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Characterizations of Outer Generalized Inverses

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Abstract. Let R be a ring and $b, c \in R$. In this paper, we give some characterizations of the (b, c)-inverse in terms of the direct sum decomposition, the annihilator, and the invertible elements. Moreover, elements with equal (b, c)-idempotents related to their (b, c)-inverses are characterized, and the reverse order rule for the (b, c)-inverse is considered.

1 Introduction

Moore–Penrose inverses, Drazin inverses, and group inverse, as well as classical generalized inverses, are special types of outer inverses. In [7], Drazin introduced a new class of outer inverse in a semigroup and called it (b, c)-inverse.

Definition 1.1 Let R be an associative ring and let $b, c \in R$. An element $a \in R$ is (b, c)-*invertible* if there exists $y \in R$ such that

$$y \in (bRy) \cap (yRc), \quad yab = b, \quad cay = c.$$

If such y exists, it is unique and is denoted by $a^{\|(b,c)}$.

From [7], we know that the Moore–Penrose inverse of a, with respect to an involution $*$ of R, is the (a^*, a^*)-inverse of a, the Drazin inverse of a is the (a^j, a^j)-inverse of a for some $j \in \mathbb{N}$, in particular, the group inverse of a is the (a, a)-inverse of a.

Given two idempotents e and f, Drazin introduced the Bott–Duffin (e, f)-inverse in [7], which can be considered as a particular cases of the (b, c)-inverse. In 2014, Kantún–Montiel introduced the image-kernel (p, q)-inverse for two idempotents p and q, and pointed out that an element a is image-kernel (p, q)-invertible if and only if it is Bott–Duffin $(p, 1-q)$-invertible [10, Proposition 3.4]. In [12], elements with equal idempotents related to their image-kernel (p, q)-inverses are characterized in terms of classical invertibility. The topics of research on the image-kernel (p, q)-inverse and the Bott–Duffin (e, f)-inverse attract wide interest (see [2–4, 6, 7, 9, 10, 12]).

This article is motivated by the papers [7,12]. In [7], as a generalization of (b, c)-inverse, hybrid (b, c)-inverse, and annihilator (b, c)-inverse were introduced. In Section 3, it is shown that if the (b, c)-inverse of a exists, then both b and c are regular.

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Further, under the natural hypothesis of both b and c regular, some characterizations of the (b,c)-inverse are obtained in terms of the direct sum decomposition, the annihilator, and the invertible elements. In particular, we will prove that (b,c)-inverse, hybrid (b,c)-inverse, and annihilator (b,c)-inverse are coincident when cab is regular. Some results of the image-kernel (p,q)-inverse in [12] are generalized.

If a has a (b,c)-inverse, then both $a^{\|(b,c)}a$ and $aa^{\|(b,c)}$ are idempotents. These will be referred as to the (b,c)-idempotents associated with a. In [5], Castro-González, Koliha, and Wei characterized matrices with the same spectral idempotents corresponding to the Drazin inverses of these matrices. Koliha and Patrício [11] extend the results to the ring case. A similar question for the Moore–Penrose inverse was considered in [13]. In [12], Mosić gave some characterizations of elements that have the same idempotents related to their image-kernel (p,q)-inverses. It is of interest to know whether two elements in the ring have equal (b,c)-idempotents. In Section 4, some characterizations of those elements with equal (b,c)-idempotents are given. Moreover, the reverse order rule for the (b,c)-inverse is considered.

2 Preliminaries

Let R be an associative ring with unit 1. Let $a \in R$. Recall that a is a regular element if there exists $x \in R$ such that $a = axa$. In this case, the element x is called an *inner inverse* for a, and we will denote it by a^-. If the equation $x = xax$ is satisfied, then we say that a is *outer generalized invertible*, and x is called an *outer inverse* for a. An element x that is both inner and outer inverse of a and commutes with a, when it exist, must be unique and is called the *group inverse* of a, denoted by $a^{\#}$. From now on, $E(R)$ and $R^{\#}$ stand for the set of all idempotents and the set of all group invertible elements in R. For the sake of convenience, we introduce some necessary notation.

