pi$*.*

Proof. (i) Let $\theta_1, \theta_2 \in E$ with $\theta_1 \neq \theta_2$ (mod π), $\theta_1 < \theta_2$ and K_1, K_2 be the points on $\partial\Omega$ such that $(\theta_j, K_j) \in \Pi$, $j = 1, 2$. We claim that $[\theta_1, \theta_2] \subset E$. For the case $K_1 = K_2 = K$, we prove that $(\theta, K) \in \Pi$ for all $\theta \in (\theta_1, \theta_2)$. Set

$$\begin{cases} \dfrac{q(x)}{w(x)} + rp(x) - K = R(x, r, K)\, \mathrm{e}^{i\gamma(x,r,K)},\ R(x, r, K) = \left|\dfrac{q(x)}{w(x)} + rp(x) - K\right|, \\ \gamma_j(x, r, K) = \gamma(x, r, K) + \theta_j,\ r > 0,\ x \in (a, b),\ j = 1, 2. \end{cases}$$

It follows from the definition of Π that for $j = 1, 2$ and $r \geqslant 0$,

$$\mathrm{Re}\left\{\mathrm{e}^{i\theta_j}\left(\frac{q(x)}{w(x)} + rp(x) - K\right)\right\} \geqslant 0$$

on (a, b), or equivalently, $\cos\gamma_j(x, r, K) \geqslant 0$. Without any confusion, we write $\gamma(x, r, K)$ (resp. $\gamma_j(x, r, K)$) as γ (resp. γ_j). If we set

$$J = \sin(\theta_2 - \theta_1) \neq 0,\ J_1(x) = \begin{vmatrix} \cos\gamma_1 & \sin\theta_1 \\ \cos\gamma_2 & \sin\theta_2 \end{vmatrix},\ J_2(x) = \begin{vmatrix} \cos\gamma_1 & \cos\theta_1 \\ \cos\gamma_2 & \cos\theta_2 \end{vmatrix},$$

then $\cos\gamma$ and $\sin\gamma$ can be expressed as

$$\cos\gamma = \frac{J_1(x)}{J},\ \sin\gamma = \frac{J_2(x)}{J} \tag{2.9}$$

by using the formulae $\cos\gamma_j = \cos\gamma\cos\theta_j - \sin\gamma\sin\theta_j$ for $j = 1, 2$. This equality gives that

$$\begin{aligned} \cos(\theta + \gamma) &= \cos\gamma\cos\theta - \sin\gamma\sin\theta = \frac{J_1}{J}\cos\theta - \frac{J_2}{J}\sin\theta \\ &= \frac{1}{J}\left[\cos\gamma_1\sin(\theta_2 - \theta) + \cos\gamma_2\sin(\theta - \theta_1)\right] \end{aligned} \tag{2.10}$$

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for $\theta \in (\theta_1, \theta_2)$. Since $\cos\gamma_j \geqslant 0$ for $j = 1, 2$ implies $-\pi/2 \leqslant \gamma_1, \gamma_2 \leqslant \pi/2 (\text{mod } 2\pi)$, we have from $\pi \geqslant \theta_2 > \theta_1 > -\pi$, $\theta_2 - \theta_1 \neq \pi$ and $\theta_2 - \theta_1 = \gamma_2 - \gamma_1$ that $0 < \theta_2 - \theta_1 < \pi$. Consequently, for $\theta \in (\theta_1, \theta_2)$

$$0 < \theta_2 - \theta,\ \theta - \theta_1 < \theta_2 - \theta_1 < \pi.$$

Therefore, each term in the right-hind side of (2.10) is nonnegative, so $\cos(\theta + \gamma) \geqslant 0$ on (a, b). That is, for $r \geqslant 0$ and $x \in (a, b)$,

$$\mathrm{Re}\left\{\mathrm{e}^{i\theta}\left(\frac{q(x)}{w(x)} + rp(x) - K\right)\right\} = R(x, r, K)\cos(\gamma(x, r, K) + \theta) \geqslant 0,$$

which implies $(\theta, K) \in \Pi$ for $\theta \in (\theta_1, \theta_2)$.

In case $K_1 \neq K_2$, we choose $\mu_0 \in \Lambda_{\theta_1, K_1} \cap \Lambda_{\theta_2, K_2}$. Then it holds that

$$\mathrm{Re}\left\{\mathrm{e}^{i\theta_j}\left(\frac{q(x)}{w(x)} + rp(x) - \mu_0\right)\right\} = R(x, r, \mu_0)\cos(\gamma(x, r, \mu_0) + \theta_j) > 0$$

on (a, b) for $j = 1, 2$ and $r \geqslant 0$ by the definition of $\Lambda_{\theta,K}$, or $\cos(\gamma(x, r, \mu_0) + \theta_j) > 0$. Then, the similar proof in (2.9) and (2.10) yields that

$$\mathrm{Re}\left\{\mathrm{e}^{i\theta}\left(\frac{q(x)}{w(x)} + rp(x) - \mu_0\right)\right\} > 0,\ \theta \in (\theta_1, \theta_2). \tag{2.11}$$

Let L be the line defined by

$$L = \{\lambda \in \mathbb{C} :\ \mathrm{Re}\left\{\mathrm{e}^{i\theta}(\lambda - \mu_0)\right\} = 0\} \tag{2.12}$$

for fixed $\theta \in (\theta_1, \theta_2)$. One sees from (2.11), (2.12) and the definition of Ω that $L \subset \mathbb{C} \setminus \Omega$. Set $d = \mathrm{dist}(L, \partial\Omega)$ and let $K \in \partial\Omega$ be a point such that $d = \mathrm{dist}(K, L)$. Since

$$\mathrm{dist}(\mu, L) = \mathrm{Re}\left\{\mathrm{e}^{i\theta}(\mu - \mu_0)\right\},\ \mu \in \Omega, \tag{2.13}$$

we have that $\mathrm{Re}\left\{\mathrm{e}^{i\theta}(\mu - \mu_0)\right\} = \mathrm{dist}(\mu, L) \geqslant \mathrm{dist}(K, L) = \mathrm{Re}\left\{\mathrm{e}^{i\theta}(K - \mu_0)\right\}$ for $\mu \in \Omega$, hence $\mathrm{Re}\left\{\mathrm{e}^{i\theta}(\mu - K)\right\} = \mathrm{Re}\left\{\mathrm{e}^{i\theta}(\mu - \mu_0)\right\} - \mathrm{Re}\left\{\mathrm{e}^{i\theta}(K - \mu_0)\right\} \geqslant 0$, or

$$\mathrm{Re}\left\{\mathrm{e}^{i\theta}\left(\frac{q(x)}{w(x)} + rp(x) - K\right)\right\} \geqslant 0$$

on (a, b) for $r \geqslant 0$ and $\theta \in (\theta_1, \theta_2)$, which means $(\theta, K) \in \Pi$, or $\theta \in E$. This proves I of lemma 2.5.

