THE HULLS OF REPRESENTABLE l-GROUPS AND f-RINGS

Dedicated to the memory of Hanna Neumann

PAUL CONRAD

(Received 11 September 1971)

Communicated by G. B. Preston

1. Introduction and statements of the main results

A lattice-ordered group ("l-group") G will be called
a *P-group* if $G = g'' \oplus g'$ for each $g \in G$ (projectable)
an *SP-group* if $G = C \oplus C'$ for each polar C of G (strongly projectable)
an *L-group* if each disjoint subset has a 1. u. b. (laterally complete)
an *O group* if it is both an L-group and a P-group (orthocomplete).
G is *representable* if it is an l-subgroup of a cardinal product of totally ordered groups. It follows that a P-group must be representable and hence SP-groups and O-groups are also representable.

G is a *large l-subgroup* of an l group H or H is an *essential extension* of G if G is an l-subgroup of H and for each non-zero convex l-subgroup S of H we have $S \cap G \neq 0$.

We show that if G is a large l-subgroup of an X-group H, where $X = P$, SP, L or O, then the intersection K of all l-subgroups of H that contain G and are X-groups is an X-group. Thus K is a minimal essential extension of G that is an X-group and we shall call such an extension of G an *X-hull* of G.

THEOREM 2.6. *There exists a unique X-hull G^X of a representable l-group G. Moreover, G is dense in G^X, G^X is representable and if G is archimedean or abelian, then so is G^X.*

We then show that if G is a representable l-group then each $0 < g \in G^O$ is the join of a disjoint subset of G^P. Thus

$$G \subseteq G^P \subseteq G^{SP} \subseteq (G^{SP})^L = (G^P)^L = G^O \text{ and}$$
$$G \subseteq G^L \subseteq (G^L)^P = (G^L)^{SP} \subseteq G^O.$$

but $(G^L)^{SP}$ need not equal G^O.

https://doi.org/10.1017/S1446788700015391 Published online by Cambridge University Press

A rather natural direct limit construction provides the existence and uniqueness of G^X.

If G is a D_f-module, f-ring or f-algebra then there is a unique way of extending the multiplication so that G^X is a D_f-module, f-ring or f-algebra that contains G as a submodule, subring or subalgebra. Thus the multiplicative structure of G^X is completely determined by its additive structure. This phenomenon is due to the fact that each polar preserving endomorphism ("p-endomorphism") of G has a unique extension to a p endomorphism of G^X.

If G is a vector lattice then G^P is the p-extension of G defined by Amemiya [1], but Amemiya's definition of a p extension is fairly complicated and so are his proofs of the existence and uniqueness of G^P. However, he does mention that G^P is the minimal P-group in which G is dense.

Now suppose that G is a representable l-group. Then G^P is the Stone extension $\Sigma(G)$ of G that is defined by Speed [21]. His definition of $\Sigma(G)$ is categorical, but the maps involved are rather special l-homomorphisms. Speed also defines G^O categorically and makes a rather thorough investigation of P-groups. G^L is the lateral completion of G defined in [9]. There the definition required that G be dense in G^L. Finally G^O is the orthocompletion of G defined by Bernau [3]. Here again the definition of G^O is somewhat complicated being modelled after the definition used by Amemiya for countably laterally complete vector lattice p extensions.

If F is a (real) f-algebra then Amemiya remarks that his p-extension is also an f-algebra. Bernau proves that if G is an f-ring or a vector lattice then so is its orthocompletion.

Vecksler [23] outlines a method for constructing the P-hull and the SP-hull of an f-ring. In [24] he corrects his definition of an SP-hull.

An archimedean l group A is a

d-group if it is divisible

v group if it is a vector lattice

c group if it is a (conditionally) complete lattice

e group if it is essentially closed in the class of archimedean l groups.

If A is a large l-subgroup of an archimedian y group H, where $y = d, v, c$ or e, then the intersection K of all l subgroups of H that contain A and are y-subgroups is a y group. Thus K is a minimal essential extension of A that is a y group. We shall call such an extension of A a *y hull.*

THEOREM 5.2. *Each archimedean l-group A admits a unique y-hull A^y for $y = d, v, c$ or e. A^c is the Dedekind-MacNeille completion of A and A is dense in A^c. A^v is the l subspace of $(A^d)^c$ that is generated by A. $A^e = ((A^d)^c)^L$ is the essential closure of A.*

Once again if A is an f-ring then there is a unique extension of the multiplication of A to a multiplication of A^y so that A^y is an f-ring and A is a subring of A^y. Thus the multiplicative structure of A^y is completely determined by its additive structure.

In Section 6 we completely characterize the structure of an archimedean essentially closed f-ring and this gives quite a bit of information about the structure of an arbitrary f-ring.

In Section 7 we get a nice representation of the orthocompletion of an f-ring with a basis and this leads to information about the structure of an arbitrary f-ring with a basis.

NOTATION. Throughout G will denote an l-group and for each $0 < g \in G$, $G(g)$ will dencte the convex l-subgroup of G generated by g. G is a *dense* l-subgroup of an l-group H if for each $0 < h \in H$ we have $0 < g \leqq h$ for some $g \in G$. ΠA_λ will denote the cardinal product of l-groups A_λ and ΣA_λ will denote the cardinal sum. The cardinal sum of a finite number of l groups will be denotedby $A_1 \oplus \cdots \oplus A_n$. For each subset S of G

$$S' = \{g \in G \,|\, |g| \wedge |s| = 0 \text{ for all } s \in S\}$$

is the *polar* of S. Sik [20] has shown that the set $P(G)$ of all polars in G is a complete Boolean algebra and that an l-group is representable if and only if each polar is normal.

2. The existence and uniqueness of X-hulls

LEMMA 2.1. *If G is a P-group and L-group then G is an SP-group.*

PROOF. If $C \in P(G)$ and $\{a_\lambda \,|\, \lambda \in \Lambda\}$ is a maximal disjoint subset of C then $a = \vee a_\lambda$ is a weak order unit in C and so $a'' = C$. Thus

$$G = a'' \oplus a' = C \oplus C'.$$

G is an *$\mathscr{L}$-subgroup* of an l-group H if G is an l-subgroup of H and for each disjoint subset $\{a_\lambda \,|\, \lambda \in \Lambda\}$ of G for which $\vee_G a_\lambda$ exists we have $\vee_G a_\lambda = \vee_H a_\lambda$. Note that the intersection of laterally complete $\mathscr{L}$ subgroups of H is a laterally complete $\mathscr{L}$-subgroup.

LEMMA 2.2. *If G is a large l-subgroup of an l group H then G is an $\mathscr{L}$-subgroup of H.*

PROOF. Suppose that $\{a_\lambda \,|\, \lambda \in \Lambda\}$ is a disjoint subset of G and $a = \vee_G a_\lambda$ exists. If h is an upper bound for the a_λ in H then $a \geqq a \wedge h = k \geqq a_\lambda$ and so it suffices to show that $a = k$. For each $\lambda \in \Lambda$, $a^\lambda = \vee_G a_\alpha$ $(\alpha \neq \lambda)$ exists, $a_\lambda \wedge a^\lambda = 0$ and $a = a_\lambda + a^\lambda$. Thus

$$H(a) = H(a_\lambda) \oplus H(a^\lambda).$$

https://doi.org/10.1017/S1446788700015391 Published online by Cambridge University Press

Now $k = k_\lambda + k_\lambda$, where $k_\lambda \in H(a_\lambda)$ and $k^\lambda \in H(a^\lambda)$ and since $a \geqq k \geqq a_\lambda$ we have $a_\lambda \geqq k_\lambda \geqq a_\lambda$. Therefore $a - k = a^\lambda - k^\lambda \in \bigcap_\Lambda H(a^\lambda) = K$. But $K \cap G = \bigcap_\Lambda G(a^\lambda) \subseteq G(a)$ and so if $0 \leqq x \in K \cap G$ then $x \wedge a_\lambda = 0$ for all $\lambda \in \Lambda$. Thus $x \wedge a = x \wedge \bigvee_G a_\lambda = \bigvee_G x \wedge a_\lambda = 0$ and since a is a unit in $G(a)$, $x = 0$. Therefore $K \cap G = 0$ and since G is large in H, $K = 0$.

Let G be an l-subgroup of H and denote the polar operation in G (H) by $'(*)$. For $B \in P(G)$ and $C \in P(H)$ define

$$B\mu = (B')^* \text{ and } C\nu = C \cap G.$$

1) $B\mu\nu = (B')^* \cap G = B^{**} \cap G = B^{**}\nu = B.$

PROOF. Since $B' \subseteq B^*$ we have $(B')^* \supseteq B^{**} \supseteq B$ and so $(B')^* \cap G \supseteq B^{**} \cap G \supseteq B$. If $0 < x \in (B')^* \cap G$ then $x \in G$ and $x \wedge B' = 0$ and so $x \in B'' = B$.

2) If ν is one-to-one then $B\mu = B^{**}$.

3) ([9] p. 455). If G is large in H then μ is an isomorphism of $P(G)$ onto $P(H)$ and ν is the inverse.

4) ([10] p. 156). If H is archimedean then the following are equivalent.

i) G is large in H.

ii) ν is an isomorphism of $P(H)$ into $P(G)$ and μ is the inverse.

iii) If $0 \neq C \in P(H)$ then $C \cap G \neq 0$.

iv) If $0 < h \in H$ then $h'' \cap G \neq 0$.

5) If G is large in H and X is an l subgroup of G or just a non-void subset of G then

i) $(X'')^{**} = X^{**}$ and $X^{**} \cap G = X''$

ii) $(X')^{**} = X^*$ and $X^* \cap G = X'$.

PROOF. Since $X \subseteq X''$ we have $X^{**} \subseteq (X'')^{**}$. Also $X^{**}\nu$ is a polar of G that contains X and so $X^{**}\nu = X^{**} \cap G \supseteq X''$. Thus $X'' \subseteq X^{**}$ and hence $(X'')^{**} \subseteq X^{**}$.

$$X^{**} \cap G = (X'')^{**} \cap G = X''\mu\nu = X''.$$

From (i) and (2) we have $X^* = (X'')^* = (X')'^* = (X')^{**}$. Finally $X^* \cap G = \{g \in G \mid |g| \wedge X = 0\} = X'$ holds for any l-subgroup G of H.

6) If α is an l-automorphism of H that induces the identity on $P(G)$ then α induces the identity on $P(H)$ provided that G is large in H.

PROOF. If $C \in P(H)$ then $C\nu = C\nu\alpha = (G \cap C)\alpha = G\alpha \cap C\alpha = G \cap C\alpha = C\alpha\nu$, so that $C = C\alpha$ by (3).

PROPOSITION 2.3. *Let G be a convex l-subgroup of an l-group H.*

i) *If H is an SP-group so is G.*

ii) *If H is a P-group so is G.*

PROOF. (i) If $A \in P(G)$ then $H = A^{**} \oplus A^*$ and hence $G = (A^{**} \cap G) \oplus (A^* \cap G) = A \oplus (A^* \cap G) = A \oplus A'$.

(ii) Pick $g \in G$. Then $H = g^{**} \oplus g^*$ and so $G = (G \cap g^{**}) \oplus (G \cap g^*) = g'' \oplus g'$. For $g' \subseteq g^*$ implies $(g'')'^* = g'^* \supseteq g^{**}$ and so $g'' = (G \cap (g'')'^* \supseteq G \cap g^{**} \supseteq g''$.

Note that a polar in an L-group is an L-group, but an l-ideal C of an L-group G need *not* be an L-group.

EXAMPLE. $C = \sum_{i=1}^{\infty} R_i \subseteq \prod_{i=1}^{\infty} R_i = G$.

This also shows that an l-ideal of an O group need not be an O-group.

THEOREM 2.4. *If H is an X-group and an essential extension of G and $\{H_\lambda \mid \lambda \in \Lambda\}$ is the set of all l-subgroups of H that contain G and are X-groups then $K = \bigcap_\Lambda H_\lambda$ is an X-hull of G, where $X = P, SP, L$* or *O.*

PROOF. If H is an L-group then by Lemma 2.2 each H_λ is a laterally complete $\mathscr{L}$-subgroup of H and so K is an L-group.

Suppose that H is a P-group, $0 < k \in K$ and denote the polar operation in H, K, and H_λ by $*$, $\#$ and $^\lambda$ respectively. If $0 < x \in K \subseteq H_\lambda$ then $x = x_1 + x_2 \in k^\lambda \oplus k^{\lambda\lambda}$ and by (5) $k^\lambda = k^* \cap H_\lambda$ and $k^{\lambda\lambda} = k^{**} \cap H^\lambda$. Thus $x_1 + x_2$ is the unique decomposition of x in $H = k^* \oplus k^{**}$. This holds for all λ so x_1, $x_2 \in \cap H_\lambda = K$. Thus $x_1 \in K \cap k^* = k^\#$ and $x_2 \in K \cap k^{**} = k^{\#\#}$. Therefore $x \in k^\# \oplus k^{\#\#}$ and hence $K = k^\# \oplus k^{\#\#}$.

If H is an SP-group then an entirely similar argument shows that K is also an SP-group.