For an element $a \in R$ and $X \subseteq R$, we define

$$aR := \{ax : x \in R\}, \qquad Ra := \{xa : x \in R\};$$
$$l(X) := \{y \in R : yx = 0 \text{ for any } x \in X\}, \quad r(X) := \{y \in R : xy = 0 \text{ for any } x \in X\}.$$

In particular,

$$l(a) := \{y \in R : ya = 0\}, \qquad r(a) := \{y \in R : ay = 0\},$$
$$rl(a) = \{y : xy = 0, x \in l(a)\}, \qquad lr(a) = \{y : yx = 0, x \in r(a)\}.$$

Let $p, q \in E(R)$. An element $a \in R$ has an image-kernel (p,q)-inverse [10, 12] if there exists an element $c \in R$ satisfying

$$cac = c, \quad caR = pR, \quad (1 - ac)R = qR.$$

The image-kernel (p,q)-inverse is unique if it exists, and it will be denoted by $a^{\times}$. A generalization of the original Bott–Duffin inverse [1] was given in [7]: let $e, f \in E(R)$, an element $a \in R$ is Bott–Duffin (e,f)-invertible if there exist $y \in R$ such that $y = ey = yf$, $yae = e$, and $fay = f$. When $e = f$, the element y, if any, is given by $y = e(ae + 1 - e)^{-1}$, as for the original Bott–Duffin inverse.

The above-mentioned generalized inverses are particular cases of the (b,c)-inverse, where b and c are both idempotents. Hence, the research of (b,c)-inverses has great significance in the development of generalized inverse theory.

For future reference we state two known results.

Lemma 2.1 ([7, Theorem 2.2]) *For any given $a, b, c \in R$, there exists the (b, c)-inverse y of a if and only if $Rb = Rt$ and $cR = tR$, where $t = cab$.*

Lemma 2.2 ([7, Proposition 6.1]) *For any given $a, b, c \in R$, y is the (b, c)-inverse of a if and only if $yay = y$, $yR = bR$, and $Ry = Rc$.*

3 Some Characterizations of the Existence of (b, c)-inverses

First, we will give some lemmas that will be used in the sequel.

Lemma 3.1 *Let $a, y \in R$ such that y is an outer inverse of a. Then*

(i) $r(a) \cap yR = \{0\}$;
(ii) $l(a) \cap Ry = \{0\}$;
(iii) $Ray = Ry$;
(iv) $yaR = yR$.

Proof (i) Let $x \in r(a) \cap yR$. Then $ax = 0$ and there exists $g \in R$ such that $x = yg$. This gives that $ayg = 0$ and, thus, $yayg = yg = 0$. Therefore, $x = 0$.

(ii) Let $x \in l(a) \cap Ry$. Then $xa = 0$ and there exists $h \in R$ such that $x = hy$. It leads to $hya = 0$. Then $hyay = hy = 0$ and, thus, $x = 0$.

(iii) and (iv) From $yay = y$ it follows that $yaR = yR$ and $Ry = Ray$. ■

Lemma 3.2 *Let $a \in R$ be regular and $b \in R$. Then*

(i) *b is regular in case $Ra = Rb$;*
(ii) *$rl(a) = aR$ and $lr(a) = Ra$.*

Proof (i) Since $Ra = Rb$, there exist some $g, h \in R$ such that $a = gb$ and $b = ha$. Hence, using that a is regular, one can see $b = (ha)a^-a = ba^-a = ba^-gb$, which means that b is regular.

(ii) It is easy to check that $aR \subseteq rl(a)$. Note that $l(a) = l(aa^-) = R(1 - aa^-)$. For any $x \in rl(a)$, one can get $R(1 - aa^-)x = l(a)x = 0$. This gives $x = aa^-x \in aR$ and $rl(a) = aR$. Similar considerations apply to prove that $lr(a) = Ra$. ■

Proposition 3.3 *If a has a (b, c)-inverse, then b, c, and $t = cab$ are all regular.*

Proof Let y be the (b, c)-inverse of a. In view of Definition 1.1, one can see $b = yab \in (bRy)ab \subseteq bRb$. This gives that b is regular. In the same manner one can obtain that c is regular. Now, on account of Lemma 2.1, we have $Rb = Rt$ and $cR = tR$ since the (b, c)-inverse of a exists. From Lemma 3.2, we conclude that t is regular. ■

In what follows, we will give necessary and sufficient conditions for the existence of the (b, c)-inverse when $t = cab$ is regular.