(ii) For $\lambda_0 \in \mathbb{C} \setminus \Omega$, choose $(\theta_0, K_0) \in \Pi$ and $\delta_0 > 0$ such that $\lambda_0 \in \Lambda_{\theta_0, K_0}$ and

$$\mathrm{Re}\left\{\mathrm{e}^{i\theta_0}(K_0 - \lambda_0)\right\} = \delta_0 > 0.$$

Since E has more than one point, we can choose $\widetilde{\theta} \in E$ such that $\widetilde{\theta} \neq \theta_0 (\text{mod}\, \pi)$. Without loss of generality, we suppose that $\widetilde{\theta} > \theta_0$. It follows from the conclusion of

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(i) that $(\theta_0, \widetilde{\theta}) \subset E$. For each $\theta \in (\theta_0, \widetilde{\theta})$, there exists a point $K(\theta) \in \partial\Omega$ such that $(\theta, K(\theta)) \in \Pi$. By the definition of Π we have that

$$\operatorname{Re}\left\{e^{i\theta_0}(K(\theta) - K_0)\right\} \geqslant 0, \ \operatorname{Re}\left\{e^{i\theta}(K_0 - K(\theta))\right\} \geqslant 0. \tag{2.14}$$

If we set $r(\theta) = |K(\theta) - K_0|$ and $K(\theta) - K_0 = r(\theta)\, e^{i\eta(\theta)}$, then (2.14) means that

$$\cos(\theta_0 + \eta(\theta)) \geqslant 0, \ \cos(\theta + \eta(\theta)) \leqslant 0.$$

This together with $\theta > \theta_0$ gives that $\theta + \eta(\theta) \geqslant \pi/2 \geqslant \theta_0 + \eta(\theta) \pmod{2\pi}$, hence $\theta_0 + \eta(\theta) \to \pi/2 \pmod{2\pi}$ as $\theta \to \theta_0 + 0$. We claim that $r(\theta)$ is bounded in a right-neighbourhood of θ_0. Suppose on the contrary, there exists a sequence, say, $\{\theta_n\}$ such that $\theta_n \to \theta_0 + 0$ and $r_n = r(\theta_n) \to +\infty$ as $n \to \infty$. Choose $\eta_0 \in (\theta_0, \widetilde{\theta})$ such that $\eta_0 - \theta_0 + \pi/2 < \pi$ and a corresponding point $K(\eta_0) \in \partial\Omega$ such that $(\eta_0, K(\eta_0)) \in \Pi$, we have that

$$\begin{aligned} 0 \leqslant \frac{1}{r_n}\operatorname{Re}\left\{e^{i\eta_0}(K(\theta_n) - K(\eta_0))\right\} &= \frac{1}{r_n}\operatorname{Re}\left\{e^{i\eta_0}\left(K(\theta_n) - K_0 - [K(\eta_0) - K_0]\right)\right\} \\ = \operatorname{Re}\left\{e^{i\eta_0}\left(e^{i\eta(\theta_n)} - [K(\eta_0) - K_0]/r_n\right)\right\} &\to \cos(\eta_0 - \theta_0 + \pi/2) < 0, \end{aligned}$$

which is a contradiction. Since $r(\theta)$ is bounded and $\theta + \eta(\theta) \to \pi/2 \pmod{2\pi}$ as $\theta \to \theta_0 + 0$, we have that $\operatorname{Re}\left\{e^{i\theta}(K_0 - K(\theta))\right\} = -r(\theta)\cos(\theta + \eta(\theta)) \to 0$ as $\theta \to \theta_0 + 0$. Hence

$$\begin{aligned} \operatorname{Re}\left\{e^{i\theta}(\lambda_0 - K(\theta))\right\} &= \operatorname{Re}\left\{e^{i\theta}(\lambda_0 - K_0)\right\} + \operatorname{Re}\left\{e^{i\theta}(K_0 - K(\theta))\right\} \\ &\to \operatorname{Re}\left\{e^{i\theta_0}(\lambda_0 - K_0)\right\} = -\delta_0 < 0 \end{aligned}$$

as $\theta \to \theta_0 + 0$. Therefore, there exists $\xi \in (\theta_0, \widetilde{\theta})$ such that for all $\theta \in (\theta_0, \xi)$, $\operatorname{Re}\left\{e^{i\theta}(\lambda_0 - K(\theta))\right\} < 0$. This means that $\lambda_0 \in \Lambda_{\theta,\,K}$ for $\theta \in (\theta_0, \xi)$. This completes the proof. □

Let T be an operator in Hilbert space H, the numerical range of operator T is denoted by $\Theta(T)$ as follows:

$$\Theta(T) := \left\{ \frac{(Tu, u)}{(u, u)} \,:\, 0 \neq u \in D(T) \right\}.$$

Let $\sigma \in (0, \frac{\pi}{2})$ and let S_σ denote the closed sector

$$S_\sigma := \{0\} \cup \{z \in \mathbb{C} : |\arg(z)| \leqslant \sigma\},$$

in the right complex half-plane, cf. figure 1.

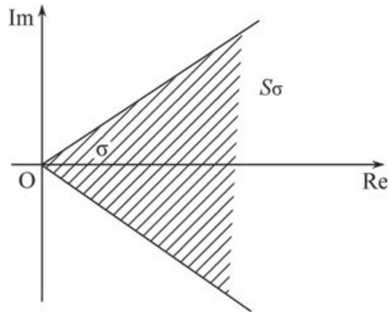

Figure 1. A sector S_σ.

An operator T in Hilbert space H is said to be accretive if the numerical range $\Theta(T)$ is a subset of the right half-plane, that is, if

$$\mathrm{Re}(Tu, u) \geqslant 0 \text{for all } u \in D(T).$$

If T is closed, then $\mathrm{def}(T - \zeta) = \mu$ is constant for $\mathrm{Re}\zeta < 0$. If $\mu = 0$, the left open half-plane is contained in the resolvent set $P(T)$ with

$$\left.\begin{array}{l}(T+\lambda)^{-1} \in \mathcal{B}(H), \\ \|(T+\lambda)^{-1}\| \leqslant \mathrm{Re}\lambda^{-1}\end{array}\right\} \text{ for Re } \lambda > 0. \tag{2.15}$$

An operator T satisfying (2.15) will be said to be m-accretive. An m-accretive operator T is maximal accretive, in the sense that T is accretive and has no proper accretive extension.

We shall say that T is quasi-accretive if $T + \alpha$ is accretive for some scalar α. This is equivalent to the condition that $\Theta(T)$ is contained in a half-plane of the form $\mathrm{Re}\zeta \geqslant$const. In the same way we say that T is quasi-m-accretive if $T + \alpha$ is m-accretive for some α. Like an m-accretive operator, a quasi-m-accretive operator is maximal quasi-accretive and densely defined.

A linear operator T in a Hilbert space H is said to be sectorial with vertex γ and semi-angle θ if the numerical range $\Theta(T - \gamma I)$ lies in a sector S_σ for some $\gamma \in \mathbb{R}$, T is said to be m-sectorial if it is sectorial and quasi-m-accretive.

Now, we turn to define operators by the formal differential operator τ. For any compact interval $[\alpha, \beta] \subset [a, b)$, using integration by parts, we obtain the so-called Green's formula.

$$\int_\alpha^\beta \left[\overline{\psi}\tau\phi - \phi\overline{\tau^+\psi}\right] = [\phi, \psi](\beta) - [\phi, \psi](\alpha),$$

where $\tau^+\psi = -(\overline{p}\psi')' + \overline{q}\psi = \overline{\tau}\psi$, $[\phi, \psi] = (\phi\overline{p\psi'} - \overline{\psi}p\phi')(x)$. Since $\tau^+ = \overline{\tau}$, we know τ is formally J-symmetric. Set

$$D_{\max} = \left\{y \in L^2_w[a, b) : py' \in \mathrm{AC}_{\mathrm{loc}}[a, b), \text{ and } \tau y \in L^2_w[a, b)\right\},$$

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we define the 'maximal operator' $T_M = T_M(\tau)$ by

$$T_M : D_{\max} \to L^2_w[a,b)$$
$$y \mapsto T_M y = \tau y = \frac{1}{w}[-(py')' + qy].$$

Similarly, we define

$$D_0' = \{y \in D_{\max} : y(a) = p(a)y'(a) = 0, \text{ and } y = 0 \text{ outside a compact subset of } [a,b)\},$$