LEMMA 2.5. *An L-hull K of a representable l-group G is representable.*

PROOF. Theorem 2.8 in [9] asserts that if G is dense in K then K is also representable. The only place in the proof where the hypothesis of denseness is used is to infer that if $(-a_\alpha + (a_\alpha \wedge b) + a_\alpha) \wedge (a_\alpha \wedge b) = 0$ and $a_\alpha \wedge b > 0$ then $a_\alpha \wedge b \geqq g > 0$ for some $g \in G$ and so $(-a_\alpha + g + a_\alpha) \wedge g = 0$. But since G is large in K we *can* conclude that $n(a_\alpha \cap b) \geqq g > 0$ for some $n > 0$ and $g \in G$. Thus $0 = n((-a_\alpha + (a_\alpha \wedge b) + a_\alpha) \wedge (a_\alpha \wedge b)) = (-a_\alpha + n(a_\alpha \wedge b) + a_\alpha) \wedge n(a_\alpha \wedge b) \geqq (-a_\alpha + g + a_\alpha) \wedge g \geqq 0$ and so $(-a_\alpha + g + a_\alpha) \wedge g = O$,

COROLLARY. *An X-hull of a representable l-group is representable, where $X = P$, SP, L or O.*

THEOREM 2.6. *There exists a unique X-hull G^X of a representable l-group G for $X = P$, SP, L or O. Morover G is dense in G^X and G^X is representable and if G is abelian or archimedean then so is G^X.*

PROOF. The existence follows from Theorem 2.4 provided that we can embed G as a large l-subgroup in an X-group. In order to do this we make use of the direct limit construction developed in [9].

Let $D(G)$ be the set of all maximal disjoint subsets of the Boolean algebra $P(G)$ of polars of G. If $\mathscr{A}, \mathscr{C} \in D(G)$ then we define $\mathscr{A} \leqq \mathscr{C}$ if each $A \in \mathscr{A}$ is contained in some $C \in \mathscr{C}$. Then $D(G)$ is a lower directed partially ordered set. For each $\mathscr{C} \in D(G)$ let $G_{\mathscr{C}}$ be the l-group

$$G_{\mathscr{C}} = \prod_{C \in \mathscr{C}} G/C'.$$

If $\mathscr{A} \leqq \mathscr{C} \in D(G)$ and $C \in \mathscr{C}$ then $C = (\cap A_\lambda{}')'$ the polar join of the $A_\lambda \in \mathscr{A}$ that are contained in C. Thus $C' = \cap A_\lambda{}'$ and so the natural map

$$G/C \rightarrow \prod G/A_\lambda{}'$$

is an l-isomorphism. Thus there is a natural l-isomorphism $\pi_{\mathscr{C}\mathscr{A}}$ of $G_{\mathscr{C}}$ into $G_{\mathscr{A}}$ obtained by combining the above maps for each G/C', where $C \in \mathscr{C}$. Let $\mathcal{O}(G)$ be the direct limit of the l-groups G with connecting l-isomorphisms $\pi_{\mathscr{C},\mathscr{A}}$. Define $k \in \mathcal{O}(G)$ to be positive if $k = 0$ or $k_{\mathscr{C}} > 0$ for some $\mathscr{C} \in D(G)$. For each $g \in G$ let $\tilde{g}$ be the element in $\mathcal{O}(G)$ with $\tilde{g}_{\mathscr{C}} = (\cdots, C' + g, \cdots)$ for each $\mathscr{C} \in D(G)$.

In [9] it is shown that $\mathcal{O}(G)$ is a representable laterally complete l-group and if G is abelian or archimedean then so is $\mathcal{O}(G)$. Also the map $g \rightarrow \tilde{g}$ is an l-isomorphism of G into $\mathcal{O}(G)$ and $\tilde{G}$ is dense in $\mathcal{O}(G)$. Thus to complete the proof of existence it suffices to show that $\mathcal{O}(G)$ is a P-group. Thus we must show that if $\theta < l \in \mathcal{O}$ then $\mathcal{O} = l^{**} \oplus l^*$.

Consider $\theta < k \in \mathcal{O}(G)$ and pick $\mathscr{C} \in D(G)$ such that $l_{\mathscr{C}} \neq 0 \neq k_{\mathscr{C}}$. Then $l_{\mathscr{C}} = (\cdots, C' + l(C), \cdots)$, where $0 \leqq l(C) \in G$. Let $\overline{l(C)}$ be the convex l-subgroup of G that is generated by $l(C)$ and pick $\mathscr{C} \geqq \mathscr{A} \in D(G)$ so that each $(C \cap \overline{l(C)})'' \neq 0$ belongs to $\mathscr{A}$.

$$\begin{aligned} G_{\mathscr{A}} &= \prod G/(C \cap \overline{l(C)})' \oplus \Pi\, G/A_\lambda' \\ k_{\mathscr{A}} &= \qquad x_{\mathscr{A}} \qquad\quad + \quad y_{\mathscr{A}} \end{aligned}$$

Let x (y) be the element in $\mathcal{O}(G)$ with $\mathscr{A}$-th component $x_{\mathscr{A}}$ if $x_{\mathscr{A}} \neq 0$ ($y_{\mathscr{A}}$ if $y_{\mathscr{A}} \neq 0$) and θ otherwise. Then $k = x + y$. It is shown in [9] that the only non-zero components of $l_{\mathscr{A}}$ are of the form $(C \cap \overline{l(C)})' + l(C)$. Thus $l_{\mathscr{A}} \wedge y_{\mathscr{A}} = 0$ and so $y \in l^*$. Thus we need only prove that $x \in l^{**}$. Consider $\theta < t \in \mathcal{O}(G)$ such that $l \wedge t = \theta$. To complete the proof of existence we need to show that $x \wedge t = \theta$.

Pick $\mathscr{D} \in D(G)$ so that $0 \neq t_{\mathscr{D}} = (\cdots, D' + t(D), \cdots)$. Now ([9] p. 456) $(C \cap \overline{l(C)})'' \cap (D \cap \overline{t(D)})'' = 0$ and so we may choose a $\mathscr{B} \in D(G)$ that contains the $(C \cap \overline{l(C)})'' \neq 0$ and the $(D \cap \overline{t(D)})'' \neq 0$. Let

$$\mathscr{A} \cap \mathscr{B} = \{A \cap B \neq 0 \mid A \in \mathscr{A} \text{ and } B \in \mathscr{B}\}$$

Then $\mathscr{A} \cap \mathscr{B} \in D(G)$ and so we have

$$\begin{tikzcd}[column sep=small] G_{\mathscr{A}} \arrow[dr] & & G_{\mathscr{B}} \arrow[dl] \\ \pi_{a,a\cap\mathscr{B}} & G_{\mathscr{A}\cap\mathscr{B}} & \pi_{\mathscr{B},\mathscr{A}\cap\mathscr{B}} \end{tikzcd}$$

Now $x_{\mathscr{A}}$ has nonzero components of the form $(C \cap \overline{l(C)})' + z$ and $t_{\mathscr{B}}$ has nonzero components of the form $(D \cap \overline{t(D)})' + t(D)$. These do not change under the maps into $G_{\mathscr{A}\cap\mathscr{B}}$ and so $x \wedge t = \theta$. Thus there exists an X-hull of G.

Let H be an X-hull of G and let $\alpha(\beta)$ the the natural l-isomorphisms of G (H) into $\mathscr{O}(G)$ $(\mathscr{O}(H))$. We complete the proof by showing that α can be extended to an l-isomorphism ρ of H onto the X-hull K of $G\alpha = \tilde{G}$ in $\mathscr{O}(G)$.

$$\begin{tikzcd} H \arrow[d] \arrow[dr, "\rho"] & \\ G \arrow[r, "\alpha"] & \tilde{G} \subseteq K \subseteq \mathscr{O}(G) \end{tikzcd}$$

Thus if H_1 and H_2 are X-hulls of G then $\rho_1\rho_2{}^{-1}$ is an l-isomorphism of H_1 onto H_2 that induces the identity on G. It follows from Theorem 2.7 that $\rho_1\rho_2{}^{-1}$ is unique.

Since G is large in H for each $C \in P(G)$ we have $C = G \cap C^{**}$ and $C' = G \cap C^*$. Thus $C' + g \dashrightarrow C^* + g$ is an l isomorphism of G/C' into H/C^*. For each $\mathscr{C} \in D(G)$ let $\bar{\mathscr{C}} = \{C^{**} \mid C \in \mathscr{C}\}$. Then $\bar{\mathscr{C}} \in D(H)$ and thus there is a natural l-isomorphism $\tau_{\mathscr{C}}$ of $G_{\mathscr{C}}$ onto $H_{\bar{\mathscr{C}}}$. Moreover if $\mathscr{A} \leqq \mathscr{C}$ in $D(G)$

$$\begin{tikzcd} G_{\mathscr{C}} \arrow[r, dashed, "\tau_{\mathscr{C}}"] \arrow[d, "\pi_{\mathscr{C}\mathscr{A}}"'] & H_{\bar{\mathscr{C}}} \arrow[d, "\pi_{\bar{\mathscr{C}}\bar{\mathscr{A}}}"] \\ G_{\mathscr{A}} \arrow[r, dashed, "\tau_{\mathscr{A}}"] & H_{\bar{\mathscr{A}}} \end{tikzcd}$$

commutes, where $\pi_{\bar{\mathscr{C}}\bar{\mathscr{A}}}$ is the l-isomorphism used in the construction of $\mathscr{O}(H)$. Thus (see [9]) the $\tau_{\mathscr{C}}$ determine an l-isomorphism τ of $\mathscr{O}(G)$ into $\mathscr{O}(H)$

$$\begin{tikzcd} H \arrow[rr, dashed, "\beta"] & & \mathscr{O}(H) \\ G \arrow[r, dashed, "\alpha"] & G\alpha = \tilde{G} \subseteq K \subseteq \mathscr{O}(G) \arrow[ur, "\tau"'] & \end{tikzcd}$$

If $g \in G$ and $\bar{\mathscr{C}} \in D(H)$ then $(g\alpha\tau)_{\bar{\mathscr{C}}} = (g\alpha)_{\mathscr{C}}\tau_{\mathscr{C}} = (\cdots, C' + g, \cdots)\tau_{\mathscr{C}} = (\cdots, C^* + g, \cdots)$ $= (g\beta)_{\bar{\mathscr{C}}}$. Thus $g\alpha\tau = g\beta$ and hence $G\beta = G\alpha\tau \subseteq \mathscr{O}(G)\tau$ which is an X group and $G\beta$ is large in $\mathscr{O}(H)$. Thus $H\beta \cap \mathscr{O}(G)\tau$ is an X-group and contains $G\beta$ and so since H_β is an X-hull of $G\beta$ we have

$$G\alpha\tau = G\beta \subseteq H\beta \subseteq \mathcal{O}(G)\tau \subseteq \mathcal{O}(H).$$

Thus $H\beta\tau^{-1}$ is an X-group that contains G_α and so

$$G\alpha = G\beta\tau^{-1} \subseteq K \subseteq H\beta\tau^{-1} \subseteq \mathcal{O}(G)$$

and since $H\beta\tau^{-1}$ is an X-hull of $G\beta\tau^{-1}$ we have $K = H\beta\tau^{-1}$. This completes the proof of Theorem 2.6.

REMARK. We can, of course, define countably laterally complete l-groups in the obvious way and then it follows from the above proof that each representable l-group admits a unique CL-hull. Also G admits a unique minimal essential extension H that is both a P-group and a CL-group. For the vector lattice case H is the "completion" of Amemiya [1]. See also Vulich [25].

THEOREM 2.7. *If α is an l-isomorphism of G_1 onto G_2, where the G_i are representable l-groups, then there exists a unique extension of α to an l-isomorphism of G_1^X onto G_2^X for $X = P$, SP, L or O.*

PROOF. α induces an isomorphism of $P(G_1)$ onto $P(G_2)$ and hence an isomorphism of $D(G_1)$ onto $D(G_2)$. Also for $C \in P(G_1)$ we have the natural map $C' + g \dashrightarrow (C\alpha)' + g\alpha$ of G_1/C' onto $G_2/(C\alpha)'$. Thus there is a natural map $\alpha_{\mathscr{C}}$ of $G_{1\mathscr{C}}$ onto $G_{2\mathscr{C}\alpha}$ such that

$$\begin{tikzcd}
G_{1\mathscr{C}} \arrow[r, dashed, "\alpha_{\mathscr{C}}"] \arrow[d, "\pi_{\mathscr{C}\mathscr{A}}"'] & G_{2\mathscr{C}\alpha} \arrow[d, "\pi_{\mathscr{C}\alpha\mathscr{A}\alpha}"] \\
G_{1\mathscr{A}} \arrow[r, dashed, "\alpha_{\mathscr{A}}"] & G_{2\mathscr{A}\alpha}
\end{tikzcd}$$

commutes. These maps $\alpha_{\mathscr{C}}$ generate an isomorphism $\bar{\alpha}$ of $\mathcal{O}(G_1)$ onto $\mathcal{O}(G_2)$ and the following diagram commutes

$$\begin{tikzcd}
 & & & G_1^X \arrow[d] & & \\
G_1 \arrow[urrr] \arrow[r, dashed] \arrow[d, "\alpha"'] & \tilde{G}_1 \arrow[r, phantom, "\subseteq"] & & \tilde{G}_1^X \arrow[r, phantom, "\subseteq"] & & \mathcal{O}(G_1) \arrow[d, "\bar{\alpha}"] \\
G_2 \arrow[drrr] \arrow[r, dashed] & \tilde{G}_2 \arrow[r, phantom, "\subseteq"] & & \tilde{G}_2^X \arrow[r, phantom, "\subseteq"] & & \mathcal{O}(G_2) \\
 & & & G_2^X \arrow[u] & &
\end{tikzcd}$$

Also it is easy to see that $\tilde{G}_1^X\bar{\alpha} = \tilde{G}_2^X$. Thus α can be extended to an l-isomorphism of G_1^X onto G_2^X.