Theorem 3.4 *Let $a, b, c \in R$. If $t = cab$ is regular, then the following statements are equivalent:*

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(i) *a has a* (b,c)*-inverse.*
(ii) $r(a) \cap bR = \{0\}$ *and* $R = abR \oplus r(c)$.
(iii) $r(t) = r(b)$ *and* $tR = cR$.
(iv) $l(t) = l(c)$ *and* $Rt = Rb$.
(v) $l(t) = l(c)$ *and* $r(t) = r(b)$.

Proof (i)⇒(ii) Suppose that y is the (b,c)-inverse of a. By Lemma 2.2, $yay = y$, $yR = bR$, and $Ry = Rc$. By Lemma 3.1(i), one can see that $r(a) \cap yR = \{0\}$; it follows that $r(a) \cap bR = \{0\}$. Since $ay \in E(R)$, we have the decomposition $R = ayR \oplus r(ay)$. From $yR = bR$ we obtain $ayR = abR$. By Lemma 3.1(iii) and $Ry = Rc$, then $Ray = Rc$ and hence $r(ay) = r(c)$. Consequently, we have $R = abR \oplus r(c)$.

(ii)⇒(iii) It is clear that $r(b) \subseteq r(t)$. For any $x \in r(t)$, we have $tx = cabx = 0$. This means that $abx \in r(c)$. Using that $r(c) \cap abR = \{0\}$, we conclude that $abx = 0$. Then $bx \in r(a) \cap bR = \{0\}$. This implies that $bx = 0$ and, thus, $x \in r(b)$. Therefore, $r(t) = r(b)$.

It is clear that $tR \subseteq cR$. Since $R = abR \oplus r(c)$, we can write $1 = abg + h$ where $g \in R$ and $h \in r(c)$. Premultiplying by c gives $c = cabg \in tR$, ensuring that $cR = tR$.

(iii)⇒(iv) Since $tR = cR$, we have $l(c) = l(t)$. It is clear the $Rt \subseteq Rb$. Using that t is regular and $r(t) = r(b)$, we obtain that $b(1 - t^- t) = 0$. Then $b = bt^- t$. Consequently, $Rt = Rb$.

(iv)⇒(v) It is clear.

(v)⇒(i) Since $r(t) = r(b)$ and t is regular, we can prove that $Rt = Rb$ as in the proof of (iii)⇒ (iv). Similarly, from $l(t) = l(c)$ and the fact that t is regular, we get $tR = cR$. On account of Lemma 2.1 we conclude that a has a (b,c)-inverse. ■

In Theorem 3.4, the implications (i)⇒(ii) and (ii)⇒(iii) are valid even if t is not regular. However, we will give a counterexample to show that (iii) does not imply (iv) in general when t is not regular.

Example 3.5 Set $R = \mathbb{Z}$, $a = b = 1$, and $c = 2$. Clearly, $tR = cR$ and $r(t) = r(b)$, but $Rb \neq Rt$.

When we replace the hypothesis that t is regular in Theorems 3.4 by the condition that both b and c are regular, we obtain the following result.

Theorem 3.6 *Let* $a, b, c \in R$. *If both* b *and* c *are regular, then the statements* (i)–(iv) *in Theorem* 3.4 *are equivalent.*

Proof We note that in item (iii) condition $tR = cR$, together with c being regular, implies that t is regular; in item (iv) $Rt = Rb$, together with b being regular, implies that t is regular. ■

Remark 3.7 The statements (v)⇒(i) in Theorem 3.4 is not true, when b and c are regular. For example, set $R = \mathbb{Z}$, $b = c = 1$, and $a = 2$. Then b and c are regular. It is easy to check that $l(t) = l(c)$ and $r(t) = r(b)$, but $t = 2$ is not regular. Then a is not (b,c)-invertible by Proposition 3.3.

As generalizations of (b,c)-inverses, hybrid (b,c)-inverses and annihilator (b,c)-inverses were introduced in [7].

Definition 3.8 Let $a,b,c,y \in R$. We say that y is a *hybrid* (b,c)*-inverse* of a if

$$yay = y, \quad yR = bR, \quad r(y) = r(c).$$

Definition 3.9 Let $a,b,c,y \in R$. We say that y is an *annihilator* (b,c)*-inverse* of a if

$$yay = y, \quad l(y) = l(b), \quad r(y) = r(c).$$

In [7], Drazin pointed out that for any given $a,b,c \in R$,

$$(b,c)\text{-invertible} \Rightarrow \text{hybrid}(b,c)\text{-invertible} \Rightarrow \text{annihilator}(b,c)\text{-invertible}.$$

In what follows, we will prove that the three generalized inverses are coincident whenever $t = cab$ is regular.