$T|_{D_0'} = T_0'$ is called the pre-minimal operator, and T_0' is closable, its closure, denoted by T_0, is called the minimal operator, its domain is denoted by D_0. By the book of Edmund and Evans [6, P144, theorem 10.7], we know that $\overline{D_0'} = L^2_w[a,b)$, and $T_0(\tau)^* = T_M(\overline{\tau})$, $T_0(\overline{\tau}) = T_M(\tau)^*$, so $JT_0J \subset T_0^*$, where J is the common conjugation, this means that T_0 is a J-symmetric operator. It's easy to see that for each $y \in D_0$,

$$y(a) = p(a)y'(a) = 0, \text{ and } [y,z](b) = 0 \text{ for all } z \in D_{\max}.$$

Lemma 2.6. *Assume that E has more than one point. We have that T_0 is a densely defined closed sectorial operator in $L^2_w[a,b)$.*

Proof. The denseness of T_0 follows from [6, theorem 10.5]. Since E has more than one point, we choose $\theta_1, \theta_2 \in E$ such that $\theta_1 \neq \theta_2$. Choose $\lambda_0 \in \Lambda_{\theta_1,K_1} \cap \Lambda_{\theta_2,K_2}$. By the definition of T_0, we see that for $u, v \in D(T_0)$

$$\begin{aligned}\langle (T_0 - \lambda)u, v\rangle &= \int_a^b \bar{v} w (T_0 - \lambda_0)u \\ &= \int_a^b \bar{v}[-(pu')' + (q - \lambda_0 w)u] \\ &= \int_\alpha^\beta (pu'\overline{v'} + (q - \lambda_0 w)u\bar{v}),\end{aligned}$$

Set $r(x) = |q(x) - \lambda_0 w(x)|$ and

$$q(x) - \lambda_0 w(x) = r(x)\,\mathrm{e}^{i\alpha(x)}, \ \alpha_1(x) = \theta_1 + \alpha(x), \ \alpha_2(x) = \theta_2 + \alpha(x). \tag{2.16}$$

Then lemma 2.1 ensures that $\cos\alpha_1, \cos\alpha_2 \geqslant 0$ on $[a,b)$. Since

$$\begin{aligned}\sin^2(\theta_2 - \theta_1) &= \sin^2(\alpha_2 - \alpha_1) \\ &= \cos^2\alpha_2 + \cos^2\alpha_1 - 2\cos\alpha_2\cos\alpha_1\cos(\alpha_1 - \alpha_2) \\ &\leqslant (\cos\alpha_2 + \cos\alpha_1)^2,\end{aligned}$$

so we obtain

$$\cos\alpha_1 + \cos\alpha_2 \geqslant |\sin(\theta_1 - \theta_2)| = \delta_0 > 0. \tag{2.17}$$

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Set

$$p_\theta = [\mathrm{e}^{i\theta_1} + \mathrm{e}^{i\theta_2}]p, (q - \lambda_0 w)_\theta = [\mathrm{e}^{i\theta_1} + \mathrm{e}^{i\theta_2}](q - \lambda_0 w). \tag{2.18}$$

Then (2.17) implies

$$\mathrm{Re}\,\{(q - \lambda_0 w)_\theta\} \geqslant \delta_0 |q - \lambda_0 w|. \tag{2.19}$$

Similarly, we have that

$$\mathrm{Re}\,\{p_\theta\} \geqslant \epsilon_0 |p|, \tag{2.20}$$

where $\epsilon_0 > 0$ is a constant sufficiently small. Hence we have for $y \in D(T_0)$,

$$\begin{aligned}
&\mathrm{Re}\,\left\{(\mathrm{e}^{i\theta_1} + \mathrm{e}^{i\theta_2})\langle (T_0 - \lambda_0)\, y, y\rangle\right\} \\
&\quad = \mathrm{Re}\int_a^b p_\theta\, |y'|^2 + \mathrm{Re}\int_a^b (q - \lambda_0 w)_\theta\, |y|^2 \\
&\quad \geqslant \int_a^b \left[\epsilon_0 |p||y'|^2 + \delta_0 |q - \lambda_0 w||y|^2\right].
\end{aligned}$$

Similarly we have that

$$\mathrm{Im}\,\left\langle (\mathrm{e}^{i\theta_1} + \mathrm{e}^{i\theta_2})(T_0 - \lambda_0)\, y, y\right\rangle \leqslant \int_a^b \left[2|p|\,|y'|^2 + 2|q - \lambda_0\, w|\,|y|^2\right]. \tag{2.21}$$

So $\tilde{T} = (\mathrm{e}^{i\theta_1} + \mathrm{e}^{i\theta_2})(T_0 - \lambda_0 I)$ is a sectorial operator. Since $\tilde{T}$ is obtained by contraction and rotation of T_0, T_0 is a sectorial operator. It is obvious that the operator T_0 is a closed operator. This completes the proof. □

2.2. Sesquilinear forms and Friedrichs extension

The sesquilinear forms in Hilbert space are closely related to the associated operators, bounded forms and bounded operators are equivalent, there is no such obvious relationship for unbounded forms and operators. However, there exists a closed theory on the relationship between semi-bounded symmetric forms and semi-bounded self-adjoint operators, this theory is extended to non-symmetric forms and operators within certain restrictions. Among these restrictions, the sectorial operators and sectorial forms are necessary.

A sesquilinear form $\mathsf{t}[u, v]$ is defined for u, v both belonging to a linear manifold D of a Hilbert space H, $\mathsf{t}[u, v]$ is complex-valued and linear in $u \in D$ for each fixed $v \in D$ and semilinear in $v \in D$ for each fixed $u \in D$. D is called the domain of t and is denoted by $D(\mathsf{t})$; t is densely defined if $D(\mathsf{t})$ is dense in H; $\mathsf{t}[u] = \mathsf{t}[u, u]$ is called the quadratic form associated with $\mathsf{t}[u, v]$; a sesquilinear form t is said to be symmetric if $\mathsf{t}[u, v] = \overline{\mathsf{t}[v, u]}$.

For a sesquilinear form t, its numerical range is defined by

$$\Theta(\mathsf{t}) = \{\mathsf{t}(u) | u \in D(\mathsf{t}), ||u|| = 1\}.$$

A nonsymmetric sesquilinear form t is said to be a sectorially bounded form from the left (or simply sectorial) if $\Theta(\mathsf{t})$ is a subset of a sectorial of the form

$$|\arg(\zeta - \gamma)| \leqslant \theta, \quad 0 \leqslant \theta < \frac{\pi}{2}, \gamma \text{ is real},$$

γ is called the vertex and θ is corresponding semi-angle of the form t.

If T is a sectorial operator, then the form defined by

$$\mathsf{t}[u, v] = \langle Tu, v\rangle \text{ with } D(\mathsf{t}) = D(T) \tag{2.22}$$

is also a sectorial form. By [**14**, theorem 1.27, P318], we know that a sectorial operator is form-closeable, i.e., the form t defined above is closable, it closure is denoted by $\tilde{\mathsf{t}}$. Particularly, If T is a densely defined, sectorial operator, by the first representation theorem ([**14**, theorem 2.1, P322]), the closure $\tilde{\mathsf{t}}$ of t generated by (2.22), generates an m-sectorial operator $T_{\tilde{\mathsf{t}}}$, $T_{\tilde{\mathsf{t}}}$ is called the Friedrichs extension of T, we denote this operator by T_F for convenience, i.e., the Friedrichs extension of a sectorial operator is the form extension of the corresponding sectorial form.

For the minimal operator T_0, by direct calculation, we deduce that the corresponding sesquilinear form is

$$\mathsf{s}[u, v] = \int_a^b [pu'\overline{v'} + qu\overline{v}], u, v \in D(\mathsf{s}) = D_0.$$

We know by lemma 2.6 that T_0 is a closed sectorial operator, so the Friedrichs extension exists. In the next two sections, we will characterize the Friedrichs extensions of T_0.