For the uniqueness it suffices to show that if α is an l-automorphism of G^X that induces the identity on G then α is the identity. Since α induces the identity on $P(G)$ it must also induce the identity on $P(G^X)$. Thus we may assume that α is an l-automorphism of $\mathcal{O}(G)$ that induces the identity on $\tilde{G}$ and $P(\mathcal{O}G)$). Consider $l \in \mathcal{O}(G)$ with $l_{\mathscr{C}} = (\cdots, C' + g, \cdots)$ and suppose (by way of contradiction that $(l\alpha)_{\mathscr{C}} = (\cdots, C' + x, \cdots)$, where $C' + x \neq C' + g$. Then

$$|g - l|_{\mathscr{C}} \wedge (0, \cdots, 0, C' + |g - x|, 0, \cdots, 0) = 0 \text{ but}$$

$$(|g - l|\alpha)_{\mathscr{C}} \wedge (0, \cdots, 0, C' + ||g - x|, 0, \cdots, 0) \neq 0.$$

Thus α does not induce the identity on $P(\mathcal{O}(G))$, a contradiction.

PROPOSITION 2.8. *Suppose that G is a representable l-group, α is an l-automorphism of G^0 and $X = P$, SP, L or 0.*

i) *$G^X\alpha = (G\alpha)^X$ and so if $G\alpha = G$, then $G^X\alpha = G^X$.*

ii) *If $G\alpha \subseteq G$ then $G^X\alpha \subseteq G^X$.*

PROOF. $G\alpha$ is large in G^O and hence in $G^X\alpha$. Also $G^X\alpha$ is an X-group. If $G\alpha \subseteq K \subset G^X\alpha$, where K is an l-subgroup of $G^X\alpha$ and an X-group then $G \subseteq K\alpha^{-1} \subset G^X$ which contradicts the minimality of G^X. Thus $G^X\alpha$ is the X-hull of $G\alpha$ and so $G^X\alpha = (G\alpha)^X$. If $G\alpha \subseteq G$ then $G^X\alpha = (G\alpha)^X \subseteq G^X$. The following example shows that we may or may not have equality.

EXAMPLE. Let G be the l-ideal in $\prod_{i=1}^{\infty} R_i$ generated by $(1, 2, 3, \cdots)$. Then $G^O = \prod R_i$. Let α be the multiplication of G^O by $(1, 1/2, 1/3, \cdots)$. Then $G\alpha$ is the l-ideal of G^O generated by $(1, 1, 1, \cdots)$. Thus $G\alpha \subset G$ and both G and $G\alpha$ are SP-groups.

$$G^P\alpha = G\alpha \subset G = G^P \text{ and}$$

$$G^L\alpha = (G\alpha)^L = G^O = G^L.$$

COROLLARY. *If α is an l-endomorphism of G^X that induces an automorphism on G then α is an automorphism of G^X.*

PROOF. Since G is large in G^X it follows that α is one-to-one on G^X and by the minimality of $G^X\alpha$ must be an l-automorphism of G^X.

THEOREM 2.9. *If G is a P-group then each $\theta < l \in \mathcal{O}(G)$ is the join of a disjoint subset of $\tilde{G}$. In particular, $\tilde{G}^L = \mathcal{O}(G)$ and hence G^L is an SP-group.*

PROOF. Consider $\theta < l \in \mathcal{O}$ and $l_{\mathscr{C}} \neq 0$. In each $C \in \mathscr{C}$ pick a maximal disjoint set $\{a_\alpha | \alpha \in A\}$ of elements of G. Then $C = (\cap a_\alpha')' = (\cup a_\alpha'')''$ and so there is a partition $\mathscr{A} \leqq \mathscr{C}$ that consists of principal polars of G.

$$\mathscr{A} = \{a_\lambda'' | \lambda \in \Lambda\}$$

Thus $0 \neq l_{\mathcal{A}} = (\cdots, a_\lambda' + l(\lambda), \cdots)$. Now $G = a_\lambda'' \oplus a_\lambda'$ and so we may assume that $0 \leqq l(\lambda) \in a_\lambda''$ for each $\lambda \in \Lambda$. In particular, the $l(\lambda)$ are disjoint in G.

$$\widetilde{l(\lambda)}_{\mathcal{A}} = (0, \cdots, 0, a_\lambda' + l(\lambda), 0, \cdots, 0).$$

Thus $\vee \widetilde{l(\lambda)}_{\mathcal{A}} = l_{\mathcal{A}}$ and so $\vee \widetilde{l(\lambda)} = l$.

Corollary I. *If G is an O-group then $\tilde{G} = \mathcal{O}(G)$.*

Corollary II. *If G is a representable l-group then*

$$\tilde{G} \subseteq \tilde{G}^P \subseteq \tilde{G}^{SP} \subseteq (\tilde{G}^{SP})^L = (\tilde{G}^P)^L = \tilde{G}^O = \mathcal{O}(G)$$

where the indicated X-hulls are all in $\mathcal{O}(G)$. In particular, $G^O = \mathcal{O}(G)$ and so G^O is the orthocompletion defined by Bernau.

Proof. Clearly $\tilde{G} \subseteq \tilde{G}^P \subseteq \tilde{G}^{SP} \subseteq (\tilde{G}^P)^L \subseteq (\tilde{G}^{SP})^L \subseteq \tilde{G}^O \subseteq \mathcal{O}(G)$ and so it suffices to show that $(\tilde{G}^P)^L = \mathcal{O}(G)$. Let H be the P-hull of G and let α, β, τ be as in the proof of Theorem 2.6.

$$\begin{array}{ccc} H & \xrightarrow{\ \beta\ } & \tilde{H} \subseteq \tilde{H}^L = \mathcal{O}(H) \\ & & \Big\downarrow \tau \\ G & \xrightarrow{\ \alpha\ } & \tilde{G} \subseteq \tilde{G}^P(\tilde{G}^P)^L \subseteq \mathcal{O}(G) \end{array}$$

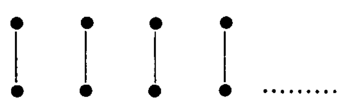
Then $\tilde{H} = \tilde{G}^P\tau \subseteq (\tilde{G}^P)^L\tau \subseteq \mathcal{O}(H)$ and $(\tilde{G}^P)^L\tau$ is an L-group. Thus $(\tilde{G}^P)^L\tau = \mathcal{O}(H)$ and so $(\tilde{G}^P)^L = \mathcal{O}(G)$.

Also it follows that

$$\tilde{G} \subseteq \tilde{G} \subseteq (\tilde{G}^L)^P \subseteq (\tilde{G}^L)^{SP} \subseteq \tilde{G}^O = \mathcal{O}(G)$$

but as the next example shows $(\tilde{G}^L)^{SP}$ need not equal $\tilde{G}^O$. Thus the operators SP and L need not commute.

Example. Let Λ be the po-set



Denote the set of maximal (minimal) elements in Λ by A (B). Let V be the set of all functions from Λ into the reals. Then V is a real vector lattice if we define addition pointwise and define $v \in V$ to be positive if each non-zero maximal component is positive. Next let

$$G = \{v \in V \mid v \text{ is constant on } A\}.$$

Note that G is laterally complete but not a P-group. Let

$$H = \{v \in V \mid v \text{ restricted to } A \text{ has finite range}\}.$$

Then H is not laterally complete and $H^L = V$. We show that

$$H = G^{SP} = G^P.$$

Clearly G is large in H and H is an SP-group. Suppose that $G \subseteq K \subseteq H$, where K is a P-group. Let $'$ $(^*)$ denote the polars in K (H). Let S be a subset of B and let $s \in G$ be the characteristic function on S. Let $a \in G$ be the characteristic function on A.

$$K = s'' \oplus s', \; H = s^{**} \oplus s^* \text{ and } s^{**} \cap K = s'' \text{ and } s^* \cap K = s'.$$

Thus $a = a_1 + a_2 \in s'' \oplus s' = K$ and this is also the decomposition in $H = s^{**} \oplus s^*$. Thus a_1 is the characteristic function of the elements in A above S, but such elements generate the group of functions on A with finite range. Therefore $K = H$ and hence $H = G^P$.

Proposition 2.10. *If G is a representable l-group then $(G^L)^P = (G^L)^{SP}$.*

Proof. Take $C \in P((G^L)^P$; then $C \cap G^L = Cv \in P(G^L)$: so as in Lemma 2.1, $Cv = a''$, and thus $C = Cv\mu = a''\mu = (a'')^{**} = a^{**}$, by (3) and (5). Thus $(G^L)^P$ is an SP-group and so $(G^L)^P = (G^L)^{SP}$.

Corollary. Let G be a representable l-group.

i) $(G^O)^X = (G^X)^O$ for $X = P$, SP or L and $(G^P)^{SP} = (G^{SP})^P = G^{SP}$.

ii) $(G^L)^P = (G^L)^{SP} \subseteq (G^P)^L = (G^{SP})^L$ and equality need not hold.

3. The X-hulls of D_f-modules and f-rings

A *p-endomorphism* of an l-group G is an endomorphism α of the group such that

$$x \wedge y = 0 \text{ implies } x\alpha \wedge y = 0 \text{ for all } x, y \in G.$$

It is easy to show that this is equivalent to $G^+\alpha \subseteq G^+$ and $C\alpha \subseteq C$ for each $C \in P(G)$ (see [13]). Thus the p-endomorphisms of G are the l-endomorphisms that preserve polars. In Section 4 we shall show that each p-endomorphism of a representable l-group G has a unique extension to the X-hull G^X of G.

Let D be a directed po-ring. G is a *D_f-module* (see [22]) if G is an abelian l-group and a D-module such that for each $d \in D^+$ the map

$$g \dashrightarrow gd \text{ for all } g \in G$$

is a p-endomorphism of G. Steinberg [22] shows that such a G is isomorphic to a subdirect sum of totally ordered modules. Note that each polar of G is a submodule. Note also that each abelian l-group A is a D_f-module with respect

https://doi.org/10.1017/S1446788700015391 Published online by Cambridge University Press

to the ring Z of integers and also with respect to the directed ring D of all polar preserving endomorphisms of A.

PROPOSITION 3.1. *If G is a vector lattice over a totally ordered division ring D then G is a D_f-module.*

PROOF. We are given that G is an abelain l-group and $G^+D^+ \subseteq G^+$. If $d \in D^+$ and $g \in G$ then $(g \vee 0)d = gd \vee 0$. For $(g \vee 0)d \geqq gd$ and 0 and if $z \geqq gd$ and 0 then $zd^{-1} \geqq g$ and 0 and so $zd^{-1} \geqq g \vee 0$. Therefore $z \geqq (g \vee 0)d$.

Now suppose that $x \wedge y = 0$, where $x, y \in G$ and $d \in D^+$. If $1 \geqq d$ then $x \geqq xd$ and hence $0 = x \wedge y \geqq xd \wedge y = 0$. If $d > 1$ then $1 > d^{-1}$ and so $x \wedge yd^{-1} = 0$. Thus $0 = (x \wedge yd^{-1})d = xd \wedge y$.

Suppose that G is a D_f-module. Then each $C \in P(G)$ is a submodule and hence G/C' is a D_f-module. Thus each of the l-groups $G_{\mathscr{C}} = \prod G/C'$ used in the construction of $\mathcal{O}(G)$ is an D_f-module and each of the connecting l-isomorphisms $\pi_{\mathscr{C}\mathscr{A}}$ also preserves scalar multiplication by elements of D. Consider $\mathscr{L} \in \mathcal{O}(G)$ and $\mathscr{C} \in D(G)$ such that

$$0 \neq \mathscr{L}_{\mathscr{C}} = (\cdots, C' + \mathscr{L}(C), \cdots) \text{ where } \mathscr{L}(C) \in G.$$

Define $\mathscr{L}d$ to be the element in $\mathcal{O}(G)$ with $(\mathscr{L}d)_{\mathscr{C}} = (\cdots, C' + \mathscr{L}(C)d, \cdots)$. It follows that $\mathcal{O}(G)$ is a D_f-module and the natural map $g \dashrightarrow \tilde{g}$ of G into $\mathcal{O}(G)$ also preserves scalar multiplication by elements of D.

THEOREM 3.2. *There exists a unique minimal essential extension G^{X_D} of the D_f-module G that is an X-group and also a D_f-module. G^{X_D} is isomorphic to the intersection of all X-subgroups of $\mathcal{O}(G)$ that contain G and are D_f-modules.*

The proof is analogous to the proof of Theorem 2.6. We shall show that $G^X = G^{X_D}$ as l-groups and there exists a unique extension of the scalar multiplication of G to a scalar multiplication of G^X by D.

Recall that an f-ring G is a lattice ordered ring such that

$$x \wedge y = 0 \text{ implies } xd \wedge y = dx \wedge y = 0 \text{ for all } x, y, d \in G^+.$$

Thus each polar of G is a ring ideal and so it follows that $\mathcal{O}(G)$ is also an f-ring and the natural l-isomorphism of G into $\mathcal{O}(G)$ is a ring isomorphism.