Theorem 3.10 *Let $a,b,c,y \in R$. If t is regular, then the following conditions are equivalent:*

(i) *y is the (b,c)-inverse of a.*
(ii) *y is the hybrid (b,c)-inverse of a.*
(iii) *y is the annihilator (b,c)-inverse of a.*

Proof (i)⇒(ii)⇒(iii) These implications are clear.

(iii)⇒(i) By Definition 3.9, we have $1-ay \in r(y) = r(c)$ and $1-ya \in l(y) = l(b)$. This implies that $c = cay$ and $b = yab$. Next, we will prove that $r(t) = r(b)$ and $l(t) = l(c)$. Combining with Theorem 3.4(v), we can find that

$$a \text{ is annihilator } (b,c)\text{-invertible} \Rightarrow a \text{ is } (b,c)\text{-invertible}.$$

It is clear that $r(b) \subseteq r(t)$. Let $w \in r(t)$. Then $cabw = 0$ and hence $abw \in r(c) = r(y)$. This implies that $yabw = 0$. Then $bw = 0$, since $yab = b$. This shows $r(t) \subseteq r(b)$. Therefore, $r(t) = r(b)$. Similarly, we can prove that $l(c) = l(t)$. Since a has a (b,c)-inverse z, a has the annihilator (b,c)-inverse z, and by the uniqueness we have $z = y$. ∎

Theorem 3.11 *Let $a,b,c \in R$. If both b and c are regular, then the statements* (i)–(iii) *in Theorem 3.10 are equivalent.*

Proof We only need to prove that (iii)⇒(i). If y is the annihilator (b,c)-inverse of a, then $l(y) = l(b)$; this gives that $rl(y) = rl(b)$. Since b and y are regular, we have $rl(b) = bR$ and $rl(y) = yR$ by Lemma 3.2(ii). This implies that $yR = bR$. Similarly, we can obtain that $Ry = Rc$. Thus, it follows that y is the (b,c)-inverse of a by Lemma 2.2. ∎

The following lemma it is well known.

Lemma 3.12 ([8, 14]) *Let $a \in R$ and $e \in E(R)$. Then the following conditions are equivalent:*

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(i) $e \in eaeR \cap Reae$.
(ii) $eae + 1 - e$ *is invertible (or* $ae + 1 - e$ *is invertible).*

Theorem 3.13 *Let* $a, b, c, d \in R$ *such that the* (b, c)*-inverse of* a *exists. Let* $e = bb^-$ *where* b^- *are fixed, but arbitrary inner inverses of* b*. Then the following statements are equivalent:*

(i) d *has a* (b, c)*-inverse.*
(ii) $e \in ea^{\|(b,c)}deR \cap Rea^{\|(b,c)}de$.
(iii) $a^{\|(b,c)}de + 1 - e$ *is invertible.*

In this case,

$$d^{\|(b,c)} = (a^{\|(b,c)}de + 1 - e)^{-1}a^{\|(b,c)}. \tag{3.1}$$

Proof First, as $a^{\|(b,c)}$ exists, we have $a^{\|(b,c)} \in bR \cap Rc$ by Lemma 2.2. Therefore,

$$a^{\|(b,c)} = bb^-a^{\|(b,c)} = a^{\|(b,c)}c^-c. \tag{3.2}$$

From Definition 1.1 we have that $b = a^{\|(b,c)}ab$. Combining with (3.2), we can write

$$b = ea^{\|(b,c)}c^-cab. \tag{3.3}$$

(i)⇒(ii) Suppose that $d^{\|(b,c)}$ exists. By Definition 1.1, we also have $c = cdd^{\|(b,c)}$. Substituting this into (3.3) yields

$$b = ea^{\|(b,c)}c^-(cdd^{\|(b,c)})ab = ea^{\|(b,c)}dd^{\|(b,c)}ab.$$

Multiplying on the right by b^-, we obtain $e = ea^{\|(b,c)}dd^{\|(b,c)}ae$. Since $d^{\|(b,c)} = ed^{\|(b,c)}$, which follows by interchanging $a^{\|(b,c)}$ and $d^{\|(b,c)}$ in (3.2), we get $e = ea^{\|(b,c)}ded^{\|(b,c)}ae$. This implies that $e \in ea^{\|(b,c)}deR$. Similarly, we can prove that $e \in Rea^{\|(b,c)}de$.