3. The Friedrichs extension under case I

Firstly, we give some spectral results on Hamiltonian differential system

$$\begin{cases} u' = A\,u + B\,v + \xi W_2\,v, \\ v = C\,u - A^*\,v - \xi W_1\,u \end{cases} \text{on } [a, b) \tag{3.1}$$

on the $\mathbb{C}^{2n}$ valued (column) functions $Y = (u^T, v^T)^T$, where u, v are $\mathbb{C}^n$ valued functions, u^T is the transpose of u, A, B, C, W_1 and W_2 are locally integrable, complex-valued $n \times n$ matrices on $[a, b)$, B, C, W_1, W_2 are Hermitian matrices and $W_1(t) > 0, W_2(t) \geqslant 0$ on $[a, b)$, ξ is the so-called spectral parameter. Assume that the **definiteness condition** (see, e.g., [**1**, chapter 9, p253]) holds:

$$\int_a^b Y^*WY > 0 \text{ for each non-trivial solution } Y \text{ of (3.1)},$$

where $W = \text{diag}(W_1, W_2)$. Let $L^2_W := L^2_W[a, b)$ be the space of Lebesgue measurable $2n$-dimensional functions f satisfying $\int_a^b f^*(s)W(s)f(s)ds < \infty$. We say that (3.1) is in the **limit point case** at b if there exists exactly n's linearly independent solutions of (3.1) belong to L^2_W for $\xi = \pm i$. Particularly, if $n = 1$ and A, B, C are real functions, then (3.1) is in the limit point case at b if and only if there exists a unique solution of (3.1) belonging to L^2_W for $\xi = i$ or $\xi = -i$.

Let D be the maximal domain associated to (3.1), i.e. $(u^T, v^T)^T \in D$ if and only if $(u^T, v^T)^T \in AC_{\mathrm{loc}} \cap L^2_W$ and there exists an element $(f^T, g^T)^T \in L^2_W$ such that

$$\begin{cases} u' = A\,u + B\,v + W_2\,g, \\ v' = C\,u - A^*\,v - W_1\,f. \end{cases} \tag{3.2}$$

It is well known (cf. [**11**, **16**]) that (3.1) is in the limit point case at b if and only if

$$Y_1^*(x) J Y_2(x) \to 0, \text{ as } x \to b, \; J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \tag{3.3}$$

for all $Y_1, Y_2 \in D$, and for each $\xi \in \mathbb{C}$ with $\mathrm{Im}\xi \neq 0$ there exists a Green function $G(t, s, \xi)$ such that for $F = (f^T, g^T)^T \in L^2_W$, (3.2) has an L^2_W-solution Y given by

$$Y = \begin{pmatrix} u \\ v \end{pmatrix} = \int_a^b G(\cdot, s, \xi) W(s) F(s)\,\mathrm{d}s. \tag{3.4}$$

Now we give the asymptotic behaviours of elements of $\mathcal{D}_{\max}$ under case I.

THEOREM 3.1. *Assume that E has more than one point, and (1.1) is in case* I. *For each $y \in D_{\max}$, we obtain*

$$y \in L^2_{|q|}, \text{ and } y' \in L^2_{|p|},$$

and for all $y_1, y_2 \in D_{\max}$,

$$p(x)\, y_1(x)\, y_2'(x) \to 0, \text{ as } x \to b. \tag{3.5}$$

Proof. Since E has more than one point, we choose $(\theta_1, K_1), (\theta_2, K_2) \in \Pi$ (with $\theta_1 \neq \theta_2 (\mathrm{mod}\, \pi)$) and $\lambda_0 \in \Lambda_{\theta_1,K_1} \cap \Lambda_{\theta_2,K_2}$. Then by the definition of $\Lambda_{\theta,K}$, we obtain

$$\mathrm{Re}\left\{\mathrm{e}^{i\theta_j}(q - \lambda_0\, w)\right\} \geqslant \delta_j w, \; \mathrm{Re}\left\{\mathrm{e}^{i\theta_j} p\right\} \geqslant 0, \; j = 1, 2 \tag{3.6}$$

for some $\delta_1, \delta_2 > 0$. Set

$$\begin{aligned} &r(x) = |q(x) - \lambda_0\, w(x)|, \; q(x) - \lambda_0\, w(x) = r(x)\,\mathrm{e}^{i\alpha(x)}, \; \alpha_j(x) = \theta_j + \alpha(x), \\ &\widetilde{r}(x) = |p(x)|, \; p(x) = \widetilde{r}(x)\,\mathrm{e}^{i\beta(x)}, \; \beta_j(x) = \theta_j + \beta(x), \; j = 1, 2. \end{aligned} \tag{3.7}$$

For $j = 1$, Let y be a solution of (1.1) with $\lambda = \lambda_0$. Set $u = y$, $v = -i\,\mathrm{e}^{i\theta_1} p y'$, then (1.1) is transformed into the Hamiltonian differential system

$$u' = B\,v + \xi w_2\,v, \quad v' = C\,u - \xi w_1\,u \tag{3.8}$$

with the new spectral parameter $\xi = i$, where

$$\begin{aligned} &C(x) = r(x) \sin \alpha_1(x), \; w_1(x) = r(x) \cos \alpha_1(x), \\ &B(x) = \sin \beta_1(x)/\widetilde{r}(x), \; w_2(x) = \cos \beta_1(x)/\widetilde{r}(x). \end{aligned} \tag{3.9}$$

This is the Hamiltonian differential system (3.1) with $n = 1$, $A(x) \equiv 0$ and $\xi = i$. Clearly, $w_1 = \mathrm{Re}\left\{\mathrm{e}^{i\theta_1}(q - \lambda_0\, w)\right\} \geqslant \delta_0\, w > 0$, $w_2 = \frac{\mathrm{Re}\left\{\mathrm{e}^{i\theta_1} p(t)\right\}}{r_2^2} \geqslant 0$ by (3.6). We

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note further that the coefficients of the Hamiltonian system (3.8) are real functions. It is easy to verify that the definiteness condition holds for the system (3.8). Therefore, (1.1) is in case I or II w.r.t. $(\theta_1, K_1) \in \Pi$ if and only if (3.8) is in the limit point case at b.

Let $D(\theta_1)$ be the maximal domain associated to (3.8), i.e., $(u, v)^T \in D(\theta_1)$ if and only if $(u, v)^T \in AC_{\mathrm{loc}} \cap L^2_W$ and there exists an element $(f, g)^T \in L^2_W$ such that

$$u' = B\,v + w_2\,g, \quad v' = C\,u - w_1\,f. \tag{3.10}$$

For $y_0 \in D(\tau)$, set

$$f_0 = \tau y_0,\ \ g_0 = f_0 - \lambda_0 y_0. \tag{3.11}$$

Then y_0 satisfies

$$(\tau - \lambda_0)y = w^{-1}[-(py')' + (q - \lambda_0\,w)y] = g_0. \tag{3.12}$$

Set $u_0 = y_0$, $v_0 = -i\,\mathrm{e}^{i\theta_1} p y_0'$. Then (u_0, v_0) satisfies

$$u' = B\,v + iw_2\,v,\ \ v' = C\,u - iw_1\,u - w_1 f_1,\ \ f_1 = \frac{w}{w_1}\left(-i\,\mathrm{e}^{i\theta_1} g_0\right). \tag{3.13}$$

Conversely, if (u, v) satisfies (3.13), then $y = u$ solves (3.12).