THEOREM 3.3. *There exists a unique minimal essential extension G^{X_f} of the f-ring G that is an X-group and also an f-ring. Moreover, G^{X_f} is isomorphic to the intersection of all X-subgroups of $\mathcal{O}(G)$ that contain G and are sub-f-rings of $\mathcal{O}(G)$.*

Again the proof is analogous to the proof of Theorem 2.6. We shall show that $G^X = G^{X_f}$ as l-groups and there exists a unique f-ring structure for G^X so that G is a subring.

4. Lifting p-endomophisms from G to G^X

Let G be a representable l-group and let $\tilde{G}^X$ be the X-hull of G in $\mathcal{O}(G)$.

THEOREM A. (*Chambless* [7]) *$\tilde{G}^{SP} = \{l \in \mathcal{O}(G)\, l = \theta$ or $l_{\mathscr{E}} \neq 0$ for some finite partition of $P(G)\}$. Thus $\tilde{G}^{SP}$ is the direct limit of the groups $G_{\mathscr{E}}$ for finite $\mathscr{E} \in D(G)$ and hence is the join of the directed set of l-groups $G_{\mathscr{E}}\pi_{\mathscr{E}}$, where $\pi_{\mathscr{E}}$ is the natural map of $G_{\mathscr{E}}$ into $\mathcal{O}(G)$.*

THEOREM B. (*Chambless* [7]). *Let S be the subalgebra of $P(G)$ generated by elements of the form g' and g''. Then*

$$\tilde{G}^P = \{l \in \mathcal{O}(G) \,|\, l = \theta \text{ or } l_{\mathscr{E}} \neq 0 \text{ for some finite partition of } P(G) \text{ such that } \mathscr{E} \subseteq S\}$$

Thus $\tilde{G}^P$ is a direct limit.

Now, as we have seen, if G is an f-ring then so are the $G_{\mathscr{C}}$ and so it follows that $\tilde{G}^P$ and $\tilde{G}^{SP}$ are subrings of $\mathcal{O}(G)$. We shall also show that $\tilde{G}^L$ is a subring of $\mathcal{O}(G)$.

Amemiya [1] mentions that if G is a vector lattice or an f-ring then under his construction G^P is also a vector lattice or an f-ring.

If G is an f-ring then each minimal prime subgroup of $(G, +)$ is a ring ideal and so $T = \prod G/M$, for all minimal prime subgroups M, is an f-ring. is a subring constructs G^P in T. Here it is hard to determine whether or not G^P Speed [21] since G^P is not large in T.

LEMMA 4.1. *If σ is a polar preserving endomorphism of an l-group G, $\{a_\alpha \,|\, \alpha \in A\}$ is a disjoint subset of G and $\bigvee a_\alpha$ exists, then $\{a_\alpha \sigma \,|\, \alpha \in A\}$ is disjoint and $(\bigvee a_\alpha)\sigma = \bigvee a_\alpha \sigma$.*

PROOF. Clearly $(\bigvee a_\alpha)\sigma \geqq a_\beta\sigma$ for all $\beta \in A$. Suppose that $d \geqq a_\beta \sigma$ for all β. Then $(\bigvee a_\alpha)\sigma \geqq (\bigvee a_\alpha)\sigma \wedge d \geqq a_\beta\sigma$ for each β and hence

$$(\bigvee a_\alpha)\sigma - x = (\bigvee a_\alpha)\sigma \wedge d \geqq a_\beta\sigma$$

for all β, where $x \geqq 0$. Therefore $(\bigvee a_\alpha)\sigma \geqq a_\beta\sigma + x$ for all β. To complete the proof it suffices to show that $x = 0$. Now $(\bigvee a_\alpha)\sigma \geqq a_\beta\sigma + x \wedge a_\beta$ for all β; so $(\bigvee_{\alpha \neq \beta} a_\alpha)\sigma \geqq x \wedge a_\beta$ for each β. But $(x \wedge a_\beta) \wedge a_\gamma = 0$ for all $\gamma \neq \beta$, and so

$$0 = (x \wedge a_\beta) \wedge (\bigvee_{\alpha \neq \beta} a_\alpha) = (x \wedge a_\beta) \wedge ((\bigvee_{\alpha \neq \beta} a_\alpha)\sigma) = x \wedge a_\beta$$

for each β; hence $x \wedge (\bigvee a_\alpha) = 0$, and thus $0 = x \wedge (\bigvee a_\alpha)\sigma = x$.

COROLLARY I. *If $\{a_\alpha \,|\, \alpha \in A\}$ is a disjoint subset of a D_f-module G over a directed po-ring D, $\bigvee a_\alpha$ exists and $0 < c \in D$ then $(\bigvee a_\alpha)c = \bigvee a_\alpha c$.*

COROLLARY II. *If $\{a_\alpha \,|\, \alpha \in A\}$ is a disjoint subset of an f-ring G and $\bigvee a_\alpha$ exists then $(\bigvee a_\alpha)c = \bigvee a_\alpha c$ and $c(\bigvee a_\alpha) = \bigvee c a_\alpha$ for each $c \in G^+$.*

LEMMA 4.2. (*Henriksen and Isbell* [15]). *If Y is a multiplicative subsemigroup of an f-ring F then the l-subgroup T of $(F, +)$ that is generated by Y is a subring.*

PROOF. Let $[Y] = \{e_1 y_1 + \cdots + e_n y_n \mid y_i \in Y,\ e_i = \pm 1 \text{ and } n \geqq 0\}$ be the subgroup of $(F, +)$ generated by Y. Then

$$T = \{\bigvee_A \bigwedge_B s_{\alpha\beta} \mid s_{\alpha\beta} \in [Y] \text{ and } A \text{ and } B \text{ are finite}\}.$$

But $[Y]$ is a subring of F and if $a = \bigvee \bigwedge a_{\alpha\beta}$ and $b = \bigvee \bigwedge b_{\gamma\delta}$ belong to T then $a^+ = \bigvee \bigwedge (a_{\alpha\beta} \vee 0)$ and $b^+ = \bigvee \bigwedge (b_{\gamma\delta} \vee 0)$ and since positive elements distribute multiplicatively over $\vee$ and $\wedge$ it follows that $a^+ b^+ \in T$ and hence T is a subring of F.

PROPOSITION 4.3. *Suppose that G is an f-ring and also a subring of the f-ring H. If H is laterally complete and an essential extension of G then the lateral completion G^L of $(G, +)$ in H is a subring.*

PROOF. Consider $\{a_\alpha \mid \alpha \in A\}$ and $\{b_\beta \mid \beta \in B\}$ disjoint subsets of G. Then by Corollary II of Lemma 4.1

$$(\vee a_\alpha)(\vee b_\beta) = \vee a_\alpha b_\beta.$$

Thus the set of all such $\vee a_\alpha$ is a subsemigroup of H. It follows from Lemma 4.2 that the l-subgroup $G(1)$ of H generated by these elements $\vee a_\alpha$ is a subring. Then by transfinite induction it follows that G^L is a subring of H, (see [9]).

THEOREM 4.4. *Let G be a representable l-group and let $X = P, SP, L$ or O.*

1) *A p-endomorphism σ of G has a unique extension to a p endomorphism σ^X of G^X.*

2) *If σ is one to one then so is σ^X. If σ is onto then so is σ^X for $X = P, SP$ or O.*

3) *If α is a p endomorphism of G^O such that $G\alpha \subseteq G$ then $G^X \alpha \subseteq G^X$.*

PROOF. If $\mathscr{C} \in D(G)$ and $C \in \mathscr{C}$ then $C' + g \dashrightarrow C' + g\sigma$ is an l-endomorphism of G/C' and hence

$$(\cdots, C' + g(C), \cdots) \xrightarrow{\sigma_{\mathscr{C}}} (\cdots, C' + g(C)\sigma, \cdots)$$

is an l-endomorphism of $G_{\mathscr{C}}$. If $\mathscr{C} \geqq \mathscr{A} \in D(G)$ then

$$\begin{array}{ccc} G_{\mathscr{C}} = \prod G/C' & \xrightarrow{\ \sigma_{\mathscr{C}}\ } & G_{\mathscr{C}} \\ \downarrow{\scriptstyle \pi_{\mathscr{C}\mathscr{A}}} & & \downarrow{\scriptstyle \pi_{\mathscr{C}\mathscr{A}}} \\ G_{\mathscr{A}} = \prod G/A' & \xrightarrow{\ \sigma_{\mathscr{A}}\ } & G_{\mathscr{A}} \end{array}$$

https://doi.org/10.1017/S1446788700015391 Published online by Cambridge University Press

commutes. For $(\cdots, C' + g(C), \cdots)\sigma_{\mathscr{C}}\pi_{\mathscr{C}\mathscr{A}} = (\cdots, C' + g(C)\sigma, \cdots)\pi_{\mathscr{C}\mathscr{A}} = (\cdots, A' + g(C)\sigma, \cdots) = (\cdots, A' + g(C), \cdots)\sigma_{\mathscr{A}} = (\cdots, C' + g(C), \cdots)\pi_{\mathscr{C}\mathscr{A}}\sigma_{\mathscr{A}}$ where of course $A \subseteq C$.

Thus σ determines an l-endomorphism $\bar{\sigma}$ of $\mathcal{O}(G)$. Let π be the natural map of G onto $\tilde{G} \subseteq \mathcal{O}(G)$. Then $(g\pi)_{\mathscr{C}} = (\cdots, C' + g, \cdots)$ for all $\mathscr{C} \in D(G)$, and $\pi\bar{\sigma} = \sigma\pi$ on G and so $\bar{\sigma}$ is an extension to $\mathcal{O}(G)$ of the p endomorphism $\pi^{-1}\sigma\pi$ of $\tilde{G}$.

We next show that $\bar{\sigma}$ is a p-endomorphism of $\mathcal{O}(G)$. If $\theta \neq l, k \in \mathcal{O}(G)$ and $\wedge k = \theta$ then there exist $\mathscr{C} \in D(G)$ such that $l_{\mathscr{C}} \neq 0 \neq k_{\mathscr{C}}$ and such that their supports are disjoint. If $l_{\mathscr{C}}\sigma_{\mathscr{C}} = 0$ then $l\bar{\sigma} = 0$ and hence $l\bar{\sigma} \wedge k = \theta$. In any case the support of $l_{\mathscr{C}}\sigma_{\mathscr{C}} \subseteq$ support of $l_{\mathscr{C}}$ and hence $l_{\mathscr{C}}\sigma_{\mathscr{C}} \wedge k_{\mathscr{C}} = 0$ and so $l\bar{\sigma} \wedge k = \theta$. Therefore $\bar{\sigma}$ is a p-endomorphism of $\mathcal{O}(G)$.

We next show that if α is a p-endomorphism of $\mathcal{O}(G)$ that induces $\pi^{-1}\sigma\pi$ on $\tilde{G}$ then $\alpha = \bar{\sigma}$. Consider $l_{\mathscr{C}} = (\cdots, C' + g, \cdots)$ and suppose that $(l\alpha)_{\mathscr{C}} = (\cdots, C' + x, \cdots)$ where $C' + x \neq C' + g\sigma$. Then

$$|\tilde{g} - l|_{\mathscr{C}} \wedge (0, \cdots, 0, C' + |g\sigma - x|, 0, \cdots, 0) = 0 \text{ but}$$

$$(|\tilde{g} - l|\alpha)_{\mathscr{C}} \wedge (0, \cdots, 0, C' + |g\sigma - x|, 0, \cdots, 0) \neq 0$$

and thus α is not a p endomorphism, a contradiction.

Therefore σ has a unique extension to a p-endomorphism of G^O. Now if ρ is an extension of σ to say G^P then it can be extended to G^O and so ρ is unique. Thus to complete the proof of (1) it suffices to verify (3). So suppose that α is a p endomorphism of G^O such that $G\alpha \subseteq G$.

a) $G^L\alpha \subseteq G^L$. For if $\{a_\lambda \mid \lambda \in \Lambda\}$ is a disjoint subset of G then by Lemma 4.1 $(\vee a_\lambda)\alpha = \vee a_\lambda\alpha$ and so $G(1)\alpha \subseteq G(1)$, where $G(1)$ is the l-subgroup of G^L that is generated by all the elements $\vee a_\lambda$. Thus it follows by transfinite induction that $G^L\alpha \subseteq G^L$.

b) $G^{SP}\alpha \subseteq G^{SP}$. Here we assume that $G = \tilde{G}$ and $G^O = \mathcal{O}(G)$. Then we know exactly how α operates on $\mathcal{O}(G)$. Consider $\theta \neq l \in G^{SP}$. Then $l_{\mathscr{C}} \neq 0$ for some finite partition $\mathscr{C}$ of $P(G)$. If $(l\alpha)_{\mathscr{C}} = 0$ then $l\alpha = \theta$ and if $(l\alpha)_{\mathscr{C}} \neq 0$ then clearly $l\alpha \in G^{SP}$ by Chambless' Theorem A.

c) $G^P\alpha \subseteq G^P$. This is a simple application of Chambless' Theorem B. This completes the proof of (1) and (3).

(2) If σ is one to one then σ^X is one to one since G is large in G^X. Now suppose that σ is onto. Then the map $C' + g \dashrightarrow C' + g\sigma$ is an l-homomorphism of G/C' onto itself. Thus σ^O is clearly onto and using our representations of G^P and G^{SP} it follows that σ^P and σ^{SP} are also onto.