(ii)⇒(iii) See Lemma 3.12.

(iii)⇒(i) First, we note that $ea^{\|(b,c)} = a^{\|(b,c)}$ by (3.2). Set $x = ea^{\|(b,c)}de + 1 - e$. It is clear that $ex = xe$ and $ex^{-1} = x^{-1}e$. Write $y = x^{-1}a^{\|(b,c)}$. Next, we verify that y is the (b, c)-inverse of d.

Step 1 $ydy = y$. Indeed, using $a^{\|(b,c)} = ea^{\|(b,c)}$, we get

$$\begin{aligned} ydy &= x^{-1}a^{\|(b,c)}dx^{-1}a^{\|(b,c)} = x^{-1}ea^{\|(b,c)}dx^{-1}ea^{\|(b,c)} \\ &= x^{-1}(ea^{\|(b,c)}de + 1 - e)ex^{-1}a^{\|(b,c)} \\ &= x^{-1}ea^{\|(b,c)} = x^{-1}a^{\|(b,c)} = y. \end{aligned}$$

Step 2 $bR = yR$. On account of $a^{\|(b,c)} = ea^{\|(b,c)}$ and $(1 - e)b = 0$, one can get

$$b = x^{-1}(ea^{\|(b,c)}de + 1 - e)b = x^{-1}ea^{\|(b,c)}deb = x^{-1}a^{\|(b,c)}deb = ydeb \in yR.$$

Meanwhile, $y = x^{-1}a^{\|(b,c)} = x^{-1}ea^{\|(b,c)} = ex^{-1}a^{\|(b,c)} \in bR$. This guarantees $bR = yR$.

Step 3 $Rc = Ry$.

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From Definition 1.1, we have $c = caa^{\|(b,c)}$. This leads to $c = caxx^{-1}a^{\|(b,c)} = caxy \in Ry$. On the other hand, from (3.2) we conclude that

$$y = x^{-1}a^{\|(b,c)} = x^{-1}a^{\|(b,c)}c^{-}c \in Rc.$$

This means that $Rc = Ry$. ∎

Similarly, we can state the analogue of Theorem 3.13.

Theorem 3.14 *Let $a, b, c, d \in R$ such that the (b, c)-inverse of a exists. Let $f = c^{-}c$ where c^{-} are fixed, but arbitrary inner inverses of c. Then the following statements are equivalent:*

(i) *d has a (b, c)-inverse.*
(ii) *$f \in fda^{\|(b,c)}fR \cap Rfda^{\|(b,c)}f$.*
(iii) *$fda^{\|(b,c)} + 1 - f$ is invertible.*

In this case,

$$d^{\|(b,c)} = a^{\|(b,c)}(fda^{\|(b,c)} + 1 - f)^{-1}. \tag{3.4}$$

Remark 3.15 In case where both $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist, from Theorems 3.13 and 3.14, it can be concluded that

$$\begin{aligned}(a^{\|(b,c)}de + 1 - e)^{-1} &= d^{\|(b,c)}ae + 1 - e,\\ (fda^{\|(b,c)} + 1 - f)^{-1} &= fad^{\|(b,c)} + 1 - f.\end{aligned} \tag{3.5}$$

Indeed, since $d^{\|(b,c)} = (a^{\|(b,c)}de+1-e)^{-1}a^{\|(b,c)}$, we have $(a^{\|(b,c)}de+1-e)d^{\|(b,c)} = a^{\|(b,c)}$. Hence,

$$(a^{\|(b,c)}de + 1 - e)(d^{\|(b,c)}ae + 1 - e) = a^{\|(b,c)}ae + 1 - e = 1,$$

where the last identity is due to the fact that $a^{\|(b,c)}ae = e$, because $b = a^{\|(b,c)}ab$. Interchanging the roles of a and d in Theorem 3.13, it follows that

$$(d^{\|(b,c)}ae + 1 - e)(a^{\|(b,c)}de + 1 - e) = 1,$$

and, in consequence, the first identity in (3.5) holds. The second identity in (3.5) can be proved in the same manner.