Let g_0 be given in (3.11). Consider the equation (3.13), we get from (3.4) that (3.13) has a solution $(u_1, v_1)^T$ such that $u_1 \in L^2_{w_1}, v_1 \in L^2_{w_2}$ and $v_1 = -i\,\mathrm{e}^{i\theta_1} p\,u_1'$. Set $y_1 = u_1$. Then y_1 satisfies (3.12), hence $(\tau - \lambda_0)(y_0 - y_1) = 0$. Note that $y_1 = u_1 \in L^2_{w_1}$ and $w_1 \geqslant \delta w$ implies that $y_1 \in L^2_w$. Thus, $y_1 - y_0$ is an L^2_w-solution of $\tau y = \lambda_0 y$. Since τ is in case I w.r.t. (θ_1, K_1), it follows from (2.8) that $y_1 - y_0 \in L^2_{w_1}$ and $v_1 - v_0 \in L^2_{w_2}$. This together with $y_1 \in L^2_{w_1}$ and $v_1 \in L^2_{w_2}$ gives that $y_0 \in L^2_{w_1}$ and $v_0 \in L^2_{w_2}$. In fact, we have proved that for $y \in D(\tau)$,

$$\int_a^b |q - \lambda_0 w| \cos\alpha_1 |y|^2 < \infty,\ \ \int_a^b |p| \cos\beta_1 |y'|^2 < \infty, \tag{3.14}$$

where α_1 and β_1 are defined in (3.7). Recall that $g_0 \in L^2_w$, or $-i\,\mathrm{e}^{i\theta_1} g_0 \in L^2_w$ and $w_1 \geqslant \delta w$ implies $f_1 \in L^2_{w_1}$. It follows (3.13) that (y_0, v_0) satisfies (3.10) with $f = iy_0 + f_1$ and $g = iv_0$. This yields that

$$y \in D(\tau) \quad \Longrightarrow (y, v)^T \in D(\theta_1) \text{ with } v = -i\,\mathrm{e}^{i\theta_1} py'. \tag{3.15}$$

Note that $Y \in D(\theta_1)$ if and only if $\overline{Y} \in D(\theta_1)$. Then for $\overline{y} \in D(\tau)$, we have from (3.15) that $(\overline{y}, \overline{v})^T \in D(\theta_1)$ with $\overline{v} = -i\,\mathrm{e}^{i\theta_1} p\overline{y}'$, hence

$$\overline{y} \in D(\tau) \Longrightarrow (y, v)^T \in D(\theta_1) \text{ with } v = i\,\mathrm{e}^{-i\theta_1} \overline{p} y'. \tag{3.16}$$

Let $y_1, \overline{y}_2 \in D(\tau)$. Since (3.8) is in the limit point case at b and $(y_j, v_j)^T \in D(\theta_1)$ for $j = 1, 2$ with $v_1 = -i\,\mathrm{e}^{i\theta_1} py_1'$ and $v_2 = i\,\mathrm{e}^{-i\theta_1} \overline{p} y_2'$ by (3.15) and (3.16), respectively, we get from (3.3) that

$$(\overline{y}_2, \overline{v}_2)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}y_1\\ v_1\end{pmatrix} = i\,\mathrm{e}^{i\theta_1}\,p(y_1'\,\overline{y}_2 - y_1\,\overline{y}_2') \to 0 \tag{3.17}$$

as $x \to b$. Furthermore, for $y_1, y_2 \in D(\tau)$, since $(y_j, v_j)^T \in D(\theta_1)$ by (3.15) with $v_j = -ie^{i\theta_1}py_j'$, $j = 1, 2$, (3.3) also gives that

$$(\overline{y}_2, \overline{v}_2)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}y_1\\ v_1\end{pmatrix} = i\,\mathrm{e}^{-i\theta_1}(\overline{p}\,\overline{y}_2'\,y_1 + \mathrm{e}^{2i\theta_1}p\,y_1'\,\overline{y}_2) \to 0$$

as $x \to b$, or

$$\overline{p}\,\overline{y}_2'\,y_1 + \mathrm{e}^{2i\theta_1}p\,y_1'\,\overline{y}_2 \to 0,\ \ y_1, y_2 \in D(\tau) \tag{3.18}$$

as $x \to b$. Similarly, for $j = 2$, the above methods also give that (3.18) holds for $y_1, y_2 \in D(\tau)$ with θ_1 replaced by θ_2. Therefore, we have

$$\left(\mathrm{e}^{2i\theta_1} - \mathrm{e}^{2i\theta_0}\right)p\,y_1'\,\overline{y}_2 \to 0 \tag{3.19}$$

as $x \to b$. This clearly gives (3.5) for $y_1, y_2 \in D(\tau)$ since $\theta_1 \neq \theta_2 \pmod{\pi}$.

Finally, we prove $y \in L^2_{|q|}$ and $y' \in L^2_{|p|}$ for $y \in D(\tau)$. In fact, the similar proof as in (3.14) gives that for $y \in D(\tau)$,

$$\int_a^b |q - \lambda_0 w| \cos\alpha_2 |y|^2 < \infty,\ \int_a^b |p| \cos\beta_2 |y'|^2 < \infty. \tag{3.20}$$

Therefore, (3.14), (3.20) and (2.17) together yield that $\int_a^b |q - \lambda_0 w||y|^2 < \infty$ and $\int_a^b |p||y'|^2 < \infty$, or $y \in L^2_{|q|}$ and $y' \in L^2_{|p|}$ since $y \in L^2_w$. This completes the proof. □

Theorem 3.2. *Assume E has more than one point, and (1.1) is in case* I. *Then the J-self-adjoint extension T defined by*

$$\begin{aligned} T : D(T) &\to L^2_w[a, b) \\ y \mapsto Ty = \tau y &= \frac{1}{w}[-(py')' + qy] \end{aligned} \tag{3.21}$$

is the Friedrichs extension of T_0, where $D(T) = \{y \in D_{\max} : y(a) = 0\}$.

Proof. Since E has more than one point, we choose $(\theta_1, K_1), (\theta_2, K_2) \in \Pi$ (with $\theta_1 \neq \theta_2 (\mathrm{mod}\,\pi)$) and $\lambda_0 \in \Lambda_{\theta_1, K_1} \cap \Lambda_{\theta_2, K_2}$. Since (1.1) is in case I, we know that T is a J-self-adjoint extension of T_0 by theorem 4.4 in paper [**3**]. Next, we define

$$\mathsf{t}[u, v] = \int_a^b [pu'\overline{v'} + qu\overline{v}], u, v \in D(\mathsf{t}), \tag{3.22}$$

$$D(\mathsf{t}) = \left\{u \in AC_{\mathrm{loc}}[a, b) : u(a) = 0, u \in L^2_w[a, b) \cap L^2_{|q-\lambda_0 w|}[a, b), u' \in L^2_{|p|}[a, b)\right\}. \tag{3.23}$$

Since (1.1) is in case I, we see by theorem 3.1 that $D(T_0) \subseteq D(\mathsf{t})$, so t is densely defined. Similar method as in lemma 2.6 can deduce t as a sectorial operator. Now we turn to prove that t is closed.

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Suppose that $y_n \in D(\mathsf{t})$, $y_n \to y$ in $L^2_w[a,b)$ and

$$\mathsf{t}[y_n - y_m] = \int_a^b [p|y_n' - y_m'|^2 + q|y_n - y_m|^2] \to 0$$

for $n, m \to \infty$. Let $y_{nm} = y_n - y_m$ for convenience. Since $\mathsf{t}[y_{nm}] \to 0$, we obtain $(\mathrm{e}^{i\theta_1} + \mathrm{e}^{i\theta_2})\mathsf{t}[y_{nm}] \to 0$, and

$$\lambda_0 \int_a^b w|y_{nm}|^2 \leqslant 2|\lambda_0| \int_a^b w\left(|y_n - y|^2 + |y_m - y|^2\right) \to 0.$$

So we obtain that

$$\int_a^b [p_\theta |y_{nm}'|^2 + (q - \lambda_0 w)_\theta |y_{nm}|^2] \to 0. \tag{3.24}$$

Since $\mathrm{Re}\,\{(q - \lambda_0 w)_\theta\} \geqslant \delta_0 |q - \lambda_0 w|$ and $\mathrm{Re}\,\{p_\theta\} \geqslant \epsilon_0 |p|$ for some constants $\delta_0, \epsilon_0 > 0$ by (2.19) and (2.20), and noticing t is a sectorial operator, we obtain

$$\epsilon_0 \int_a^b |p||y_{nm}'|^2 + \delta_0 \int_a^b |q - \lambda_0 w||y_{nm}|^2 \to 0.$$