Question. *Is σ^L onto provided that σ is onto?*

Theorem 4.5. *If G is a D_f-module over the directed po-ring D then there* *exists a unique extension of the scalar multiplication by elements of D so that G^X is also a D_f-module. Moreover G^X with this scalar multiplication equals G^{X_D} for $X = P$, SP, L or O.*

PROOF. The first part follows from the fact that each p-endomorphism of G has a unique extension to a p endomorphism of G^X. Now (without loss of generality) $G \subseteq G^X \subseteq G^{X_D} \subseteq \mathcal{O}(G)$ and G^X is a submodule of G^{X_D}. Therefore $G^X = G^{X_D}$.

THEOREM 4.6. *If G is an f-ring then there is a unique multiplication on G^X so that G^X is an f-ring and G is a subring. Moreover, G^X with this ring structure equals G^{X_f} for $X = P$, SP, L or O.*

PROOF. We first verify the result for $X = O$. Now as we have seen $\mathcal{O}(G)$ is a ring and the natural map $g \dashrightarrow \tilde{g}$ is a ring l-isomorphism. So all we need show is that the multiplication of $\mathcal{O}(G)$ is uniquely determined by that of $\tilde{G}$. Suppose that $\cdot$ is a multiplication on $\mathcal{O}(G)$ so that $\mathcal{O}(G)$ is an f-ring and $\cdot$ induces the given multiplication on $\tilde{G}$.

If $0 < \tilde{g} \in \tilde{G}$ then the right multiplication of $\tilde{G}$ by $\tilde{g}$ is a p-endomorphism of $\tilde{G}$ and so has a unique extension to a p endomorphism of $\mathcal{O}(G)$. Therefore

$$x \cdot \tilde{g} = x\tilde{g} \text{ for all } x \in \mathcal{O}(G).$$

Suppose that $x_{\mathscr{C}} = (0, \cdots, 0, C' + t, 0, \cdots, 0)$. Now

$$\begin{aligned} \tilde{g}_{\mathscr{C}} &= (0, \cdots, 0, C' + g, 0, \cdots, 0) + \text{(the other non-zero components)} \\ &= \qquad\qquad a \qquad\qquad + \qquad\qquad b. \end{aligned}$$

Now $x_{\mathscr{C}} \cdot b = 0$ since they are disjoint and so $(0, \cdots, 0, C' + tg, 0, \cdots, 0) = x_{\mathscr{C}}\tilde{g}_{\mathscr{C}} = x_{\mathscr{C}} \cdot (a + b) = x_{\mathscr{C}} \cdot a = (0, \cdots, 0, C' + t, 0, \cdots, 0) \cdot (0, \cdots, 0, C' + g, 0, \cdots, 0)$.

Now consider $x, y \in \mathcal{O}(G)$ with $x_{\mathscr{C}} \neq 0 \neq y_{\mathscr{C}}$.

$$x_{\mathscr{C}} = (\cdots, C' + x(C), \cdots) = \bigvee x_C, \text{ where } x_C = (0, \cdots, 0, C' + x(C), 0, \cdots, 0)$$

$$y_{\mathscr{C}} = (\cdots, C' + y(C), \cdots) = \bigvee y_C, \text{ where } y_C = (0, \cdots, 0, C' + y(C), 0, \cdots, 0).$$

Thus by Lemma 4.1 and the above

$$x_{\mathscr{C}} \cdot y_{\mathscr{C}} = \bigvee x_C \cdot \bigvee y_C = \bigvee x_C \cdot y_C = \bigvee x_C y_C = x_{\mathscr{C}} y_{\mathscr{C}}.$$

Therefore $\cdot$ is the natural multiplication on $\mathcal{O}(G)$ and so there is a unique f-ring structure on G^O so that G is a subring of the f-ring G^O.

Finally we have shown that $\tilde{G}^P$, $\tilde{G}^{SP}$ and $\tilde{G}^L$ are all subrings of $\mathcal{O}(G)$. Also any ring structure on G^X that induces the given one on G can be extended to a ring structure on G^O. Therefore the ring structures of G^P, G^{SP} and G^L are also determined by their additive structures.

5. The y-hulls of archimedean l-groups and f-rings

An archimedean l-group A is called a
d-group if it is divisible,
v-group if it is a vector lattice,
c-group if it a conditionally complete lattice,
e-group if it is essentially closed in the class of archimedean l-groups.

It is well known that an abelian l-group A is contained in a unique minimal divisible abelian l-group A^d. For there is exactly one way of extending the order of A to a lattice-order of its injective hull A^d so that $(A^d)^+ \cap A = A^+$. Also if A is archimedean then so is A^d.

THEOREM 5.1. *If A is a large l-subgroup of an archimedean y-group H, where $y = d$, v, c or e, then the intersection K of all the l-subgroups of H that contain A and are y-groups is a y-group. Thus K is a minimal essential extension of A that is a y-group and we shall call such an extension a y-hull of A.*

THEOREM 5.2. *Each archimedean l group A admits a unique y-hull A^y for $y = d, v, c$ or e. A^c is the Dedekind MacNeille completion $A^\wedge$ of A and A is dense in A^c. A^v is the l-subspace of $(A^d)^c$ that is generated by A. $A^e = ((A^d)^c)^L$ is the essential closure of A.*

REMARKS. A minimal essential extension of an archimedean l-group that is a vector lattice is necessarily archimedean [11]. Bleier [6] has shown that a minimal archimedean vector lattice that contains A is necessarily an essential extension of A and hence is A^v. Also, of course, any complete l-group is archimedean.

PROOF OF THEOREM 5.1. If $y = d$ or v then clearly the theorem holds. For the intersection of divisible subgroups (subspaces) is again divisible (a subspace). If A is a large l-subgroup of an archimedean e-group H then clearly H is an e-hull of A. To prove the theorem for $y = c$ we make use of the following two lemmas.

LEMMA 5.3. (*Bernau* [3]). *If G is a dense l-subgroup of an l-group H then all joins and intersections in G agree with those in H.*

LEMMA 5.4. *If A is a large l-subgroup of an abelian l group B then all joins and intersections in A agree with those in B.*

PROOF. A is large in B^d and so A^d is dense in B^d. Suppose that $\{a_\lambda \mid \lambda \in \Lambda\} \subseteq A$ and $\bigvee_A a_\lambda$ exists. If $\{a_\lambda \mid \lambda \in \Lambda\} \leqq y \in A^d$ then $ny \in A$ for some $n > 0$ and so $ny \geqq \bigvee_A na_\lambda = n \bigvee_A a_\lambda$. Thus $y \geqq \bigvee_A a_\lambda$ and hence $\bigvee_{A^d} a_\lambda = \bigvee_A a_\lambda$.

Next $\bigvee_{A^d} a_\lambda = \bigvee_{B^d} a_\lambda$ since A^d is dense in B^d. Finally $\bigvee_{B^d} a_\lambda = \bigvee_B a_\lambda$ since $\{a_\lambda \mid \lambda \in \Lambda\} \subseteq B$ and $\bigvee_{B^d} a_\lambda = \bigvee_A a_\lambda \in A \subseteq B$. Thus $\bigvee_A a_\lambda = \bigvee_B a_\lambda$.

COROLLARY. *If A is a large l-subgrcup of a complete l-group H, then the intersection of all c subgroups of H that contain A is a c subgroup.*

QUESTION. *Is Lemma 5.4 true for non abelian l groups?*

PROOF OF THEOREM 5.2. Clearly the theorem holds for $y = d$. In [11] it is shown that A admits a unique v hull A^v and that A^v is the l subspace of $(A^d)^\wedge$ that is generated by A.

In [10] it is shown that A admits a unique essential closure A^e and that $A^e = ((A^d)^\wedge)^L$.

The existence of A^e for a complete vector lattice A was proven by Pinsker [19] and Jakubik [16] showed that A^e can be constructed solely from the underlying lattice structure of A.

We now show that there exists a unique c hull A^c and that $A^c = A^\wedge$. Note that $A^\wedge$ is the unique minimal complete l group in which A is dense [12]. Also if A is an l-subgroup of a complete l-group H then H need *not* contain a copy of $A^\wedge$ [12].

LEMMA 5.5. *If A is a large l-subgroup of a complete l group H then $A^\wedge \subseteq H$.*

PROOF. We shall show that there exists an l-isomorphism of $A^\wedge$ into H that is the identity on A. If $x \in A^\wedge$ then

$$x = \bigvee \{\underline{x} \in A \mid \underline{x} \leqq x\} = \bigwedge \{\bar{x} \in A \mid \bar{x} \geqq x\}.$$

Since $\bar{x} \geqq \{\underline{x} \in A \mid \underline{x} \leqq x\}$ we have that $\bigvee_H \underline{x}$ exists. In particular for $0 < x \in A^\wedge$, $x = \bigvee \{\underline{x} \in A^+ \mid \underline{x} \leqq x\}$ and $\bigvee_H \{\underline{x} \in A^+ \mid \underline{x} \leqq x\}$ exists. Define

$$x\sigma = \bigvee_H \{\underline{x} \in A^+ \mid \underline{x} \leqq x\}.$$

1) If $a \wedge b = 0$ in $A^\wedge$ then $a\sigma \wedge b\sigma = 0$.

For $a = \bigvee \underline{a}$ and $b = \bigvee \underline{b}$, where $\underline{a} \wedge \underline{b} = 0$ and hence

$$0 \leqq a\sigma \wedge b\sigma = \bigvee_H \underline{a} \wedge \bigvee_H \underline{b} = \bigvee_H (\underline{a} \wedge \underline{b}) = 0.$$

2) If $a, b \in (A^\wedge)^+$ then $a\sigma + b\sigma = (a + b)\sigma$.

For $a\sigma + b\sigma = \bigvee_H \underline{a} + \bigvee_H \underline{b} = \bigvee_H (\underline{a} + \underline{b}) = \bigvee_H X$, where

$$X = \{\underline{a} + \underline{b} \mid \underline{a}, \underline{b} \in A^+, \underline{a} \leqq a \text{ and } \underline{b} \leqq b\}, \text{ and}$$

$$(a + b)\sigma = \bigvee_H \underline{a + b} = \bigvee_H Y, \text{ where}$$

$$Y = \{y \in A^+ \mid y \leqq a + b\}.$$

Now if $x \in X$ then $x = \underline{a} + \underline{b} \leqq a + b$ and so $x \in Y$. Thus $X \subseteq Y$ and hence $\bigvee_H X \leqq \bigvee_H Y$.

https://doi.org/10.1017/S1446788700015391 Published online by Cambridge University Press

If $y \in Y$ then $0 \leqq y \leqq a + b$ and hence $y = u + v$ where $u, v \in A^\wedge, 0 \leqq u \leqq a$ and $0 \leqq v \leqq b$. Thus $u = \bigvee \underline{u}$ and $v = \bigvee \underline{v}$ and hence $y = \bigvee (\underline{u} + \underline{v}) = \bigvee_{A^\wedge} S$ where $S \subseteq X \subseteq A$ and $y \in A$. Therefore $y = \bigvee_{A^\wedge} S = \bigvee_A S = \bigvee_H S$ since by Lemma 5.4 joins in A agree with those in H. Thus $y \leqq \bigvee_H X$ and so $\bigvee_H Y \leqq \bigvee_H X$.

Therefore σ is a map of $(A^\wedge)^+$ into H^+ that preserves addition and disjointness and induces the identity on A^+. For $g = a - b \in A^\wedge$, where $a, b \in (A^\wedge)^+$ define $g\tau = a\sigma - b\sigma$. Then τ is a group homorphism of $A^\wedge$ into H that preserves disjointness and so it is an l-homomorphism. Since τ induces the identity on the large l subgroup A of $A^\wedge$ it follows that τ is an l-isomorphism.

Corollary I. $A^\wedge \subseteq (A^d)^\wedge$.

Corollary II. *If A is a large l-subgroup of a complete l-group H and no proper l-subgroup of H contains A and is complete, then $H = A^\wedge$. In particular A is dense in H.*

Corollary III. $A^c = A^\wedge$ *is unique.*

This completes the proof of Theorem 5.2.

If follows at once from Lemma 5.4 that if A is a large l-subgroup of a σ-complete l-group H then the intersection K of all the σ complete l-subgroups of H that contain A is σ complete. Thus K is a *σ complete hull* of A. Since A is large in $K^\wedge$ it follows from Lemma 5.5 that $A \subseteq A^\wedge \subseteq K^\wedge$. Now $A^\wedge \cap K$ is σ-complete and contains A and so since K is minimal we have $A \subseteq K \subseteq A^\wedge$. Thus K is the intersection of all σ-complete l-subgroups of $A^\wedge$ that contain A and hence K is unique. Therefore *each archimedean l-group A admits a unique σ-complete hull A^σ.*

It is well known that A^σ is a P-group but need not be an SP-group (see for example [25] p. 85).

If each bounded disjoint subset of an archimedean vector lattice A is countable then since A is dense in A^σ it follows that each bounded disjoint subset of A^σ is also countable. Thus ([25] p. 156) A^σ is complete and hence $A^\sigma = A^\wedge$. These spaces A^σ of "countable type" were introduced by Pinsker and have many nice properties (see [25] pp. 156–160).