For any two idempotents p and q, we replace b and c by p and $1 - q$ respectively in Theorems 3.13 and 3.14, and we obtain the following corollary.

Corollary 3.16 ([12, Theorem 3.3]) *Let $p, q \in E(R)$ and let $a \in R$ be such that $a^{\times}$ exists. Then for $d \in R$ the following statements are equivalent:*

(i) *$d^{\times}$ exists.*
(ii) *$1 - p + a^{\times}dp$ is invertible.*
(iii) *$q + (1 - q)da^{\times}$ is invertible.*

4 Characterizations of Elements with Equal (b,c)-idempotents

Let $a^{\|(b,c)}$ exist. Since $a^{\|(b,c)}$ is an outer inverse of a when it exists, both $a^{\|(b,c)}a$ and $aa^{\|(b,c)}$ are idempotents. These will be referred to as the (b,c)-idempotents associated with a. We are interested in finding characterizations of those elements in the ring with equal (b,c)-idempotents.

In what follows, we will give necessary and sufficient conditions for $aa^{\|(b,c)} = dd^{\|(b,c)}$. We first establish an auxiliary result.

Lemma 4.1 *Let $a,b,c,d \in R$ such that $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist. Let $e = bb^-$ and $f = c^-c$, where b^- and c^- are fixed, but arbitrary inner inverses of b and c, respectively. Then*

(i) $d^{\|(b,c)} = d^{\|(b,c)}aa^{\|(b,c)} = a^{\|(b,c)}ad^{\|(b,c)}$;
(ii) $a^{\|(b,c)} = a^{\|(b,c)}dd^{\|(b,c)} = d^{\|(b,c)}da^{\|(b,c)}$;
(iii) $e = ed^{\|(b,c)}aa^{\|(b,c)}de = ea^{\|(b,c)}ae = ed^{\|(b,c)}de$;
(iv) $f = fda^{\|(b,c)}ad^{\|(b,c)}f = fdd^{\|(b,c)}f = faa^{\|(b,c)}f$.

Proof (i) In view of (3.1) and (3.4), with the notation $e = bb^-$ and $f = c^-c$, we have

$$\begin{aligned} d^{\|(b,c)} &= (a^{\|(b,c)}de + 1 - e)^{-1}a^{\|(b,c)} = d^{\|(b,c)}aa^{\|(b,c)} \\ &= a^{\|(b,c)}(fda^{\|(b,c)} + 1 - f)^{-1} = a^{\|(b,c)}ad^{\|(b,c)}. \end{aligned}$$

(ii) We get these equalities by interchanging the roles of $a^{\|(b,c)}$ and $d^{\|(b,c)}$ in previous results.

(iii) By Definition 1.1, we have $b = d^{\|(b,c)}db$. Multiplying on the right by b^- gives $e = d^{\|(b,c)}de$. Similarly, $e = ea^{\|(b,c)}ae$. Multiplying (i) on the right by de leads to $e = ed^{\|(b,c)}aa^{\|(b,c)}de$.

(iv) By Definition 1.1, we have $c = cad^{\|(b,c)}$ and, multiplying on the left by c^-, we get $f = fdd^{\|(b,c)}$. Similarly, $faa^{\|(b,c)}f$. Multiplying (ii) on the left by fd, one can see that $f = fda^{\|(b,c)}ad^{\|(b,c)}f$. ■

Theorem 4.2 *Let $a,b,c,d \in R$ such that $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist. Then the following statements are equivalent:*

(i) $aa^{\|(b,c)} = dd^{\|(b,c)}$.
(ii) $aa^{\|(b,c)}dd^{\|(b,c)} = dd^{\|(b,c)}aa^{\|(b,c)}$.
(iii) $ad^{\|(b,c)}da^{\|(b,c)} = da^{\|(b,c)}ad^{\|(b,c)}$.
(iv) $ad^{\|(b,c)} \in R^\#$ *and* $(ad^{\|(b,c)})^\# = da^{\|(b,c)}$.
(v) $da^{\|(b,c)} \in R^\#$ *and* $(da^{\|(b,c)})^\# = ad^{\|(b,c)}$.