So

$$\int_a^b |p||y_{nm}'|^2 \to 0, \int_a^b |q - \lambda_0 w||y_{nm}|^2 \to 0 \text{ as } n, m \to \infty.$$

This shows that $\{y_n'\}$ and $\{y_n\}$ are Cauchy sequences in $L^2_{|p|}[a,b)$ and $L^2_{|q-\lambda_0 w|}[a,b)$, respectively. We may assume $y_n' \to z$ in $L^2_{|p|}[a,b)$. For each fixed $x \geqslant a$,

$$\begin{aligned} |y_{nm}(x)| &= \left| \int_a^x y_{nm}'(s)\,\mathrm{d}s \right| \\ &\leqslant \left(\int_a^x \frac{1}{|p(s)|}\,\mathrm{d}s \right)^{1/2} \left(\int_a^x |p(s)||y_{nm}'(s)|\,\mathrm{d}s \right)^{1/2} \\ &\leqslant \left(\int_a^x \frac{1}{|p(s)|}\,\mathrm{d}s \right)^{1/2} \left(\int_a^b |p(s)||y_{nm}'(s)|\,\mathrm{d}s \right)^{1/2} \to 0 \end{aligned}$$

as $n, m \to \infty$, since $y_n \to y$ in $L^2_w[a,b)$, we obtain $y_n(x) \to y(x)$ point-wise for $x \geqslant a$ as $n \to \infty$, hence $y(0) = \lim_{n\to\infty} y_n(0) = 0$, and $y'(x) = z(x)$, so $y \in D(\mathsf{t})$, and t is a closed sesquilinear form.

By the first representation theorem [**14**, P322, theorem 2.1], there exists an m-sectorial operator $T = T_{\mathsf{t}}$ such that $D(T) \subseteq D(\mathsf{t})$, $\mathsf{t}[u,v] = \langle Tu, v\rangle$ for $u, v \in D(T)$, and $D(T)$ is a core of t. Since $T = T_{\mathsf{t}}$, so $T_F = T_{\mathsf{t}}$ is the Friedrichs extension of T_0. We prove $D(T_F) = \{y \in D_{\max} : y(a) = 0\} := \tilde{D}$ to complete the proof. For each $y \in \tilde{D}$, then $y \in D_{\max}$, by theorem 3.1, $y' \in L^2_{|p|}[a,b)$ and $y \in L^2_{|q|}[a,b)$, this together

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with $y \in L^2_w[a,b)$ implies

$$\int_a^b |q-\lambda_0 w||y|^2 \leqslant \int_a^b |q||y|^2 + |\lambda_0| \int_a^b w|y|^2 < \infty,$$

so $y \in L^2_{|q-\lambda_0 w|}[a,b)$, thus we obtain $y \in T_F$. Conversely, for each $y \in D(T_F)$, set $T_F y = f$. By the definition of T_F, we obtain $\mathfrak{t}[y,v] = \langle T_F y, v\rangle = \langle f, v\rangle$ for all $v \in D(\mathfrak{t})$, or

$$\int_a^b wf\overline{v} = \int_a^b \left[py'\overline{v'} + qy\overline{v}\right]. \tag{3.25}$$

Set $z'(x) = w(x)f(x) - q(x)y(x)$ and for arbitrary $d < b$,

$$A_d = \{v:\ v' \in L^2[a,d],\ v(a) = v(d) = 0\}.$$

Then for $v \in A_d$

$$\int_a^d z'\,\overline{v} = z\,\overline{v}\big|_a^d - \int_a^d z\,\overline{v'} = -\int_a^d z\,\overline{v}'. \tag{3.26}$$

If we set

$$v_0(x) = \begin{cases} v(x), & x \in [a,d], \\ 0, & x \in [d,b), \end{cases}$$

then $v_0 \in D(\mathfrak{t})$, hence (3.25) and (3.26) imply that

$$\int_a^d (py' + z)\overline{v'} = 0. \tag{3.27}$$

Note that

$$\{v' :\ v \in A_d\} = \{f :\ f \in L^2[a,d],\ \int_a^d f = 0\},$$

which is the set of orthogonal to 1 in $L^2[a,d]$. Therefore, $py' + z = C_d$, where C_d is a constant, hence $z'(x) = -(py')'(x)$ a.e. $x \in [a,d]$, or $-(py')' + qy = wf$ on $[a,d]$. By the arbitrary of d we know that

$$-(py')' + qy = wf \text{ on } [a,b). \tag{3.28}$$

This means $y \in \tilde{D}$. So $D(T_F) = \tilde{D}$, this completes the proof. □

4. The Friedrichs extension under case II

When τ is not in case I, the J-self-adjoint extension T_F is comparatively complicated. Even for the case where the corresponding formal differential operator is symmetric but not in the limit-point case, the characterization of Friedrichs extensions is hard to be obtained. This problem has been studied by a lot of authors both for formal symmetric arbitrary order differential operator and Hamiltonian differential operators [**7**, **8**, **25**, **26**, **31**] and Hamiltonian differential systems [**12**, **16**, **17**] and references therein. When τ is not in case I, by operator theory we know that such domain is a restriction of the maximal domain. We should impose some restrictions on the elements of the maximal domain to instruct the domain of a operator realization. The standard method is to choose suitable number independent elements from the maximal domain to construct the corresponding boundary conditions at end-points of the interval considered. The similar result is also valid for the case where $p(x)$, $q(x)$ are complex valued (see [**3**, theorem 4.1]).

We will not follow the same line as above to give the Friedrichs extensions associated to τ. The following Friedrichs extension is given by the restriction of the maximal domain on a suitable Hilbert space. The main result of this section is also a generalization of the corresponding result in [**22**] to some extent.

THEOREM 4.1. *Assume E has more than one point, and (1.1) is in case* II. *For each $y \in D_{\max}$ with $y' \in L^2_{|p|}[a, b)$, we obtain*

$$y \in L^2_{|q|}[a, b),$$

and for all $y_1, y_2 \in D_{\max}$ with $y_1', y_2' \in L^2_{|p|}[a, b)$,

$$p(x)\, y_1(x)\, y_2'(x) \to 0, \text{ as } x \to b. \tag{4.1}$$

Proof. Similar to the proof of theorem 3.1, we choose $(\theta_1, K_1), (\theta_2, K_2) \in \Pi$ (with $\theta_1 \neq \theta_2 (\mathrm{mod}\, \pi)$) and $\lambda_0 \in \Lambda_{\theta_1,K_1} \cap \Lambda_{\theta_2,K_2}$. Since (1.1) is in case II (with λ replaced by λ_0), we obtain the Hamiltonian system (3.8) is in the limit point case at b.