Theorem 5.6. *If α is a p-endomorphism of an archimedean l-group A then there exists a unique extension of α to a p endomorphism $\bar{\alpha}$ of the y-hull A^y of A, where $y = d, v, c$ or e.*

Proof. The proof for $y = c$ is contained in [13]. Suppose that $y = d$ and consider $a \in A^y$. Then $na \in A$ for some $n > 0$. Define $a\bar{\alpha} = ((na)\alpha)/n$. A straightforward computation shows that $\bar{\alpha}$ is a p endomorphism of A^y and an extension

https://doi.org/10.1017/S1446788700015391 Published online by Cambridge University Press

of α. If β is an extension of α to a p-endomorphism of A^y then

$$n(a\beta) = (na)\beta = (na)\alpha = (na)\bar{\alpha} = n(a\bar{\alpha})$$

and hence $a\beta = a\bar{\alpha}$.

Combining the above we get a unique extension of α to a p-endomorphism γ of $(A^d)^c$. Also γ is linear [13] and maps A into A. Thus γ maps the l-subspace A^v of $(A^d)^c$ that is generated by A into A^v.

Finally since $A^e = ((A^d)^c)^L$ it follows from Theorem 4.4 that α has a unique extension to a p endomorphism of A^e.

COROLLARY. *If A is an archimedean D_f-module over the directed po-ring D then there exists a unique extension of the scalar multiplication by elements of D so that A^y is also a D_f-module, where $y = d, v, c$ or e.*

REMARKS. Since A is large in A^y it follows that α is one-to-one if and only if $\bar{\alpha}$ is one-to-one. It can be shown that if $y = d, v$ or c then $\bar{\alpha}$ is onto provided that α is onto. The proof for $y = c$ is given in [13]. Bleier [6] shows that an l-automorphism of A has a unique extension to an l-automorphism of A^y.

THEOREM 5.7. *If A is an archimedean l-group and α is an l-automorphism of A then there exists a unique extension to an l-automorphism $\bar{\alpha}$ of A^y, where $y = d, v, c$ or e.*

PROOF. For $y = d$ the map $\bar{\alpha}$ defined in the proof of the last theorem is an l-automorphism of A^d. We have shown that the theorem holds for $y = L$. Thus to complete the proof it suffices to show that α can be extended uniquely to an l-automorphism of A^c. For $h \in (A^c)^+$, $h = \bigvee \{\underline{h} \in A^+ \mid \underline{h} \leqq h\}$. Define

$$h\bar{\alpha} = \bigvee \underline{h}\alpha.$$

A straightforward computation shows that $\bar{\alpha}$ determines an l-automorphism of A^c that is the unique extension of α (see the proof of Lemma 5.5).

LEMMA 5.8. (*Bernau* [2]). *If F is an archimedian f-ring, $x \in F^+$, $\{a_\lambda \mid \lambda \in \Lambda\} \subseteq F$ and $\bigvee a_\lambda$ exists then $\bigvee (xa_\lambda)$ exists and $\bigvee (xa_\lambda) = x(\bigvee a_\lambda)$, and dually.*

THEOREM 5.9. *Suppose that A is an archimedean f-ring, and A^y is the y-hull of $(A, +)$ for $y = d, v, c$ or e. Then there is a unique multiplication on A^y so that A^y is an f-ring and A is a subring. Thus the additive structure of A^y completely determines the ring structure.*

PROOF. For $a, b \in A^d$ there exists an integer $n > 0$ such that na and nb belong to A. Define

$$ab = ((na)(nb)/n^2.$$

A routine check shows that A^d is an f-ring and this is the unique extension of the multiplication of A to an f-ring multiplication of A^d.

For $a, b \in ((A^d)^c)^+$ define

$$ab = \wedge \{xy \mid x \geqq a, y \geqq b \text{ and } x, y \in A^d\}$$

and for $x = x_1 - x_2$ and $y = y_1 - y_2$ in $(A^d)^c$ where $x_i, y_i \in ((A^d)^c)^+$ define

$$xy = x_1y_1 + x_2y_2 - (x_1y_2 + x_2y_1).$$

A rather long messy computation shows that $(A^d)^c$ is an f-ring. This construction is "well known".

Now suppose that $\cdot$ and $\times$ are two multiplications of $(A^d)^c$ so that it is an f-ring and A^d is a subring and consider $a, b \in ((A^d)^{c+}$.

$$a = \wedge \{x \in A^d \mid x \geqq a\} \text{ and } b = \wedge \{y \in A^d \mid y \geqq b\}$$

and hence by Lemma 5.8

$$a \cdot b = (\wedge x) \cdot (\wedge y) = \wedge (x \cdot y) = \wedge (x \times y) = (\wedge x) \times (\wedge y) = a \times b.$$

Thus there is only one such multiplication. Of course the same result holds for A^c.

Now we have shown that the ring structure of $(A^d)^c$ has a unique extension to $((A^d)^c)^L = A^e$ (see Theorem 4.6). To complete the proof it suffices to show that A^v is a subring of A^e. Consider $x, y \in A$ and $r, s \in R$. Then $rx, sy \in A^v$ and $xy \in A$. Thus since A^e is a real algebra (see Section 6)

$$(rx)(sy) = rs(xy) \in A^v.$$

It follows that the subspace S of A^e that is generated by A is a subring of A^e. Now

$$A^v = \{\vee_U \wedge_V a_{\alpha\beta} \mid a_{\alpha\beta} \in S, \alpha \in U, \beta \in V \text{ and } U \text{ and } V \text{ are finite}\}.$$

Thus by Lemma 4.2 A^v is a subring of A^e.

REMARKS. If A is an archimedean f-ring and H is a minimal essential extension of A that is an archimedean f-ring and a y-group then $H = A^y$. For clearly $A \subseteq A^y \subseteq H$ as l-groups by Theorems 5.1 and 5.2. If $y = e$ then A^e is essentially closed and large in H and so $A^e = H$. If $y = d$ then an easy computation shows that A^d is a subring of H and so $A^d = H$.

If $y = c$ or v then a rather messy proof shows that A^y is a subring of H and so once again $A^y = H$.

6. The structure of an archimedean f-ring

Let A be an archimedean f-ring and let X be the Stone space of the complete Boolean algebra $P(A)$ of polars of A. Then X is compact, Hausdorff and extremally disconnected. Let $D(X)$ be the ring of continuous functions from X

into the extended reals $(R, \pm \infty)$ that are finite on a dense open subset of X. Then as l groups A^e and $D(X)$ are isomorphic [10]. So let us examine the ways in which $D(X)$ can be made into an f-ring with pointwise addition and order.

Suppose that $D = D(X)$ has a multiplication $\cdot$ so that it is an f-ring. Then for $a \in D^+$ the map $d \dashrightarrow d \cdot a$, for all $d \in D$, is a p-endomorphism of $(D, +)$ and so (see [13]) there is an element $\bar{a} \in D^+$ such that

$$d \cdot a = d\bar{a} \text{ for all } d \in D.$$

We investigate the map $a \dashrightarrow \bar{a}$. Consider $a, b \in D^+$.

1) $\overline{a+b} = \bar{a} + \bar{b}$.

For $d(\overline{a+b}) = d \cdot (a+b) = d \cdot a + d \cdot b = d\bar{a} + d\bar{b} = d(\bar{a} + \bar{b})$ for all $d \in D$ and so for $d = 1$, $\overline{a+b} = \bar{a} + \bar{b}$.

2) $\overline{a\bar{b}} = \bar{a}\bar{b}$.

$d(\overline{a\bar{b}}) = d(\overline{a \cdot b}) = d \cdot (a \cdot b) = (d \cdot a) \cdot b = (d\bar{a})\bar{b} = d(\bar{a}\bar{b})$.

3) $a\bar{a} = b\bar{a}$.

$b\bar{a} = b \cdot a = a \cdot b = a\bar{b}$. Here we use the fact that an archimedean f-ring is commutative.

4) Put $\bar{1} = p$; then for $u, v \in D^+$, $u \cdot v = uvp$.

For, for $a \in D^+$, we have $\bar{a} = 1\bar{a} = a\bar{1} = ap$. Now, $v = a - b$, where $a, b \in D^+$, and so $u \cdot v = u \cdot (a - b) = u \cdot a - u \cdot b = u\bar{a} - u\bar{b} = uap - ubp = u(a-b)p = uvp$.

5) If $\cdot$ is a multiplication on $D(X)$ such that $D(X)$ is an f-ring with componentwise addition and order then there exists an element $p \in D^+$ so that $a \cdot b = abp$ for all $a, b \in D$, and conversely.

Now D is complete and hence a P group. Thus

$$D = p'' \oplus p'.$$

Clearly p'' is a subring with respect to the $\cdot$ multiplication and p' is a zero subring. Consider $d = u + v \in p'' \oplus p'$ and define

$$d\tau = pu + v.$$

Then for $d_1 = u_1 + v_1$ and $d_2 = u_2 + v_2$ in D we have

$$(d_1 \cdot d_2)\tau = (pu_1u_2)\tau = pu_1pu_2 = d_1\tau d_2\tau$$

and so we have an l-isomorphism of the f-ring $(D, +, \cdot, \leqq)$ onto the f-ring $D = p'' \oplus p'$, where p'' is a ring with respect to the pointwise multiplication of D and p' has the zero multiplication.

THEOREM 6.1. *Let X be a Stone space and suppose that $D(X)$ is an f-ring with componentwise addition and order. Then there exist clopen subsets Y and*

Z of X such that $X = Y \cup Z$, $Y \cap Z = \varnothing$ and $D(X) = D(Y) \oplus D(Z)$, where $D(Y)$ has the pointwise multiplication and $D(Z)$ has the zero multiplication.

Thus we have the structure of an arbitrary essentially closed archimedean f-ring. Recall that the radical of an f-ring A consists of the nilpotent elements.

COROLLARY I. (*Henricksen and Isbell* [15]). *An archimedian f-ring is a subdirect sum of a ring with zero multiplication and one with radical zero.*

COROLLARY II. *If A is an archimedean f-ring then rad $A = \{a \in A \mid aA = 0\}$ the set of annihilators of A. In particular, rad A is a polar.*

PROOF. $A \subseteq D(Y) \oplus D(Z)$ and if $a = u + v \in A$ is nilpotent, where $u \in D(Y)$ and $v \in D(Z)$ then $u = 0$ and so $a = v$ is an annihilator. Thus rad $A = A \cap D(Z)$. Now $D(Z)$ is a polar in $D(X)$ and A is large in $D(X)$. Thus rad A is a polar in A.

COROLLARY III. *If A is an archimedean f-ring and also an SP-group, then rad A is a cardinal summand. In particular, rad A is a cardinal summand of a complete f-ring A.*

Note that Corollaries II and III follow directly from Corollary I.

COROLLARY IV. *If A is an archimedean f-ring with a weak order unit u and also a P-group, then rad A is a cardinal summand.*

PROOF. Since A is large in A^e, u is also a weak unit of A^e and without loss of generality we may assume that as l-groups $A^e = D(X)$ and $1 = u \in A$. Then $1 \cdot 1 = p \in A$ and so $A = p'' \oplus p'$, where the polars are taken in A.

COROLLARY V. *For an archimedean f-ring A the following are equivalent.*

i) *rad $A = 0$.*
ii) *A^e contains an identity.*
iii) *rad $A^e = 0$.*

PROOF. $(\text{rad } A^e) \cap A = \text{rad } A$ and hence since A is large in A^e it follows that i) and iii) are equivalent. From the Theorem iii) and ii) are equivalent.

Let A be an archimedean f-ring with identity u. Then u is a weak unit in A ($u \wedge a = 0$ implies $a = ua = 0$) and hence in A^e. Let X be the Stone space of $P(A) = P(A^e)$. Then there is a l-group isomorphism of A^e onto $D(X)$ so that u maps upon 1. Thus without loss of generality, $1 \in A \subseteq A^e = D(X)$ as l-groups. It follows from the next theorem that A and A^e are both subrings of $D(X)$. Thus, once again, the additive structure of A determines the ring structure.

THEOREM 6.2. *Suppose that A is an l-subgroup of $(D(X), +)$ and $1 \in A$, where X is a Stone space. If A is an f-ring with identity* 1 *then A is a subring of $D(X)$.*

PROOF. Let $\cdot$ be the multiplication in A. Then by (6)

$$1 = 1 \cdot 1 = 1p = p.$$

Thus $\cdot$ agrees with the pointwise multiplication of $D(X)$.

COROLLARY I. (*Birkhoff and Pierce* [5]). *An archimedean f-ring with identity has radical zero.*

COROLLARY II. *If A is an archimedean f-ring with identity u then u is also an identity for the f-ring A^y, where $y = d, v, c$ or e.*

COROLLARY III. *If A is an archimedean f-ring with identity then each p-endomorphism of A is a multiplication by a positive element.*

PROOF. We may assume that A is a subring of $D(X)$, where $D(X)$ has the pointwise multiplication, and $1 \in A$. Thus any p-enomorphism of A has a unique extension to a p-endomorphism of $D(X)$, but each p-endomorphism of $D(X)$ is a multiplication by an element $d \in D^+$ [13]. Thus since $1 \in A$ it follows that $d \in A$.

We give two examples of archimedean f-rings for which the radical is not a cardinal summand.