Proof (i)⇔(ii)⇔(iii) From Lemma 4.1 we obtain

$$\begin{aligned} aa^{\|(b,c)} &= aa^{\|(b,c)}dd^{\|(b,c)} = ad^{\|(b,c)}da^{\|(b,c)}; \\ dd^{\|(b,c)} &= dd^{\|(b,c)}aa^{\|(b,c)} = da^{\|(b,c)}ad^{\|(b,c)}. \end{aligned}$$

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This leads to

$$aa^{\|(b,c)} = dd^{\|(b,c)} \Leftrightarrow aa^{\|(b,c)}dd^{\|(b,c)} = dd^{\|(b,c)}aa^{\|(b,c)}$$
$$\Leftrightarrow ad^{\|(b,c)}da^{\|(b,c)} = da^{\|(b,c)}ad^{\|(b,c)}.$$

(iii)⇔(iv) Set $x = da^{\|(b,c)}$. We will prove that x is the group inverse of $ad^{\|(b,c)}$. Combining (iii) with Lemma 4.1, we get

$$xad^{\|(b,c)} = da^{\|(b,c)}ad^{\|(b,c)} = ad^{\|(b,c)}da^{\|(b,c)} = ad^{\|(b,c)}x,$$
$$ad^{\|(b,c)}xad^{\|(b,c)} = a(d^{\|(b,c)}da^{\|(b,c)})ad^{\|(b,c)} = a(a^{\|(b,c)}ad^{\|(b,c)}) = ad^{\|(b,c)},$$
$$xad^{\|(b,c)}x = xad^{\|(b,c)}da^{\|(b,c)} = xaa^{\|(b,c)} = da^{\|(b,c)}aa^{\|(b,c)} = x.$$

This implies that $ad^{\|(b,c)} \in R^{\#}$ and $(ad^{\|(b,c)})^{\#} = da^{\|(b,c)}$. Conversely, if the latter holds, then $da^{\|(b,c)}ad^{\|(b,c)} = ad^{\|(b,c)}da^{\|(b,c)}$.

(iii)⇔(v) The proof is similar to the previous equivalence. ∎

We state the result in terms of the other (b, c)-idempotent.

Theorem 4.3 *Let $a, b, c, d \in R$ such that $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist. Then the following statements are equivalent:*

(i) $a^{\|(b,c)}a = d^{\|(b,c)}d$.
(ii) $d^{\|(b,c)}da^{\|(b,c)}a = a^{\|(b,c)}ad^{\|(b,c)}d$.
(iii) $a^{\|(b,c)}dd^{\|(b,c)}a = d^{\|(b,c)}aa^{\|(b,c)}d$.
(iv) $a^{\|(b,c)}d \in R^{\#}$ *and* $(a^{\|(b,c)}d)^{\#} = d^{\|(b,c)}a$.
(v) $d^{\|(b,c)}a \in R^{\#}$ *and* $(d^{\|(b,c)}a)^{\#} = a^{\|(b,c)}d$.

Next, we consider conditions under which the reverse order rule for the (b, c)-inverse of the product ad, $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$ holds.

Theorem 4.4 *Let $a, b, c, d \in R$ such that $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist. Then the following statements are equivalent:*

(i) *ad has a (b, c)-inverse of the form* $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$.
(ii) $d^{\|(b,c)} = d^{\|(b,c)}add^{\|(b,c)}a^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}add^{\|(b,c)}$.
(iii) $a^{\|(b,c)} = a^{\|(b,c)}add^{\|(b,c)}a^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}ada^{\|(b,c)}$.

Proof (i)⇔(ii) We first assume that ad has a (b, c)-inverse given by $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$. Then Lemma 4.1 is true for $(ad)^{\|(b,c)}$ in place of $a^{\|(b,c)}$. It follows that

$$d^{\|(b,c)} = d^{\|(b,c)}ad(ad)^{\|(b,c)} = (ad)^{\|(b,c)}add^{\|(b,c)}.$$

Substituting $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$ yields

$$d^{\|(b,c)} = d^{\|(b,c)}add^{\|(b,c)}a^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}add^{\|(b,c)}.$$