Let $D(\theta_1)$ denote the maximal domain associated to (3.8), we obtain that $y \in D_{\max}$ with $y' \in L^2_{|p|}[a, b)$ yields $(y, v)^T \in D(\theta_1)$, where$v = -ie^{i\theta_1} py'$, and $\overline{y} \in D_{\max}$ with $\overline{y}' \in L^2_{|p|}[a, b)$ yields $(y, v)^T \in D(\theta_1)$, where$v = ie^{-i\theta_1} \overline{p} y'$. Then as $x \to b$, we obtain

$$\overline{p}\, \overline{y}_2'\, y_1 + \mathrm{e}^{2i\theta_1} p\, y_1'\, \overline{y}_2 \to 0 \quad \text{for all } y_1, y_2 \in D_{\max} \quad \text{with } y_1', y_2' \in L^2_{|p|}[a, b).$$

Similar method implies

$$\overline{p}\, \overline{y}_2'\, y_1 + \mathrm{e}^{2i\theta_2} p\, y_1'\, \overline{y}_2 \to 0 \quad \text{for all } y_1, y_2 \in D_{\max} \quad \text{with } y_1', y_2' \in L^2_{|p|}[a, b).$$

The above two formula imply (4.1) holds, so we complete the first part of theorem 4.1. Similar argument as theorem 3.1 deduce $y \in L^2_{|q|}[a, b)$ for all $y \in D_{\max}$ with $y' \in L^2_{|p|}$. This completes the proof. □

THEOREM 4.2. *Suppose that E has more than one point and (1.1) is in case* II. *Then the J-self-adjoint extension T defined by*

$$\begin{aligned} T : D(T) &\to L^2_w[a, b) \\ y \mapsto Ty &= \tau y = \frac{1}{w}[-(py')' + qy] \end{aligned} \tag{4.2}$$

is the Friedrichs extension of T_0, where $D(T) = \{y \in D_{\max} : y(a) = 0, y' \in L^2_{|p|}[a, b)\}$.

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Proof. Since E has more than one point, we choose $(\theta_1, K_1), (\theta_2, K_2) \in \Pi$ (with $\theta_1 \neq \theta_2 (\mathrm{mod}\, \pi)$) and $\lambda_0 \in \Lambda_{\theta_1,K_1} \cap \Lambda_{\theta_2,K_2}$. We define the sesquliniear form $\mathfrak{t}$ as (3.22), then the similar method implies $\mathfrak{t}$ is a densely defined closed sectorial operator. By the first representation theorem [**14**, P322, theorem 2.1], the m-sectorial operator $T = T_{\mathfrak{t}}$ defined by $\mathfrak{t}[u, v] = \langle Tu, v\rangle$ for $u, v \in D(T)$ is the Friedrichs extension of T_0. Now, using the similar method as theorem 3.2, we obtain

$$D(T_F) = \{y \in D_{\max} : y(a) = 0, y' \in L^2_{|p|}[a, b)\} := \tilde{D}.$$

Finally, we turn to prove T_F is a J-self-adjoint operator by three steps.

(1) $JT_F J \subset T_F^*$**.** It is equivalent to prove that

$$\overline{y}_0 \in \tilde{D} \Longrightarrow y_0 \in D(T_F^*), JT_F J y_0 = T_F^* y_0. \tag{4.3}$$

Note that for $Jy_0 = \overline{y}_0 \in \tilde{D}$ and all $y \in \tilde{D}$

$$\begin{aligned}
\langle T_F y, y_0\rangle - \langle y, JT_F J y_0\rangle &= \int_a^b \left(\overline{y_0} w\tau y - y w\overline{\tau^+ y_0}\right) \\
&= \int_a^b \left(\overline{y_0}\left[-(py')' + qy\right] - y\left[\overline{-(\overline{p}y_0')' + \overline{q}y_0}\right]\right) \\
&= \int_a^b \left(y\overline{(\overline{p}y_0')'} - \overline{y_0}(py')'\right) = \int_a^b \left(y(p\overline{y_0'})' - \overline{y_0}(py')'\right) \\
&= p\left(y\overline{y_0'} - \overline{y_0}y'\right)\Big|_a^b.
\end{aligned}$$

This equality together with the boundary condition $y(a) = \overline{y_0}(a) = 0$ gives

$$\langle T_F y, y_0\rangle - \langle y, JT_F J y_0\rangle = p\left(y\overline{y_0'} - \overline{y_0}y'\right)(b), \tag{4.4}$$

(here we denote $\lim_{x\to b} p(y\overline{y_0'} - \overline{y_0}y')(x)$ by $p(y\overline{y_0'} - \overline{y_0}y')(b)$ since this limit always exists). Since E has more than one point and (1.1) is in case II, we know from theorem 4.1 that $p(y\overline{y_0'} - \overline{y_0}y')(b) = 0$, and hence we have $y_0 \in D(T_F^*)$ and $JT_F J y_0 = T_F^* y_0$ by (4.4). This proves that $T_F \subset JT_F^* J$.

(2) T_F **is closed operator.** Since for $u, v \in D(\alpha)$, we find that

$$\langle (T_F - \lambda_0)\, u, v\rangle = \int_a^b \left[p\, u'\, \overline{v}' + (q - \lambda_0 w)\, u\, \overline{v}\right].$$

Suppose $y_n \in \tilde{D}$ such that $y_n \to y_0$ and $T_F y_n \to f_0$, or

$$y_n \to y_0,\ (T_F - \lambda_0 I)y_n \to g_0 = f_0 - \lambda_0 y_0 \quad \text{in } L^2_w[a, b).$$

Let $y_{nm} = y_n - y_m$, as (3.24), we have that $\{y_n\}$ and $\{y_n'\}$ are Cauchy sequences in $L^2_{|q|}[a, b)$ and $L^2_{|p|}[a, b)$, respectively. Since $y_n \to y$ in $L^2_w[a, b)$ as $n \to \infty$ and $\{y_n'\}$

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is convergent in $L^2_{|p|}[a,b)$, we see that,

$$y_n(x) = \int_a^x \frac{1}{p} p y_n' \to y_0(x)$$

pointwise in x as $n \to \infty$ on $[a,b)$. It follows from $\tau y_n = T_F y_n = f_n$ that

$$y_n(x) = \int_a^x \frac{1}{p} \left(p y_n'(a) + \int_a^s [w\, f_n - q\, y_n] \right).$$

Note that the convergence of each sequence in the above equality except for $\{p y_n'(a)\}$ implies $\{p y_n'(a)\}$ is convergent, and hence by letting $n \to \infty$ we have that

$$y(x) = \int_a^x \frac{1}{p} \left(\xi + \int_a^s [w\, f - q\, y] \right),$$

where $p y_n'(a) \to \xi$ as $n \to \infty$. This means that $\tau y = f$ and $y(a) = 0$, and hence $y \in \tilde{D}$ and $T_F y = f$. This proves the closedness of T_F.

(3) T_F **is a** J**-self-adjoint operator**. By the definition of an m-sectorial operator we know $\operatorname{def} T_F = 0$. So

$$\operatorname{def}(T_F - \lambda_0 I) = \operatorname{def} T_F = 0,$$

where $\lambda_0 \in \Lambda_{\theta_1,K_1} \cap \Lambda_{\theta_2,K_2}$. Then it follows from [**6**, p115, theorem 5.5] that T_F is a J-self-adjoint extension of T_0. □

Remark 4.3. If E has more than one point and (1.1) is in case III for at least two $\theta_1 \neq \theta_2 (\operatorname{mod} \pi)$, then the Hamiltonian system (3.8) is in the limit circle case at b, by classical Friedrichs characterization of the Dirichlet boundary conditions at regular endpoint and principal solutions conditions at the singular endpoint as in paper [**10**], we can obtain the Friedrichs extension of (1.1) in case III. Here we omit the details.

5. Spectral properties of the Friedrichs extensions

In this section, we give some of the properties of the Friedrichs extensions and their applications.

Since T_F is an m-sectorial extension of T_0, we know that T_F has the smallest form-domain (that is, the domain of the associated form $\mathfrak{t}$ is contained in the domain of any other J-self-adjoint extension operator T), and T_F is the only $m-$sectorial extension of S with domain contained in $D(\mathfrak{t})$. The next theorem characterizes the spectral properties of T_F.