I. Let $A = C[0, 1]$ and let

$$p(x) = \begin{cases} -x + \frac{1}{2} \text{ if } 0 \leqq x \leqq \frac{1}{2} \\ \quad 0 \text{ if } \frac{1}{2} \leqq x \leqq 1. \end{cases}$$

Define $g \cdot f = gfh$ *for* $g, f \in A$. Then A is an f-ring with

$$\operatorname{rad} A = \{f \in A \,|\, f(x) = 0 \text{ for } 0 \leqq x \leqq \tfrac{1}{2}\}$$

but $(A, +)$ is cardinally indecomposable and so $\operatorname{rad} A$ is not a summand.

II. Let $H = \prod_{i=1}^{\infty} Q_i$, where Q_i is the additive group of rationals. In the even components use zero multiplication and in the odd components use the natural multiplication. Let $a = (1/2, 1/4, 1/8, \cdots, 1/2^n, \cdots)$, and let S be the subring generated by a. Thus S is the ring of polynomials without constant terms in a and with integral coefficients. Let A be the subring of H generated by S and $\sum Q_i$.

$A = \{h \in H \,|\, h$ is a polynomial in a except at a finite number of places$\}$. Then A is an f-ring with a basis and a strong order unit, a but $\operatorname{rad} A$ is not a cardinal summand. Note that $a^2 = (1/4, 0, 1/64, 0, \cdots)$ but a does not split into a "zero part and a radical zero part".

The next two examples show the well known fact that the class of f-rings with zero radical is not equationally definable.

III. Let S be the semigroup of negative integers. Let A be the semigroup ring of S over the integers and define an element in A to be positive if its largest non-zero component is positive. Then A is a totally ordered integral domain

https://doi.org/10.1017/S1446788700015391 Published online by Cambridge University Press

and so $\operatorname{rad} A = 0$. Let J be the set of elements in A with support included in $-2, -3, \cdots$. Then J is a convex ring ideal and A/J is a zero ring. Thus $\operatorname{rad} A/J = A/J$.

IV. Let A be the set of all bounded rational sequences with cardinal order. Then $\operatorname{rad} A = 0$. Let $a = (1, 1/4, 1/9, \cdots, 1/n^2, \cdots)$ and

$$\langle a \rangle = \{x \in A \mid |x| < na \text{ for some } n > 0\}.$$

Then $J/\langle a \rangle$ is an f-ring and $0 \neq \langle a \rangle + (1, 1/2, 1/3, \cdots) \in \operatorname{rad} J/\langle a \rangle$.

The following example is due to Roger Bleier and shows that if G is an l-subgroup of an essentially closed archimedean l-group H then H need not contain a copy of the essential closure G^e of G.

V. Pick a Stone space Y so that $D(Y)$ cannot be represented as a subdirect sum of reals. Let $C(Y)$ be the l-group of all continuous real valued functions on Y. Then $C(Y) \subseteq \prod R_y$ and $C(Y)^e = D(Y) = C(Y)^L$.

7. The structure of an f-ring with a basis

A strictly positive element s in an f-ring A is called *basic* if s'' is totally ordered or equivalently if A/s' is a totally ordered ring. A *basis* for A is a maximal disjoint subset $\{s_\lambda \mid \lambda \in \Lambda\}$ where in addition each s_λ is basic. Let $S = \{s_\lambda \mid \lambda \in \Lambda\}$ be a basis for A. Then there exists a natural ring l-isomorphism σ of A into $K = \prod A/s_\lambda'$

$$a \xrightarrow{\ \ \sigma\ \ } (\cdots, s_\lambda' + a, \cdots).$$

THEOREM 7.1. *$K = (A\sigma)^O$ and if S is finite then $K = (A\sigma)^P$. In either case A is dense in A^O.*

PROOF. Consider $0 < x = (\cdots, s_\lambda' + x_\lambda, \cdots) \in K$ with say $s_\alpha' + x_\alpha > s_\alpha'$. Then we may assume $0 < x_\alpha \notin s_\alpha'$ and so $0 < a = x_\alpha \wedge s_\alpha \in (\cap_{\lambda \neq \alpha} s_\lambda') \setminus s_\alpha'$. Thus $0 < a\sigma \leqq x$ and so $A\sigma$ is dense in K. Thus since K is a P-group

$$A\sigma \subseteq (A\sigma)^P \subseteq K.$$

We next show that $\overline{s_\alpha' + x_\alpha} = (0, \cdots, 0, s_\alpha' + x_\alpha, 0, \cdots, 0) \in (A\sigma)^P$ and hence $(A\sigma)^P \supseteq \sum A/s_\lambda'$. Let $*$ ($\#$) be the polar operation in $(A\sigma)^P$ (K).

$$\begin{aligned} (A\sigma)^P &= \overline{s_\alpha' + s_\alpha}^{**} \oplus \overline{s_\alpha' + s_\alpha}^{*} = s_\alpha\sigma^{**} \oplus s_\alpha\sigma^{*} \\ x_\alpha\sigma &= \qquad c \qquad + \qquad d \end{aligned}$$

but this is also the unique decomposition of $x_\alpha\sigma$ in

$$K = \overline{s_\alpha' + s_\alpha}\#\# \oplus \overline{s_\alpha' + s_\alpha}\# = A/s_\alpha' \oplus \prod_{\lambda \neq \alpha} A/s_\lambda\,.$$

Thus $c = \overline{s_\alpha' + x_\alpha} \in (A\sigma)^P$.

https://doi.org/10.1017/S1446788700015391 Published online by Cambridge University Press

Clearly K is the lateral completion of $\sum A/s_\lambda'$ and hence of $(A\sigma)^P$. Thus K is the orthocompletion of $A\sigma$. If S is finite then $K = \sum A/s_\lambda'$ and so $(A\sigma)^P = K$.

COROLLARY I. *Each s_λ' is a prime ring ideal if and only if* $\operatorname{rad} A = 0$.

PROOF. $(\rightarrow)$ Each stalk A/s_λ' is an integral domain and so $\operatorname{rad} A = 0$.

$(\leftarrow)$ Suppose that $x, y \in A$, and $xy \in s_\alpha'$, then $|x||y| = |xy| \in s_\alpha'$ and so without loss of generality $0 < x \leqq y$ and $xy \in s_\alpha'$. Then by convexity $x^2 \in s_\alpha'$. Suppose (by way of contradiction) that $x \notin s_\alpha'$. Then $0 < a = x \wedge s_\alpha \in (\cap_{\lambda \neq \alpha} s_\lambda')$ s_α' and hence $a^2 \in \cap s_\lambda' = 0$, a contradiction.

REMARK. Chambless [7] has shown that if A is an f-ring with $\operatorname{rad} A = 0$ then each minimal prime subgroup of $(A, +)$ is a prime ring ideal.

Let A be an f-ring and suppose that A satisfies

(F) each bounded disjoint subset of A is finite.

Then A has a basis $S = \{s_\lambda \mid \lambda \in \Lambda\}$ and the mapping of a onto $(\cdots, s_\lambda' + a, \cdots)$ is a ring l-isomorphism of A into $\sum A/s_\lambda'$.

COROLLARY II. $\sum A/s_\lambda' = (A\sigma)^P$.

PROOF. Since $A\sigma$ is dense in $H = \sum A/s_\lambda'$ we have $A\sigma \subseteq (A\sigma)^P \subseteq H$ and we have shown that $H \subseteq (A\sigma)^P$.

COROLLARY III. *For an f-ring A the following are equivalent.*

1) *$A = \sum A_\lambda$, where each A_λ is a totally ordered ring.*

2) *A satisfies (F) and is a P-group.*

PROOF. Clearly 1) implies 2). If 2) holds then by Corollary II we have $A \cong \sum A/s_\lambda'$.

COROLLARY IV. *For an f-ring A the following are equivalent.*

1) $A = \sum A_\lambda$, where each A_λ is a totally ordered integral domain.

2) A satisfies (F), A is a P-group and $\operatorname{rad} A = 0$.

PROOF. Once again it is clear that 1) implies 2). Suppose that 2) is true. By Corollary III, $A \cong \sum A/s_\lambda'$ and by Corollary I each stalk A/s_λ' is an integral domain.

A convex l-subgroup C of an f-ring A will be called an *L-ideal* if C is also an ideal of the ring A and a *P-ideal* if C is a ring ideal and A/C is totally ordered. If $0 < s \in A$ is basic, then s' is a P-ideal.

THEOREM 7.2. *For an f-ring the following are equivalent.*

1) *$A = \sum A_\lambda$, where each A_λ is an o-simple totally ordered integral domain.*

2) *A satisfies (F), $\operatorname{rad} A = 0$ and the P-ideals of A satisfy the DCC.*

If this is the case then the P-ideals of A are trivially ordered by inclusion.

https://doi.org/10.1017/S1446788700015391 Published online by Cambridge University Press

PROOF. $1 \to 2$. For $\lambda \in \Lambda$ let $M_\lambda = \{a \in A \mid a_\lambda = 0\}$. We shall show that these are the only P-ideals of A and hence the P-ideals are trivially ordered. For let M be a P-ideal of A. If for each $\lambda \in \Lambda$ there exists $0 < a \in M$ with $a_\lambda > 0$ then it follows that $M = \Sigma A_\lambda$ a contradiction. Thus $M \subseteq M_\lambda$ for some λ. Pick $0 < a_\lambda \in A_\lambda$. Then $a = (0, \cdots, 0, a_\lambda, 0, \cdots, 0) \notin M$ and since M is a prime subgroup of $(A, +)$ we have $M_\lambda = a' \subseteq M$. Thus $M = M_\lambda$.

$2 \to 1$. Let $\{s_\lambda \mid \lambda \in \Lambda\}$ be a basis for A. Since A satisfies (F) the mapping σ of a upon $(\cdots, s_\lambda' + a, \cdots)$ is an l-isomorphism of A into $\Sigma A/s_\lambda'$. s_λ' is a P ideal and hence the P-ideals of A/s_λ' satisfy the DCC. Let $\mathscr{I} = I/s_\lambda'$ be the minimal convex ring ideal of A/s_λ'. By Corollary I of Theorem 7.1 we have that A/s_λ' is an integral domain and hence $\mathscr{I}^2 \neq 0$. Thus by a theorem of Johnson (see [14] p. 132) A/s_λ' is o-simple and so s_λ' is a maximal L-ideal of A. Now $s_\alpha \in \bigcap_{\lambda \neq \alpha} s_\lambda' \setminus s_\alpha$ and hence since s_α' is a maximal L-ideal we have

$$A = \bigcap{}_{\lambda \neq \alpha} s_\lambda' + s_\alpha'.$$

If $0 < a \in A$ then $a = x + t$, where $x \in \bigcap_{\lambda \neq \alpha} s_\lambda'$ and $t \in s_\alpha'$. Thus $s_\alpha' + x = s_\alpha' + a$ and $s_\lambda' + x = s_\lambda'$ for all $\lambda \neq \alpha$. Therefore

$$x\sigma = (0, \cdots, 0, s_\alpha' + a, 0, \cdots, 0)$$

and so $A\sigma = \Sigma A/s_\lambda'$.

COROLLARY. (*Birkhoff and Pierce* [5]). *For an f-ring A the following are equivalent.*

1) $A = \Sigma_{i=1}^n A_i$, *where each A_i is an o-simple totally ordered integral domain.*

2) *The L-ideals of A satisfy the DCC and* $\operatorname{rad} A = 0$.

3) *There are only a finite number of L-ideals of A and* $\operatorname{rad} A = 0$.

PROOF. $1 \to 3$. If T is an L-ideal then $T = \Sigma(A_i \cap T)$ and since each A_i is o-simple $A_i \cap T = A_i$ or 0. Thus there are only a finite number of L-ideals.

$3 \to 2$. Trivial.

$2 \to 1$. Let $P_1, P_2, \cdots$ be the minimal prime subgroups of $(A, +)$. Then $P_1 \supset P_1 \cap P_2 \supset P_1 \cap P_2 \cap P_3 \supset \cdots$; for if $a_1 \in P_1 \setminus P_3$ and $a_2 \in P_2 \setminus P_3$ then $a_1 \wedge a_2 \in (P_1 \cap P_2) \setminus P_3$. Thus there are only a finite number of P_i and hence A has a finite basis and so satisfies (F).

Commutative laws for the various operators

Throughout this section y will denote d, v, c or e, X will denote P, SP, L or O and W will denote d, v, c, e, P, SP, L or O. We shall investigate when two of these operators commute.

1) For an archimedean l-group G, $(G^W)^e = (G^e)^W = G^e$.

https://doi.org/10.1017/S1446788700015391 Published online by Cambridge University Press

2) For an archimedean l-group $G, (G^W)^d \subseteq (G^d)^W$. For $W = v$, e, P or SP we have equality, but for $W = c$, L or O there need not be equality.

PROOF. G is a large l-subgroup of $(G^d)^W$ which is divisible. Thus G^W is large in $(G^d)^W$ and so $(G^W)^d \subseteq (G^d)^W$. Clearly $(G^v)^d = (G^d)^v = G^v$. If $0 < g \in (G^P)^d$ then $ng \in G^P$ for some $n > 0$ and hence $G^P = (ng)'' \oplus (ng)'$. Thus $(G^P)^d = ((ng)'')^d \oplus ((ng)')^d = (ng)^{**} \oplus (ng)^*$, where $*$ is the polar operation in $(G^P)^d$. Thus $(G^P)^d$ is a P-group and hence $(G^P)^d = (G^d)^P$.