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Conversely, if the latter identities hold then $y = d^{\|(b,c)}a^{\|(b,c)}$ is the (b,c)-inverse of ad. Indeed, since $d^{\|(b,c)}db = b$ and $c = cdd^{\|(b,c)}$, we have

$$yady = d^{\|(b,c)}a^{\|(b,c)}add^{\|(b,c)}a^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)},$$
$$yadb = d^{\|(b,c)}a^{\|(b,c)}adb = d^{\|(b,c)}a^{\|(b,c)}add^{\|(b,c)}db = d^{\|(b,c)}db = b,$$
$$cady = cadd^{\|(b,c)}a^{\|(b,c)} = cdd^{\|(b,c)}add^{\|(b,c)}a^{\|(b,c)} = cdd^{\|(b,c)} = c.$$

(ii)⇒(iii) By Lemma 4.1 we have $a^{\|(b,c)} = a^{\|(b,c)}dd^{\|(b,c)} = d^{\|(b,c)}da^{\|(b,c)}$. By (ii), one can see

$$a^{\|(b,c)} = a^{\|(b,c)}d(d^{\|(b,c)}add^{\|(b,c)}a^{\|(b,c)}) = (d^{\|(b,c)}a^{\|(b,c)}add^{\|(b,c)})da^{\|(b,c)}.$$

Hence, it is easy to get $a^{\|(b,c)} = a^{\|(b,c)}add^{\|(b,c)}a^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}ada^{\|(b,c)}$.

(iii)⇒(ii) The proof is similar to (ii)⇒(iii). ■

Theorem 4.5 *Let $a,b,c,d \in R$ such that $a^{\|(b,c)}$ and $d^{\|(b,c)}$ exist. Then the following statements are equivalent:*

(i) $a^{\|(b,c)}a = dd^{\|(b,c)}$.
(ii) $a^{\|(b,c)}dd^{\|(b,c)}a = dd^{\|(b,c)}aa^{\|(b,c)}$.
(iii) $d^{\|(b,c)}da^{\|(b,c)}a = da^{\|(b,c)}ad^{\|(b,c)}$.
(iv) $a^{\|(b,c)} = dd^{\|(b,c)}a^{\|(b,c)}$ *and* $d^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}a$.
(v) $a^{\|(b,c)}ad^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}a$ *and* $a^{\|(b,c)}dd^{\|(b,c)} = dd^{\|(b,c)}a^{\|(b,c)}$.

If any of the previous statements is valid, then $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$.

Proof (i)⇔(ii)⇔(iii) From Lemma 4.1 we obtain

$$a^{\|(b,c)}a = a^{\|(b,c)}dd^{\|(b,c)}a = d^{\|(b,c)}da^{\|(b,c)}a,$$
$$dd^{\|(b,c)} = dd^{\|(b,c)}aa^{\|(b,c)} = da^{\|(b,c)}ad^{\|(b,c)}.$$

Hence, it gives that

$$a^{\|(b,c)}a = dd^{\|(b,c)} \Leftrightarrow a^{\|(b,c)}dd^{\|(b,c)}a = dd^{\|(b,c)}aa^{\|(b,c)}$$
$$\Leftrightarrow d^{\|(b,c)}da^{\|(b,c)}a = da^{\|(b,c)}ad^{\|(b,c)}.$$

(i)⇔(iv) The necessary condition is immediate. Next, we assume that $a^{\|(b,c)} = dd^{\|(b,c)}a^{\|(b,c)}$ and $d^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}a$. Then we have $a^{\|(b,c)}a = dd^{\|(b,c)}a^{\|(b,c)}a$ and $dd^{\|(b,c)} = dd^{\|(b,c)}a^{\|(b,c)}a$. So $a^{\|(b,c)}a = dd^{\|(b,c)}$, as desired.

(v)⇔(i) The proof is similar to the above.

Finally, we show that $dd^{\|(b,c)} = a^{\|(b,c)}a$ implies that $(ad)^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}$. Since $d^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}a$, we have $d^{\|(b,c)} = d^{\|(b,c)}a^{\|(b,c)}add^{\|(b,c)}$. Moreover, since $d^{\|(b,c)} = d^{\|(b,c)}aa^{\|(b,c)}$ by Lemma 4.1, using $dd^{\|(b,c)} = a^{\|(b,c)}a$, it follows that

$$d^{\|(b,c)} = d^{\|(b,c)}aa^{\|(b,c)} = d^{\|(b,c)}aa^{\|(b,c)}aa^{\|(b,c)} = d^{\|(b,c)}add^{\|(b,c)}a^{\|(b,c)}.$$

By Theorem 4.4 our assertion is proved. ■

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