Theorem 5.1. *Let $\sigma(T_F)$ denote the spectrum of the Friedrichs extension operator T_F. Then we have $\sigma(T_F) \subseteq \Omega$.*

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Proof. Since each spectral point of given operator lies in the numerical range of the operator, we turn to prove the numerical range of T_F

$$\Theta(T_F) = \{\langle T_F u, u\rangle : u \in D(T_F), ||u|| = 1\}$$

is contained in Ω. For all $\lambda \in \Theta(T_F)$ there exists $u \in D(T_F), ||u|| = 1$ such that

$$\lambda = \langle T_F u, u\rangle = \mathsf{t}[u, u] = \int_a^b [p|u'|^2 + q|u|^2].$$

First, we assume $u \neq 0$ in any subinterval of $[a, b)$. Then

$$\begin{aligned}\lambda &= \int_a^b \left(p(x)\frac{|u'(x)|^2}{w(x)|u(x)|^2} + \frac{q(x)}{w(x)}\right) w(x)|u(x)|^2 \,\mathrm{d}x \\ &:= \int_a^b \left(p(x)r_1(x) + \frac{q(x)}{w(x)}\right) w(x)|u(x)|^2 \,\mathrm{d}x,\end{aligned}$$

where $r_1(x) = \dfrac{|u'(x)|^2}{w(x)|u(x)|^2} \in (0, \infty)$. For arbitrary $\epsilon > 0$, there exists $T' \in (a, b)$ such that

$$\left|\int_a^{T'} \left(p(x)r_1(x) + \frac{q(x)}{w(x)}\right) w(x)|u(x)|^2 \,\mathrm{d}x - \lambda\right| < \frac{\epsilon}{4},$$
$$\left|\int_{T'}^b \left(p(x)r_1(x) + \frac{q(x)}{w(x)}\right) w(x)|u(x)|^2 \,\mathrm{d}x\right| < \frac{\epsilon}{4}.$$

Since $||u|| = 1$ implies $\int_a^b w(x)|u(x)|^2 \,\mathrm{d}x = 1$, there exists $T^{''} \in (a, b)$ such that

$$\int_{T''}^b w(x)|u(x)|^2 \,\mathrm{d}x < \frac{\epsilon}{4}\left|p(T')r_1(T') + \frac{q(T')}{w(T')}\right|^{-1}.$$

Let $T = \max\{T', T^{''}\}$ and $\lambda_T = \int_a^T (p(x)r_1(x) + \frac{q(x)}{w(x)})w(x)|u(x)|^2 \,\mathrm{d}x$. Then

$$\begin{aligned}&|\lambda_T - \lambda| < \frac{\epsilon}{2}, \\ &\int_T^b w(x)|u(x)|^2 \,\mathrm{d}x < \frac{\epsilon}{4}\left[p(T')r_1(T') + \frac{q(T')}{w(T')}\right]^{-1}.\end{aligned}$$

Now we consider the constant λ_T, by the definition of integrand, for $\epsilon > 0$, there exists $\delta > 0$, and a partition $a = x_0 < x_1 < x_2 < \cdots < x_n = T$,

$$\left|\sum_{i=1}^n \left[p(\xi_i)r_1(\xi_i) + \frac{q(\xi_i)}{w(\xi_i)}\right] \Delta\left(\int_{x_{i-1}}^{x_i} w(x)|u(x)|^2 \,\mathrm{d}x\right) - \lambda_T\right| < \frac{\epsilon}{4}$$

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provided $\Delta(T) = \max_{1\leqslant i\leqslant n}\{x_i - x_{i-1}\} < \delta$, where $\xi_i \in (x_{i-1}, x_i)$. Since $\int_a^b w(x)|u(x)|^2 \,\mathrm{d}x = 1$, we obtain

$$\begin{aligned}\rho(\Delta) &:= \sum_{i=1}^{n}\left[p(\xi_i)r_1(\xi_i) + \frac{q(\xi_i)}{w(\xi_i)}\right]\Delta\left(\int_{x_{i-1}}^{x_i} w(x)|u(x)|^2\,\mathrm{d}x\right)\\ &\quad + \left[p(T')r_1(T') + \frac{q(T')}{w(T')}\right]\int_T^b w(x)|u(x)|^2\,\mathrm{d}x\\ &\in \mathrm{co}\left\{\frac{q(x)}{w(x)} + rp(x) : r > 0, x\in(a,b)\right\}.\end{aligned}$$

So we obtain

$$\begin{aligned}|\lambda - \rho(\Delta)| &\leqslant |\lambda - \lambda_T| + \left|\lambda_T - \sum_{i=1}^{n}\left[p(\xi_i)r_1(\xi_i) + \frac{q(\xi_i)}{w(\xi_i)}\right]\Delta\left(\int_{x_{i-1}}^{x_i} w(x)|u(x)|^2\,\mathrm{d}x\right)\right|\\ &\quad + \left|\sum_{i=1}^{n}\left[p(\xi_i)r_1(\xi_i) + \frac{q(\xi_i)}{w(\xi_i)}\right]\Delta\left(\int_{x_{i-1}}^{x_i} w(x)|u(x)|^2\,\mathrm{d}x\right) - \rho(\Delta)\right|\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{4} + \left[p(T')r_1(T') + \frac{q(T')}{w(T')}\right]\int_T^b w(x)|u(x)|^2\,\mathrm{d}x\\ &= \frac{3\epsilon}{4} + \left|p(T')r_1(T') + \frac{q(T')}{w(T')}\right|\int_T^b w(x)|u(x)|^2\,\mathrm{d}x < \epsilon,\end{aligned}$$

which implies

$$\lambda \in \overline{\mathrm{co}}\left\{\frac{q(x)}{w(x)} + rp(x) : r > 0, x\in(a,b)\right\} = \Omega.$$

For the case where $u(t) \equiv 0$ in some sub-interval of $[a, b)$, we define $V = \bigcup_{i=1}^{n}(a_i, b_i)$, where n is finite or $n = \infty$, (a_i, b_i) are disjoint open intervals such that $u(x) \equiv 0$ for $x \in (a_i, b_i)$, $1 \leqslant i \leqslant n$. We assume further that $a < a_i < b_i < a_{i+1}$ and the set $\{a_i\}$ has no finite accumulation point if $b = \infty$, and the only possible finite accumulation point is b if $b < \infty$.

Let $\tau = \tau(x) = m([a, x] - V)$, where m denotes the usual linear Lebesgue measure, and let $B = \tau(b)$, $A_i = \tau(a_i)$, $i = 1, 2, \cdots, n$. $[a, B)$ is obtained from $[a, b)$ by shrinking each interval (a_i, b_i) to its left endpoint.

Let D_I denote the set of piecewise continuous functions on I. We construct a class of transformation $L_V : D_{[a,b)} \to D_{[a,B)}$ as follows: Let $f \in D_{[a,b)}$. Then $F = L_V(f)$ is defined by

$$F(\tau) = f(x) \text{ if } \tau = \tau(x), \tau \neq A_i, \text{ and } F(A_i) = f(a_i).$$

The function F is obtained from f by collapsing each interval (a_i, b_i) to a point. For all $\lambda \in \Theta(T_F)$, we obtain

$$\begin{aligned}\lambda &= \int_a^b [p(x)|u'(x)|^2 + q(x)|u(x)|^2]\,\mathrm{d}x \\ &= \int_a^B \left[(L_V p)(\tau)|(L_V u)'(\tau)|^2 + (L_V q)(\tau)|(L_V u)(\tau)|^2\right] \mathrm{d}\tau.\end{aligned}$$

The remainder is similar as above, so we omit the details. In both cases, we have proved $\Theta(T_F) \subseteq \Omega$. This completes the proof. □

Corollary 5.2. *Let $\sigma(T_F)$ denote the spectrum of the Friedrichs extension operator T_F. Then for all $(\theta_0, K_0) \in \Pi$, we have*

$$\begin{aligned}&\inf\left\{\Re\left\{\mathrm{e}^{i\theta}\langle (T_0 - K_0)y, y\rangle : y \in D(T_0), ||y|| = 1\right\}\right\} \\ &\quad = \min\left\{\Re\left\{\mathrm{e}^{i\theta}(\mu - K_0) : \mu \in \sigma(T_F)\right\}\right\}.\end{aligned}$$

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