If C is a polar in $(G^{SP})^d$ then $C \cap G^{SP}$ is a polar in G^{SP} and so $G^{SP} = (C \cap G^{SP}) \oplus (C \cap G^{SP})'$. Thus

$$(G^{SP})^d = (C \cap G^{SP})^d \oplus ((C \cap G^{SP})')^d = C \oplus C^*.$$

Therefore $(G^{SP})^d$ is an SP-group and so $(G^{SP})^d = (G^d)^{SP}$.

If $G = Z$ then $(G^c)^d = Z^d = Q \subset R = Q^c = (G^d)^c$. If $G = \sum_{i=1}^{\infty} Z_i$ then $(G^d)^L = (G^d)^O = \prod_{i=1}^{\infty} Q_i$ and $G^L = G^O = \prod_{i=1}^{\infty} Z_i$. Thus $a = (1, 1/2, 1/3, \cdots)$ belongs to $(G^d)^L \setminus (G^L)^d$ since no multiple of a belongs to G^L.

From the above computation we have.

3) For an abelian l-group G, $(G^X)^d \subseteq (G^d)^X$. For $X = P$ or SP there is equality, but for $X = L$ or O there need not be equality.

For the remainder of this section G will denote an archimedean l-group.

4) $(G^W)^v \subseteq (G^v)^W$. For $W = d$, e or SP we have equality, but for $W = c, P, O$ or L there need not be equality.

PROOF. $(G^v)^W$ is a vector lattice. This is clear except for $(G^v)^L$, but if $\{a_\lambda \mid \lambda \in \Lambda\}$ is a disjoint subset of G^v and $0 < r \in R$ then $r(\bigvee a_\lambda) = \bigvee ra_\lambda$ since $x \dashrightarrow rx$ is a p endomorphism of G^v and hence has a unique extension to $(G^v)^L$. Thus it follows that $(G^v)^L$ is also a vector lattice. Now since G^W is large in the vector lattice $(G^v)^W$ we have $(G^W)^v \subseteq (G^v)^W$.

Now let $G = \prod_\Lambda Z_\lambda$, where Λ is an infinite set. Then

$$G^v = \{r_1 g_1 + \cdots + r_t g_t \mid r_i \in R, g_i \in G \text{ and } t > 0\} = T.$$

For clearly T is a subspace of ΠR_λ and hence it suffices to prove that

$$(r_1 g_1 + \cdots + r_t g_t) \vee 0 \in T.$$

Consider the λ-th component

$$(r_1 g_1 + \cdots + r_t g_t)_\lambda = (r_1 g_1)_\lambda + \cdots + (r_t g_t)_\lambda.$$

If this is negative then replace $(g_i)_\lambda$ by 0 in each of the g_i. Do this for each λ and call the new element $\bar{g}_i$. Then $(r_1 g_1 + \cdots + r_t g_t) \vee 0 = r_1 \bar{g}_1 + \cdots + r_t \bar{g}_t \in T$ and hence $(G^c)^v = G^v \subset \prod R_\lambda = (G^v)^e$. Now let $H = \sum Z_\lambda$. Then $H^L = H^O = \prod Z_\lambda$, $H^v = \sum R_\lambda$ and $(H^v)^L = (H^v)^O = \prod R_\lambda$. Thus

https://doi.org/10.1017/S1446788700015391 Published online by Cambridge University Press

$$(H^L)^v = (H^O)^v = T \subset \prod R_\lambda = (H^v)^L = (H^v)^O.$$

Next let G be the subgroup of $\prod_{i=1}^{\infty} R_i$ generated by $\sum R_i$, $a = (1, 1, \cdots)$ and $b = (\pi + 1/2, \pi - 1/3, \pi + 1/4, \pi - 1/5, \cdots)$. Then G is the direct sum of $\sum R_i$ and the cyclic groups generated by a and b. It is reasonably easy to check that G is a P-group but G^v is not a P-group.

Finally we show that $(G^{SP})^v$ is an SP-group and hence $(G^{SP})^v = (G^v)^{SP}$. For let C be a polar in $(G^{SP})^v$. Then $C \cap G^{SP}$ is a polar in G^{SP} and hence

$$G^{SP} = (C \cap G^{SP}) \oplus (C \cap G^{SP})'$$

and so since the operators ^{d} and ${}^\wedge$ preserve summands we have

$$(G^{SP})^v = (C \cap G^{SP})^v \oplus ((C \cap G^{SP})')^v.$$

But $(C \cap G^{SP})^v = C$ and so $(G^{SP})^v$ is an SP-group. For clearly $(C \cap G^{SP})^v \subseteq C$ and if $0 < c \in C$ then $c = x + y \in (C \cap G^{SP})^v \oplus ((C \cap G^{SP})')^v$. Thus $y \in C$ and so if $y \neq 0$ then $ny > g > 0$ for some $g \in G^{SP}$. Then $g \in C \cap G^{SP}$ and so $g \wedge y = 0$ a contradiction.

An element $s > 0$ in an l-group H is called *singular* if for each $a \in H$

$$0 \leqq a < s \text{ implies } a \wedge (s - a) = 0.$$

The following proposition is essentially due to Iwasawa, see [12] for a proof.

Proposition. *If G is an archimedean l-group then G^c is a vector lattice if and only if G contains no singular elements.*

Corollary. *If G is an archimedean l group with no singular elements then $(G^v)^c = (G^c)^v = G^c$.*

5) $(G^X)^c = (G^c)^X = G^c$ *for* $X = P$ *or* SP.

Proof. This follows from the fact that G^c is an SP-group (see [14] p. 91 for a proof).

6) $(G^L)^c \subseteq (G^c)^L = (G^c)^O = (G^O)^c \subseteq G^e$.

Proof. Since G^c is a P-group it follows from Theorem 2.9 that $(G^c)^L = (G^c)^O$. Now $G^L \subseteq G^O \subseteq (G^O)^c$ and since G^L is dense in G^O we have $(G^L)^c \subseteq (G^O)^c$. So we need to prove $(G^O)^c = (G^c)^O$.

We first show that $(G^O)^c$ is laterally complete and hence $(G^O)^c \supseteq (G^c)^O$. Let $\{a_\lambda \mid \lambda \in \Lambda\}$ be a disjoint subset of $(G^O)^c$. Now for each $\lambda \in \Lambda$, $(G^O)^c = a_\lambda{}^{**} \oplus a_\lambda{}^*$, and since G^O is a large P-subgroup of $(G^O)^c$ we have

$$G^O = (a_\lambda{}^{**} \cap G^O) \oplus (a_\lambda{}^* \cap G^O).$$

Now for each $\lambda \in \Lambda$ let b_λ be an upper bound for a_λ in G^O. Then without loss of generality $b_\lambda \in a_\lambda{}^{**} \cap G^O$ and hence the b_λ are disjoint in G^O and so $\vee b_\lambda$

https://doi.org/10.1017/S1446788700015391 Published online by Cambridge University Press

exists. Thus $\bigvee b_\lambda$ is an upper bound for the a_λ in G^O and so since $(G^O)^c$ is complete, $\bigvee a_\lambda$ exists.

We now show that $H = \mathcal{O}(G^c)$ is complete and so $(G^O)^c \subseteq (G^c)^O$. If $C \in P(G^c)$ then $G^c = C \oplus C'$ and so $G/C' \cong C$ is complete. Thus the groups $G^c{}_{\mathscr{C}}$ used in the construction of $\mathcal{O}(G^c)$ are complete. Also the map $\pi_{\mathscr{C}} x$ of $G^c{}_{\mathscr{C}}$ into $G^c{}_{\mathscr{A}}$ is onto a large subgroup of $G^c{}_{\mathscr{A}}$ and hence preserves all joins and intersections.

Thus without loss of generality, H is the set join of a directed set of complete l-groups $G^c{}_{\mathscr{C}}$ and if $\mathscr{A} \leqq \mathscr{C}$ then $G^c{}_{\mathscr{C}}$ is a complete l-subgroup of $G^c{}_{\mathscr{A}}$. Now let $\{a_\lambda \mid \lambda \in \Lambda\}$ be a subset of H that is bounded from above by $a \in H$. Then $a \in G^c{}_{\mathscr{C}}$ for some partition $\mathscr{C}$. By Theorem 2.9 each a_λ is the join of disjoint elements from G^c and of course each of these elements belongs to the complete l group $G^c{}_{\mathscr{C}}$ and they are bounded by a in $G^c{}_{\mathscr{C}}$. It follows that each $a_\lambda \in G^c{}_{\mathscr{C}}$ and so $\bigvee a_\lambda \in G^c{}_{\mathscr{C}} \subseteq H$.

7) $(G^c)^O = G^e$ if and only if G contains no singular elements.

PROOF. If G contains no singular elements then G^c is a vector lattice. Thus $(G^c)^L = ((G^d)^c)^L = G^e$ (see [10]). If $G^e = (G^c)^O$ then $(G^c)^O$ is a vector lattice and hence contains no singular element. If $0 < g \in G^c$ is singular in G^c and $C \in P(G^c)$ then $C' + g$ is singular in G^c/C' (see [10]). It follows that $\tilde{g}$ is singular in $\mathcal{O}(G)$. Thus G^c contains no singular elements and hence is a vector lattice. Thus G contains no singular elements.

REMARKS. If G has a basis then in [10] it is shown that $(G^L)^c = (G^c)^L$. whether or not this is always the case is an open question. In Section 2 we showed that $(G^L)^{SP} \subseteq G^O$ and equality need not hold. If G is archimedean then do we have equality? If so then $G^L \subseteq (G^L)^c \rightarrow (G^L)^{SP} \subseteq (G^L)^c \rightarrow (G^O)^c = ((G^L)^{SP})^c \subseteq (G^L)^c$ and hence $(G^c)^L = (G^L)^c$, since by (6) $(G^L)^c \subseteq (G^c)^L \subseteq (G^O)^c$.

References

[1] I. Amemiya, 'A general spectral theory in semi-ordered linear spaces', *J. Fac. Sci. Hokkaido Univ.* 12 (1953), 111–156.

[2] S. Bernau, 'Unique representations of lattice groups and normal Archimedean lattice rings', *Proc. London Math. Soc.* 15 (1965), 599–631.

[3] S. Bernau, 'Orthocompletions of lattice groups', ibid. 16 (1966), 107–130.

[4] A. Bigard, *Contribution à la théorie des groups reticules*, (Thèse Univ. Paris (1969)).

[5] G. Birkhoff and T. Pierce, 'Lattice ordered rings', *An. Acad. Brasil.* Ci 28 (1956), 41–69.

[6] R. Bleier, *Free vector lattices*, (Dissertation Tulane University (1971)).

[7] D. Chambless, *The representation and structure of lattice-ordered groups and f-rings*, (Dissertation Tulane University (1971)).

[8] D. Chambless, 'Representation of l-groups by almost-finite quotient maps', *Proc. Amer. Math. Soc.* 28 (1971), 59–62.

[9] P. Conrad, 'The lateral completion of a lattice-ordered group', *Proc. London Math. Soc.* 19 (1969), 444–486.

[10] P. Conrad, 'The essential closure of an archimedean lattice-ordered group', *Duke Math. J.* 38 (1971) 151–160.

https://doi.org/10.1017/S1446788700015391 Published online by Cambridge University Press

[11] P. Conrad, 'Minimal vector lattice covers', *Bull. Australian Math. Soc.* 5 (1971), 35–39.
[12] P. Conrad and D. McAlister, 'The completion of a lattice ordered group', *J. Australian Math. Soc.* 9 (1962), 182–208.
[13] P. Cornad, and J. Diem, 'The ring of polar preserving endomorphisms of an l-group', *Illinois J. Math.* 15 (1971), 222–240.
[14] L. Fuchs, *Partially ordered algebraic systems*, (Pergamon Press, Oxford, London, New York, Paris) (1963).
[15] M. Henriksen and J. Isbell, 'Lattice-ordered rings and function rings', *Pacific J. Math.* 12 (1962), 533–565.
[16] J. Jakubik, 'Representations and extensions of l-groups', *Czech. Math. J.* 13 (1963), 267–283.
[17] D. Johnson, 'A structure theory for a class of lattice-ordered rings', *Acta Math.* 104 (1960), 163–215.
[18] H. Nakano, *Modern spectral theory*, Tokyo Math. Book Series II, (Maruzen Tokyo) (1950).
[19] A. Pinsker, 'Extended semiordered groups and spaces', *Uchen. Zapiski Leningrad Gos. Ped. Inst.* 86 (1949), 236–265.
[20] F. Sik, 'Zur Theorie der halbgeordneten Gruppen', *Czech. Math. J.* 6 (1956), 1–25.
[21] T. Speed, 'On lattice ordered groups', (preprint).
[22] S. Steinberg, *Lattice-ordered rings and modules*, (Thesis University Illinois (1970)).
[23] A. Vecksler, 'Structural orderability of algebras and rings', *Soviet Math. Dokl.* 6 (1965), 1201–1204.
[24] A. Vecksler, 'On the partial orderability of rings and algebras', ibid. 11 (1970), 175–179.
[25] B. Vulich, *Introduction to the theory of partially ordered spaces*, (Wolters-Noordhoff, Groningen 1967).

Department of Mathematics
University of Kansas
Lawrence, Kansas 66044
U. S. A.